Piecewise Linear Wilson Lines

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Outline

Path-Ordered Exponentials

Wilson Line

$$\mathcal{U}[\mathcal{C}] = \mathcal{P}\,\mathsf{exp}\left(-\mathrm{i}g\int\limits_{\mathcal{C}}\mathrm{d}z^{\mu}\;\;A_{\mu}(z)\right)$$

$$= \mathcal{P}\,\mathsf{exp}\left(-\mathrm{i}g\int\limits_{a}^{b}\mathrm{d}\lambda\;\;\left(z^{\mu}\right)'\,A_{\mu}(\lambda)\right)$$

Path-Ordered Exponentials

Wilson Line

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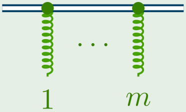
$$= \mathcal{P}\,\mathsf{exp}\left(-\mathrm{i}g\int\limits_{a}^{b}\mathrm{d}\lambda\;\;\left(z^{\mu}\right)'\,A_{\mu}(\lambda)\right)$$



$$A(\lambda_m)\cdots A(\lambda_1)$$

Path-Ordered Exponentials

Path-Ordering for Linear Lines

$$z^\mu = r^\mu + \hat{n}^\mu \lambda \qquad \lambda = a \dots b$$

$$\mathcal{P} \int\limits_{\mathcal{C}} \cdots \int\limits_{\mathcal{C}} \mathrm{d}z_1 \cdots \mathrm{d}z_m = m! \int\limits_a^b \int\limits_{\lambda_1}^b \cdots \int\limits_{\lambda_{m-1}}^b \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_m$$

$$= m! \int\limits_a^b \int\limits_a^{\lambda_m} \cdots \int\limits_a^{\lambda_2} \mathrm{d}\lambda_m \cdots \mathrm{d}\lambda_1$$

$$\lambda_m \geq \cdots \geq \lambda_1$$

Feynman Rules

Wilson Line Bounded From Above

$$\mathcal{U}_{(r\,;-\infty)} = \sum_{m=0}^{\infty} (-\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(k_m)\cdots\hat{n}\cdot A(k_1)\times$$

$$\times \int_{-\infty}^{0}\int_{-\infty}^{\lambda_m}\cdots\int_{-\infty}^{\lambda_2}\mathrm{d}\lambda_m\cdots\mathrm{d}\lambda_1\; \mathrm{e}^{-\mathrm{i}(r+\hat{n}\,\lambda_1)\cdot k_1}\ldots\mathrm{e}^{-\mathrm{i}(r+\hat{n}\,\lambda_m)\cdot k_m}$$

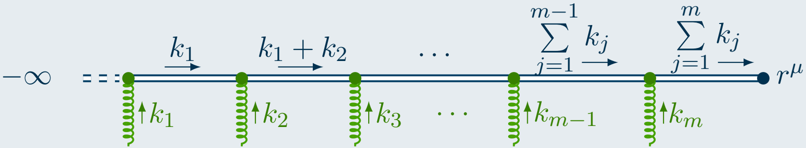
Feynman Rules

Wilson Line Bounded From Above

$$\mathcal{U}_{(r\,;\,-\infty)} = \sum_{m=0}^{\infty} (-\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(k_m)\cdots\hat{n}\cdot A(k_1)\times$$

$$K(j) = \sum_{l=1}^{j} k_l \qquad \times\, \mathrm{e}^{-\mathrm{i}r\cdot K} \prod_{j=1}^{m} \frac{\mathrm{i}}{\hat{n}\cdot K(j) + \mathrm{i}\eta}$$

Feynman Diagram



Feynman Rules

Feynman Rules for Linear Wilson Lines

1) Wilson line propagator:

$$= \frac{\mathrm{i}}{\hat{n}\cdot k + \mathrm{i}\eta}$$

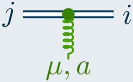
2) external point:

$$= e^{-\mathrm{i}r\cdot k}$$

3) infinite point:

$$= 1 \qquad (k=0)$$

4) Wilson vertex:



$$= -\mathrm{i}g\,\hat{n}^\mu\,(t^a)_{ij}$$

Feynman Rules

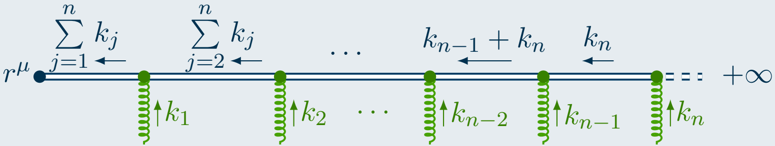
Wilson Line Bounded From Below

$$\mathcal{U}_{(+\infty\,;\,r)} = \sum_{m=0}^{\infty} (-\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(k_m)\cdots\hat{n}\cdot A(k_1)\times\cdots$$

$$\cdots\times\mathrm{e}^{-\mathrm{i}r\cdot K}\prod_{j=1}^{m}\frac{-\mathrm{i}}{\hat{n}\cdot\widetilde{K}(j)-\mathrm{i}\eta}$$

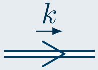
$$\widetilde{K}(j)=\sum_{l=1}^{j}k_{m-l+1}$$

Feynman Diagram

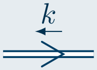


Feynman Rules

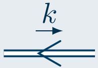
Reversals



$$= \frac{\mathrm{i}}{\hat{n}\cdot k + \mathrm{i}\eta}$$



$$= \frac{-\mathrm{i}}{\hat{n}\cdot k - \mathrm{i}\eta}$$



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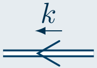
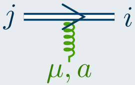
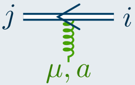


$$= \frac{\mathrm{i}}{\hat{n}\cdot k + \mathrm{i}\eta}$$



$$= -\mathrm{i}\, g\, \hat{n}^{\mu}\, (t^{a})_{ij}$$



$$= \mathrm{i}\, g\, \hat{n}^{\mu}\, (t^{a})_{ij}$$

More Types

Finite Line

$$\mathcal{U}_{(b\,;\,a)} = \sum_{n=0}^{\infty} (-\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(k_m)\cdots \hat{n}\cdot A(k_1) \times \cdots$$

$$\cdots \times \sum_{l=0}^{m} \mathrm{e}^{-\mathrm{i}a\cdot K(l)}\mathrm{e}^{-\mathrm{i}b\cdot K(m-l)} \prod_{j=1}^{l} \frac{-\mathrm{i}}{\hat{n}\cdot \widetilde{K}(j)} \prod_{j=l+1}^{m} \frac{\mathrm{i}}{n\cdot K(j)}$$

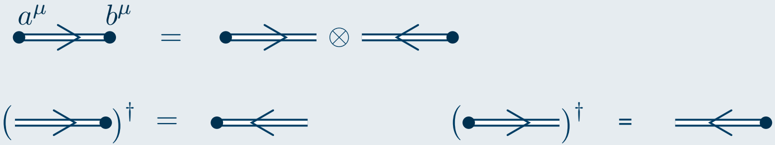
Hermitian Conjugate

$$\mathcal{U}^{\dagger}_{(r\,;\,-\infty)} = \sum_{n=0}^{\infty} (\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(k_1)\cdots \hat{n}\cdot A(k_m) \times \cdots$$

$$\cdots \times \mathrm{e}^{-\mathrm{i}b\cdot K} \prod_{j=1}^{m} \frac{\mathrm{i}}{\hat{n}\cdot K(j) + \mathrm{i}\eta}$$

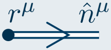
More Types

Finite Lines and Hermitian Conjugates



Different Types

Semi-Infinite Lines and Path Reversals



$$(-\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{-\mathrm{i}r\cdot K} \prod_j^m \frac{-\mathrm{i}}{\hat{n}\cdot\widetilde{K}(j) - \mathrm{i}\eta} \overset{\mathsf{N}}{=} A^m(r,\hat{n})$$

$$(\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{-\mathrm{i}r\cdot K} \prod_j^m \frac{-\mathrm{i}}{\hat{n}\cdot K(j) - \mathrm{i}\eta} \overset{\mathsf{N}}{=} B^m(r,\hat{n})$$

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$$(-\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{-\mathrm{i}r\cdot K} \prod_j^m \frac{\mathrm{i}}{\hat{n}\cdot K(j) + \mathrm{i}\eta} = B^m(r,-\hat{n})$$

Relation Between A^m and B^m

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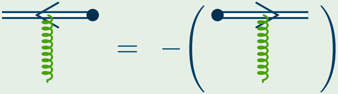
$$B^m(r,\hat{n}) = (-)^m A^m(r,\hat{n})\Big|_{(k_1,\ldots,k_m)\to(k_m,\ldots,k_1)}$$



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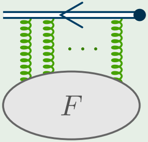


$$= (-)^m \int \left(\frac{\mathrm{d}k_i}{16\pi^4}\right)^m A^m(r,\hat{n}) F^{\mu_1\cdots\mu_m}_{a_1\cdots a_m}(k_m,\ldots,k_1)$$

(absorb gluon propagators in F)

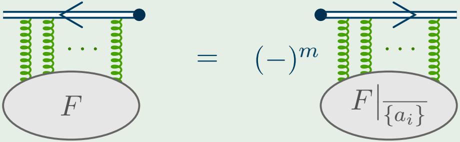
Relation Between A^m and B^m

Symmetrising the Blob

- symmetrise F simultaneously in k_i, μ_i and a_i
  (identical to making all crossings)
- because all Lorentz indices are contracted with the same $\hat{n}^\mu$,
  F is automatically symmetric in μ_i
- interchanging k_i and k_j is thus same as interchanging a_i and a_j
- sometimes F will have straightforward colour symmetry

Relation Between A^m and B^m

After Symmetrising the Blob



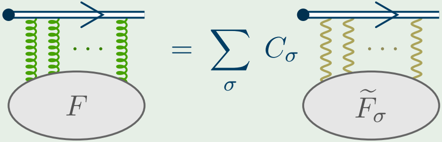
Easy Blob Example: 3-Gluon Vertex

Non-Trivial Colour Structure

Blob With Non-Trivial Colour Structure

$$F^{a_1\cdots a_m}_{\mu_1\cdots\mu_m}(k_1,\ldots,k_m)=\sum_{\text{perm}}C^{\sigma(a_1\cdots a_m)}\widetilde{F}_{\sigma(\mu_1\cdots\mu_m)}\left(\sigma(k_1,\ldots,k_m)\right)$$

Factorise Out Colour



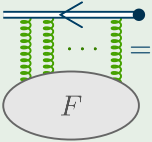
$$C_\sigma=t^{a_m}\ldots t^{a_1}C^{\sigma(a_1\cdots a_m)}$$

Non-Trivial Colour Structure

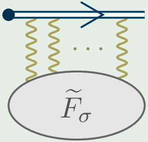
Blob With Non-Trivial Colour Structure

$$F^{a_1\cdots a_m}_{\mu_1\cdots\mu_m}(k_1,\ldots,k_m) = \sum_{\text{perm}} C^{\sigma(a_1\cdots a_m)}\,\widetilde{F}_{\sigma(\mu_1\cdots\mu_m)}\,(\sigma(k_1,\ldots,k_m))$$

Factorise Out Colour



$$= (-)^m \sum_{\sigma} C_{\overline{\sigma}}$$

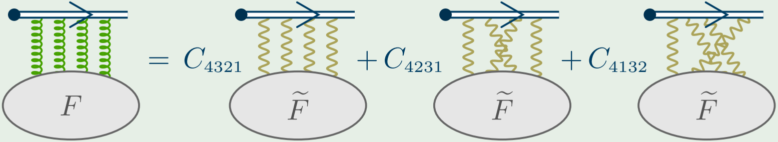


$$C_{\overline{\sigma}} = t^{a_1}\ldots t^{a_m}\,C^{\sigma(a_1\cdots a_m)}$$

Blob Example With Non-Trivial Colour Structure

$m = 4$

$$= \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \widetilde{F}_{\mu_1 \cdots \mu_4}(k_1, \ldots, k_4)$$



$$C_{ijkl} = t^{a_i} t^{a_j} t^{a_k} t^{a_l} \; f^{a_1 a_2 x} f^{x a_3 a_4}$$

Blob Example With Non-Trivial Colour Structure

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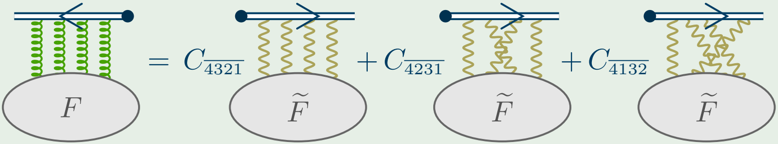


$$= C_{\overline{4321}} + C_{\overline{4231}} + C_{\overline{4132}}$$

$$C_{\overline{ijkl}} = C_{lkji} = t^{a_l} t^{a_k} t^{a_j} t^{a_i} \, f^{a_1 a_2 x} f^{x a_3 a_4} = C_{ijkl}$$

Blob Example With Non-Trivial Colour Structure

$m = 4$

Outline

Piecewise Path-Ordered Exponentials

Wilson Line With M Segments

$$\mathcal{U}(\lambda) = \begin{cases} \mathcal{U}^A(\lambda) & \lambda = a_1 \dots a_2 \\ \mathcal{U}^B(\lambda) & \lambda = a_2 \dots a_3 \\ \vdots & \\ \mathcal{U}^M(\lambda) & \lambda = a_M \dots a_{M+1} \end{cases}$$

Result for Full Wilson Line

$$\mathcal{U}_1 = \sum_{J=1}^{M} \mathcal{U}_1^J$$

$$\mathcal{U}_2 = \sum_{J=1}^{M} \mathcal{U}_2^J + \sum_{K=2}^{M} \sum_{J=1}^{K-1} \mathcal{U}_1^K \mathcal{U}_1^J$$

Piecewise Path-Ordered Exponentials

Result for Full Wilson Line

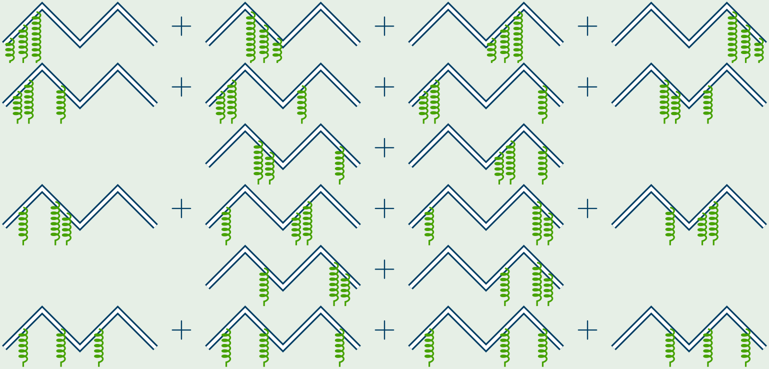
$$\mathcal{U}_3 = \sum_{J=1}^{M} \mathcal{U}_3^J + \sum_{J=2}^{M} \sum_{K=1}^{J-1} \left[\mathcal{U}_1^J \mathcal{U}_2^K + \mathcal{U}_2^J \mathcal{U}_1^K \right] + \sum_{J=3}^{M} \sum_{K=2}^{J-1} \sum_{L=1}^{K-1} \mathcal{U}_1^J \mathcal{U}_1^K \mathcal{U}_1^L$$

$$\vdots$$

$$\mathcal{U}_m = \sum_{i=1}^{m} \left[\left(\prod_{j=1}^{i} \sum_{J_j = i-j+1}^{J_{j-1}-1} \right)_{J_0 - 1 = M} \left(\begin{array}{c} \text{All terms of the form } \prod_{j=1}^{i} \mathcal{U}_{l_j}^{J_j} \\ \text{such that } \sum_{j=1}^{i} l_j = m \end{array} \right) \right]$$

Piecewise Path-Ordered Exponentials

Illustration for $\mathcal{U}_3$

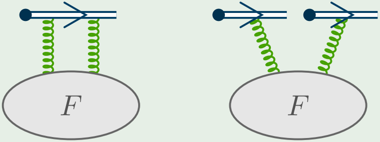


Outline

1. Linear Wilson Lines
2. Piecewise Wilson Lines
3. Piecewise Linear Wilson Lines: Methodology
4. Making Cuts
5. Example Calculations

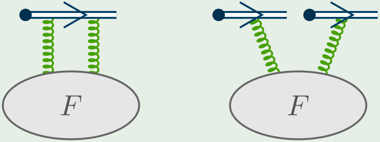
Basic Diagrams

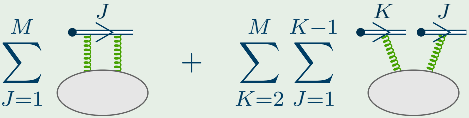
$m = 2$



Basic Diagrams

$m = 2$

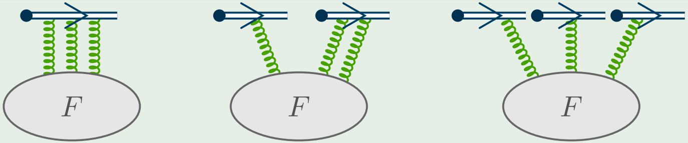




$$\sum_{J=1}^{M} \quad + \quad \sum_{K=2}^{M} \sum_{J=1}^{K-1}$$

Basic Diagrams

$m = 3$



Basic Diagrams

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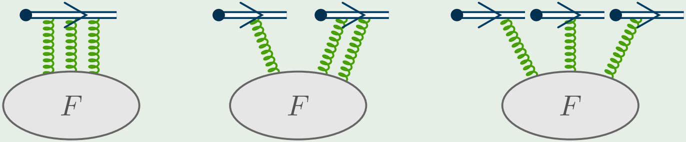
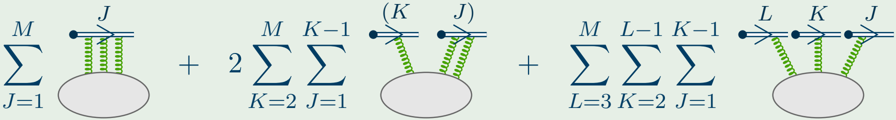
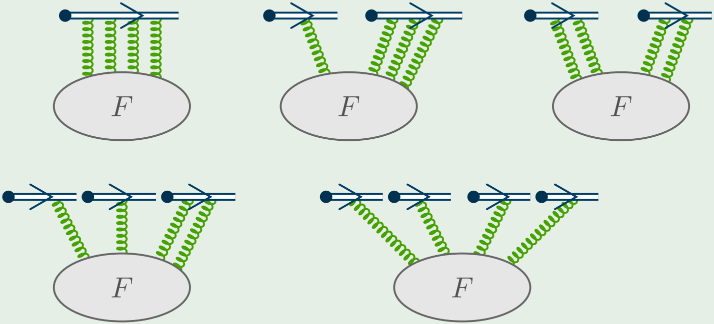




Basic Diagrams

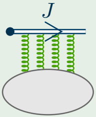
$m = 4$



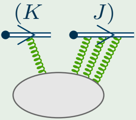
Basic Diagrams

$m = 4$

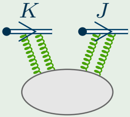
$\sum_{J=1}^{M}$



$+ \sum_{K=2}^{M}\sum_{J=1}^{K-1}\Bigg(2$

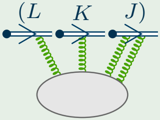


$+$

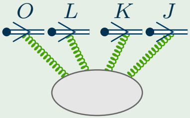


$\Bigg)$

$+ \; 3\sum_{L=3}^{M}\sum_{K=2}^{L-1}\sum_{J=1}^{K-1}$

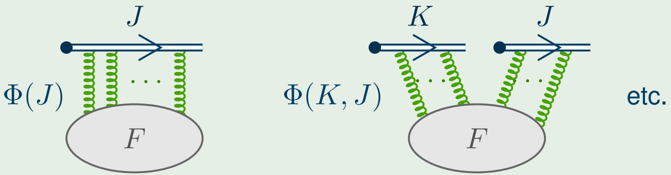


$+ \sum_{O=4}^{M}\sum_{L=3}^{O-1}\sum_{K=2}^{L-1}\sum_{J=1}^{K-1}$



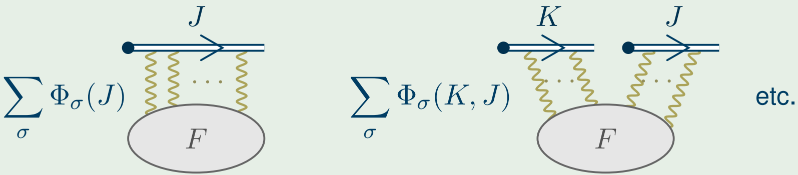
Path Constants

For a Blob With Trivial Colour Structure



$\Phi(J)$ $\Phi(K, J)$ etc.

For a Blob With Non-Trivial Colour Structure

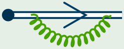
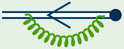


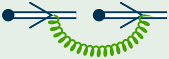
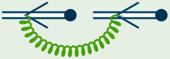
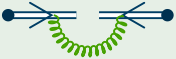
$\sum_{\sigma} \Phi_{\sigma}(J)$ $\sum_{\sigma} \Phi_{\sigma}(K, J)$ etc.

Blob Examples

$m = 2$

 $= \delta^{ab}\,\delta^{(4)}(k_1 - k_2)\,D_{\mu\nu}(k_1)$

 $=$  $\Rightarrow \Phi(J) = +1$

 $=$  $= -$ 

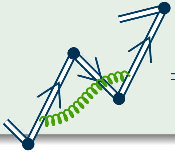
$$\Rightarrow \Phi(K, J) = (-)^{\Phi_K + \Phi_J}$$

$$\phi_J = \begin{cases} 0 \\ 1 \end{cases}$$

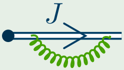
Blob Examples

$m = 2$

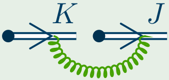
$$\mathcal{U}_2 = \sum_{J=1}^{M} \mathcal{U}_2^J + \sum_{K=2}^{M} \sum_{J=1}^{K-1} \mathcal{U}_1^K \mathcal{U}_1^J$$



$$= \sum_{J}^{M}$$



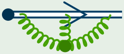
$$+ \sum_{K}^{M} \sum_{J}^{K-1} (-)^{\phi_K + \phi_J}$$



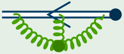
Blob Examples

$m = 3$

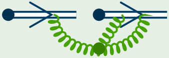
$$= g f^{abc} D^{k_1}_{\mu_1\nu_1} D^{k_2}_{\mu_2\nu_2} D^{k_3}_{\mu_3\nu_3}\, g^{\nu_1\nu_2}\,(k_1 - k_2)^{\nu_3}$$ +cross.



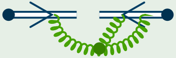
$=$



$\Rightarrow \Phi(J) = +1$



$= -$

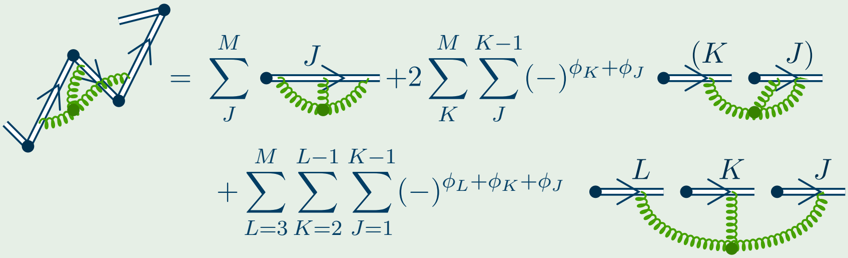


$\Phi(K, J) = (-)^{\phi_K + \phi_J}$

Blob Examples

$m = 3$

$$\mathcal{U}_3 = \sum_{J=1}^{M} \mathcal{U}_3^J + \sum_{K=2}^{M} \sum_{J=1}^{K-1} \left[\mathcal{U}_1^K \mathcal{U}_2^J + \mathcal{U}_2^K \mathcal{U}_1^J\right] + \sum_{L=3}^{M} \sum_{K=2}^{L-1} \sum_{J=1}^{K-1} \mathcal{U}_1^L \mathcal{U}_1^K \mathcal{U}_1^J$$

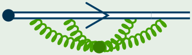
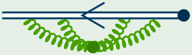


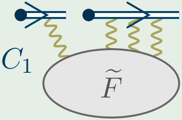
$$= \sum_{J}^{M} \quad + 2 \sum_{K}^{M} \sum_{J}^{K-1} (-)^{\phi_K + \phi_J} \quad + \sum_{L=3}^{M} \sum_{K=2}^{L-1} \sum_{J=1}^{K-1} (-)^{\phi_L + \phi_K + \phi_J}$$

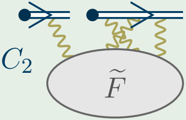
Blob Example With Non-Trivial Colour Structure

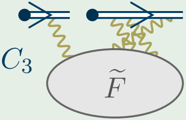
$m = 4$

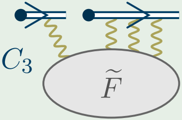
 $= \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \widetilde{F}_{\mu_1 \cdots \mu_4}(k_1, \ldots, k_4)$

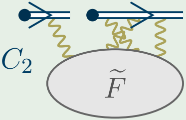
 $=$ 

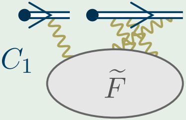
 $= C_1$ 

$+ C_2$ 

$+ C_3$ 

 $= C_3$ 

$+ C_2$ 

$+ C_1$ 

Outline

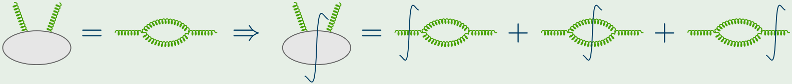
1. Linear Wilson Lines
2. Piecewise Wilson Lines
3. Piecewise Linear Wilson Lines: Methodology
4. Making Cuts
5. Example Calculations

Final-State Cut

Cutting

- cutting happens at infinity ⇒ no momentum cut in Wilson line
- cut line doesn't cross segment, only *between* segments

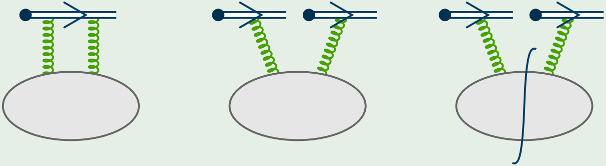
  ⇒ only blob is cut
- define cut blob as sum of possible cuttings, e.g.:



- denote number of segments before the cut by M_c

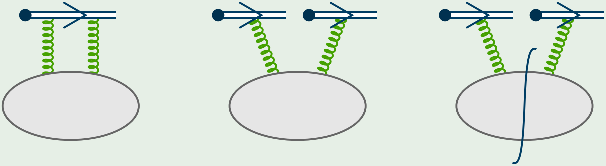
Expanding the Set of Basic Diagrams

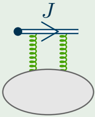
$m = 2$

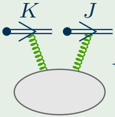


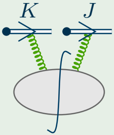
Expanding the Set of Basic Diagrams

$m = 2$



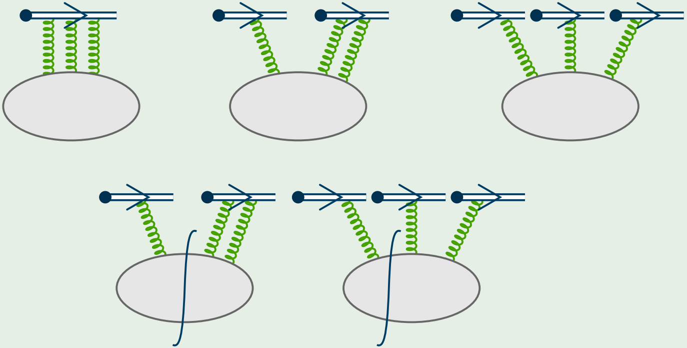
$\sum_{J=1}^{M}$ 

$+\left(\sum_{K=2}^{M_c}\sum_{J=1}^{K-1}+\sum_{K=M_c+2}^{M}\sum_{J=M_c+1}^{K-1}\right)$ 

$+\sum_{K=M_c+1}^{M}\sum_{J=1}^{M_c}$ 

Expanding the Set of Basic Diagrams

$m = 3$



Outline

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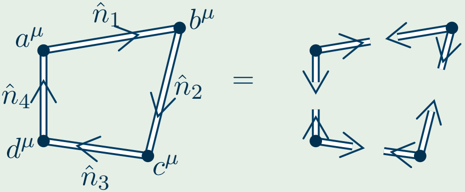
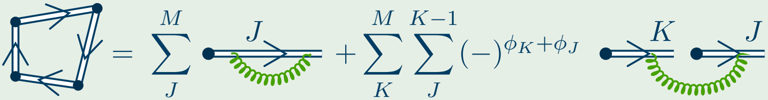
Quadrilateral Wilson Loop

Quadrilateral Wilson Loop



Quadrilateral Wilson Loop

First Order



Result for Light-Like Loop

$$\mathcal{U}_2 = \frac{\alpha_s C_F}{\pi} (-2\pi\mu^2)^\epsilon \, \Gamma(1-\epsilon) \times \cdots$$

$$\cdots \times \left[\frac{1}{\epsilon^2} \left((b-d)^2 - \mathrm{i}\eta \right)^\epsilon + \frac{1}{\epsilon^2} \left((c-a)^2 - \mathrm{i}\eta \right)^\epsilon \right]$$

Conjecture

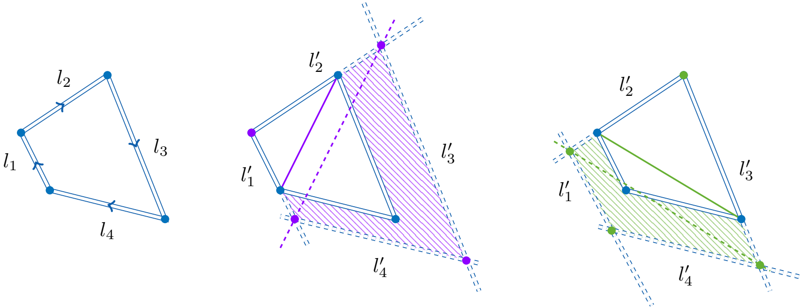
Geometric Evolution of Light-Like Quadrilateral

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\left\langle\frac{\delta}{\delta\ln\Sigma}\right\rangle\ln\mathcal{W}_{\gamma}=-\sum_{\text{cusps}}\Gamma_{\text{cusp}}$$

Gamma cusp at NLO:

$$\Gamma_{\text{cusp}}(g)=\frac{\alpha_s}{\pi}C_F+\left(\frac{\alpha_s}{\pi}\right)^2C_F\left(C_A\left(\frac{67}{36}-\frac{\pi^2}{12}\right)-N_f\frac{5}{18}\right)$$

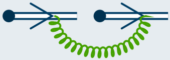
Conjecture



Reusability

Example

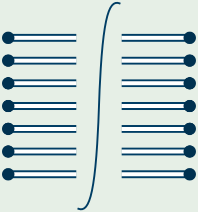
$$= \mathrm{i}\, g^2 \frac{C_F}{16\pi} \frac{\hat{n}^2}{\eta} \left(-4\pi\mu^2\right)^\epsilon \Gamma(\epsilon) X(\hat{n}^2, \epsilon)$$



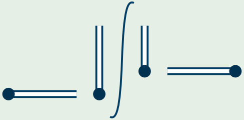
$$= g^2 \frac{C_F}{16\pi} \frac{\hat{n}_1 \cdot \hat{n}_2}{\hat{n}_1 \sqrt{\widetilde{n}_2 \cdot \hat{n}_2}} \left(2\pi \mathrm{i}\, \mu^2 \frac{1}{\eta} \sqrt{\frac{R}{N_2}}\right)^\epsilon Y_\epsilon(\mathrm{i}\eta\sqrt{R N_2})$$

Example

$pp \to 5$ jets in eikonal approximation



TMD Wilson Line Self Energy



Example

$\Rightarrow$ difference only in path parameters $r_{J_i}, \hat{n}_{J_i}$ and path constants $\Phi(J_i)$!

Conclusions

Conclusions & Outlook

- framework to minimize number of diagrams for piecewise linear Wilson lines
- lesser diagrams in exchange for more general (and thus more complicated) integrals
- only interesting for $M > 2$

- clean up result & calculate higher orders
- try framework for TMD Wilson line structure

Conclusions

Thank You!