Wigner Distributions and Orbital Angular Momentum of Quarks

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- Wigner distribution for the quarks
- Reduced wigner distributions in position and momentum space
- Model calculations for a dressed quark
- Quark orbital angular momentum
- Summary and conclusions

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Wigner distributions for quarks in the nucleon

• In quantum mechanics position and momentum operators do not commute and they cannot be measured simultaneously : one cannot define a joint position and momentum space distribution of quarks inside the nucleon

• Wigner distributions : not positive definite (no probabilistic interpretation); reduces to classical phase space distribution for $h \to 0$

• For a 1-D quantum system with wave function $\psi(x)$, Wigner distribution can be defined as

$$W(x,p) = \int dy e^{ip \cdot y} \psi^*(x - y/2)\psi(x + y/2)$$

• Matrix elements of Wigner operator for a nucleon state can be interpreted as distributions of the partons in 6-D space (3 position and 3 momentum)

X. Ji, PRL (2003); Belitsky, Ji, Yuan, PRD (2004)

Wigner Distributions (contd.)

• 5-D Wigner distributions (by integrating one position variable) can be studied in infinite momentum frame or light-front formalism

Lorce, Pasquini (2011)

• Advantage : boost invariant definition of Wigner distributions

• Integrating over $k_\perp$ Wigner distributions reduce to the Fourier transform of GPDs

• Integrating over transverse position $b_\perp$ they reduce to TMD correlators

• Wigner distributions are related to GPCFs and GTMDs

• Contains informations on both GPDs and TMDs

• Are related to the orbital angular momentum carried by quarks in the nucleon

• Model studies : constituent quark model and chiral quark soliton model in the above reference

• We calculate the reduced Wigner distributions for a quark dressed with a gluon in light-front Hamiltonian perturbation theory : relativistic composite spin $1/2$ state

Wigner Distributions : Definitions

Wigner distributions for the quarks are defined as

$$\rho^{[\Gamma]}(\vec{b}_\perp, \vec{k}_\perp, x, \sigma) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\Delta_\perp . b_\perp} W^{[\Gamma]}(\Delta_\perp, \vec{k}_\perp, x, \sigma);$$

$\Delta_\perp$ is momentum transfer of dressed quark in transverse direction and $\vec{b}_\perp$ is impact parameter conjugate to $\Delta_\perp$

$W^{[\Gamma]}$ is the quark-quark correlator given by

$$W^{[\Gamma]}(\vec{\Delta}_\perp, \vec{k}_\perp, x, \sigma) == \frac{1}{2} \int \frac{dz^- d^2 z_\perp}{(2\pi)^3} e^{i(xp^+ z^-/2 - k_\perp . z_\perp)}$$
$$\left\langle p^+, \frac{\Delta_\perp}{2}, \sigma \middle| \overline{\psi}(-\frac{z}{2}) \Omega \Gamma \psi(\frac{z}{2}) \middle| p^+, -\frac{\Delta_\perp}{2}, \sigma \right\rangle \Bigg|_{z^+=0}.$$

Ω is the gauge link needed for color gauge invariance : we use light-front gauge and take the gauge link to be unity

Γ represents the Dirac matrix defining the types of quark densities

Wigner Distributions : Model Calculations

We calculate the above Wigner distributions in a quark state dressed with a gluon

The state can be expanded in Fock space in terms of multi-parton light-front wave functions (LFWFs)

$$\left|p^+, p_\perp, \sigma\right\rangle = \Phi^\sigma(p) b^\dagger_\sigma(p)|0\rangle + \sum_{\sigma_1\sigma_2} \int [dp_1] \int [dp_2] \sqrt{16\pi^3 p^+} \delta^3(p - p_1 - p_2)$$
$$\Phi^\sigma_{\sigma_1\sigma_2}(p; p_1, p_2) b^\dagger_{\sigma_1}(p_1) a^\dagger_{\sigma_2}(p_2)|0\rangle;$$

Used light-front gauge, took gauge link as unity and used two-component formalism introduced in

Zhang, Harindranath, PRD (1993)

$\Phi^\sigma(p)$ gives the normalization of the wave function, $\Phi^\sigma_{\sigma_1\sigma_2}(p; p_1, p_2)$ is the two-particle light-front wave function (LFWF), related to the boost invariant wave function

single particle sector contributes through the normalization of the state, which is important to get the complete contribution at $x = 1$

Wigner Distributions : Model Calculations

Two-particle LFWFs can be calculated

$$\Psi^{\sigma a}_{\sigma_1\sigma_2}(x,q_\perp) = \frac{1}{\Big[m^2 - \frac{m^2+(q_\perp)^2}{x} - \frac{(q_\perp)^2}{1-x}\Big]} \frac{g}{\sqrt{2(2\pi)^3}} T^a \chi^\dagger_{\sigma_1} \frac{1}{\sqrt{1-x}}$$

$$\Big[-2\frac{q_\perp}{1-x} - -\frac{(\sigma_\perp.q_\perp)\sigma_\perp}{x} + \frac{im\sigma_\perp(1-x)}{x}\Big]\chi_\sigma(\epsilon_{\perp\sigma_2})^*.$$

χ is the two-component spinor

The state above mimicks the bound state of a two particle system. Note that for a bound state, the bound state mass M should be less than the sum of the masses of the constituents

Using this, we can calculate the Wigner distributions for given helicity of the initial and final target state in terms of overlaps of LFWFs

We use the symbol $\rho_{\lambda\lambda'}$ for Wigner distributions, where $\lambda(\lambda')$ is longitudinal polarization of target state(quark)

$+\vec{e_z}$ and $-\vec{e_z}$ correspond to helicity up and down of the target state, respectively

Wigner Distributions : Definitions

Wigner distribution of unpolarized quarks in unpolarized target state is given by

$$\rho_{UU}(\vec{b}_\perp, \vec{k}_\perp, x) = \frac{1}{2}\Big[\rho^{[\gamma^+]}(\vec{b}_\perp, \vec{k}_\perp, x, +\vec{e}_z) + \rho^{[\gamma^+]}(\vec{b}_\perp, \vec{k}_\perp, x, -\vec{e}_z)\Big]$$

Distortion due to longitudinal polarization of the target state :

$$\rho_{LU}(\vec{b}_\perp, \vec{k}_\perp, x) = \frac{1}{2}\Big[\rho^{[\gamma^+]}(\vec{b}_\perp, \vec{k}_\perp, x, +\vec{e}_z) - \rho^{[\gamma^+]}(\vec{b}_\perp, \vec{k}_\perp, x, -\vec{e}_z)\Big]$$

Distortion due to the longitudinal polarization of quarks :

$$\rho_{UL}(\vec{b}_\perp, \vec{k}_\perp, x) = \frac{1}{2}\Big[\rho^{[\gamma^+\gamma_5]}(\vec{b}_\perp, \vec{k}_\perp, x, +\vec{e}_z) + \rho^{[\gamma^+\gamma_5]}(\vec{b}_\perp, \vec{k}_\perp, x, -\vec{e}_z)\Big]$$

Distortion due to the correlation between the longitudinal polarized target state and quarks :

$$\rho_{LL}(\vec{b}_\perp, \vec{k}_\perp, x) = \frac{1}{2}\Big[\rho^{[\gamma^+\gamma_5]}(\vec{b}_\perp, \vec{k}_\perp, x, +\vec{e}_z) - \rho^{[\gamma^+\gamma_5]}(\vec{b}_\perp, \vec{k}_\perp, x, -\vec{e}_z)\Big]$$

Connection with GTMDs

- All the Wigner distributions introduced are related to the Fourier transforms of GTMDs (Defs to be shown later)

- ρ_{UU} is related to F_{11}, ρ_{UL} is related to G_{11}, ρ_{LU} to F_{14} and ρ_{LL} to G_{14}

- On integration over $k_\perp$ or $b_\perp$, ρ_{UL} and ρ_{LU} give zero : there is no GPD or TMD associated with these two. They give new information not contained in the GPDs and TMDs

- ρ_{UU} can be considered as the mother distribution for the GPD H and TMD f_1

- ρ_{LL} is the mother distribution for the GPD $\tilde{H}$ and TMD g_{1L}

Wigner Distributions : Model Calculations

In our model $\rho_{LU} = \rho_{UL}$

$$\rho^{[\gamma^+]}_{UU}(b_\perp, k_\perp, x) = N \int d\Delta_x \int d\Delta_y \frac{\cos(\Delta_\perp \cdot b_\perp)}{D(q_\perp)D(q'_\perp)} \left[I_1 + \frac{4m^2(1-x)}{x^2} \right];$$

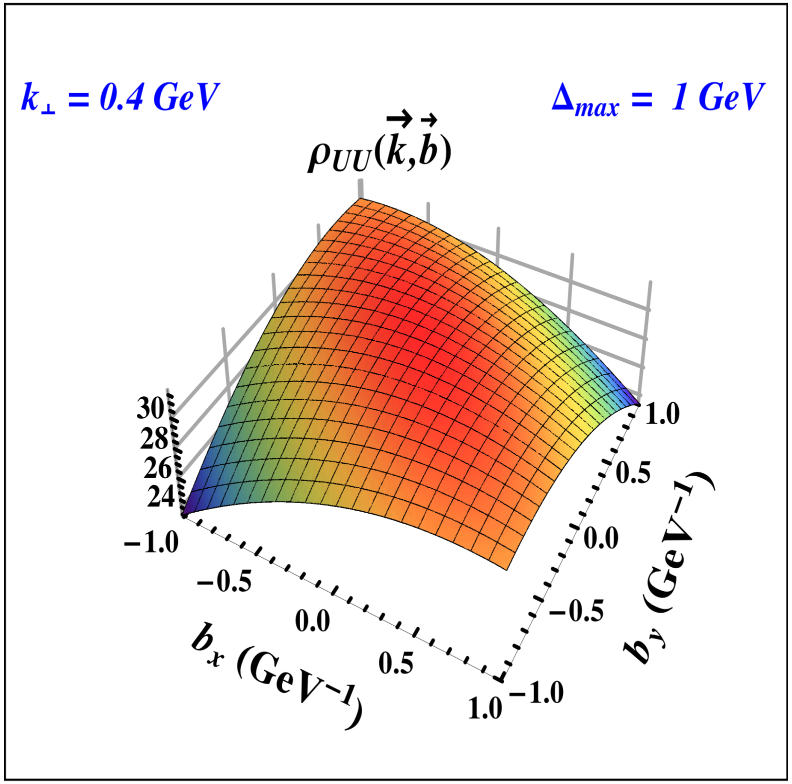
$$\rho^{[\gamma^+]}_{LU}(b_\perp, k_\perp, x) \quad = \quad N \int d\Delta_x \int d\Delta_y \frac{\sin(\Delta_\perp \cdot b_\perp)}{D(q_\perp)D(q'_\perp)} \left[4(k_x \Delta_y - k_y \Delta_x) \frac{(1+x)}{x^2(1-x)} \right];$$

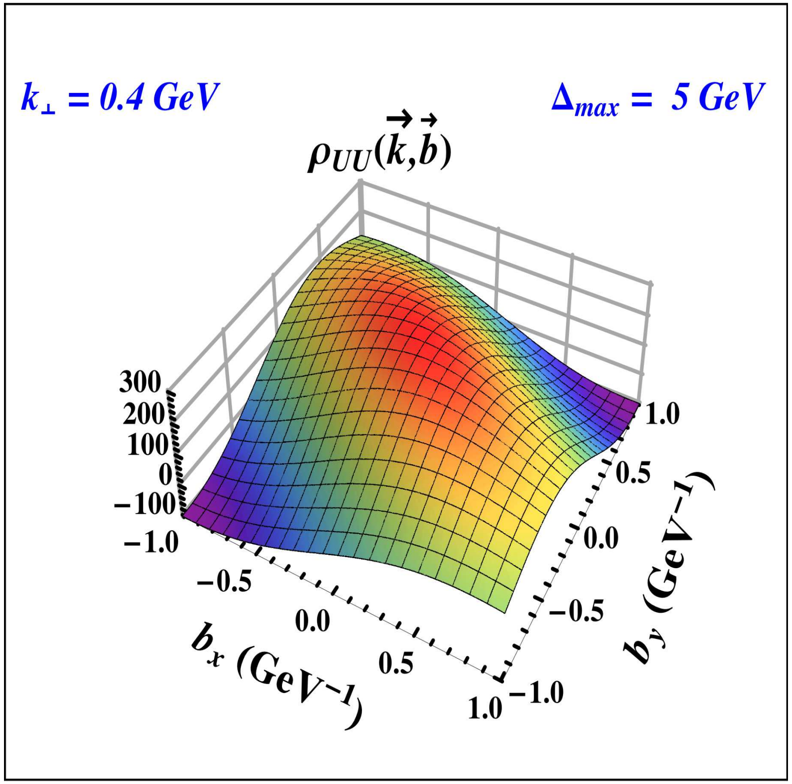
$$\rho^{[\gamma^+\gamma_5]}_{LL}(b_\perp, k_\perp, x) \quad = \quad N \int d\Delta_x \int d\Delta_y \frac{\cos(\Delta_\perp \cdot b_\perp)}{D(q_\perp)D(q'_\perp)} \left[I_1 - \frac{4m^2(1-x)}{x^2} \right];$$

$$D(k_\perp) = \left(m^2 - \frac{m^2 + (k_\perp)^2}{x} - \frac{(k_\perp)^2}{1-x} \right) \qquad I_1 = 4\left((k_\perp)^2 - \frac{\Delta_\perp^2 (1-x)^2}{4} \right) \frac{(1+x^2)}{x^2(1-x)^3}.$$

N is a normalization constant

Wigner Distributions : Results

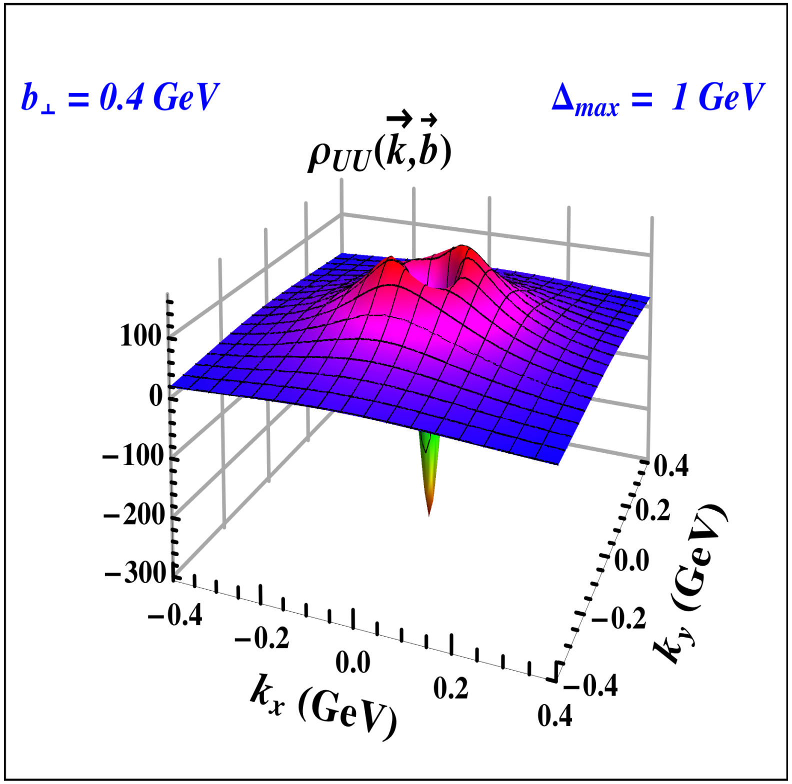


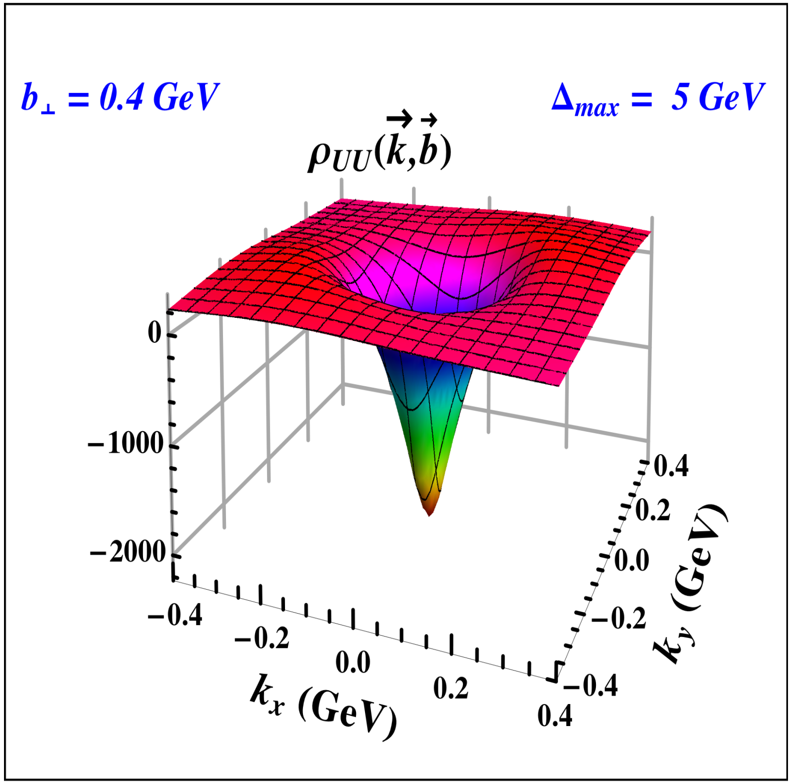


AM, Nair, Ojha, arXiv:1403.6233 [hep-ph]

- Integrated over x (all plots)
- Used an upper limit on $\Delta_\perp$ integration, $k_\perp$ in y direction
- Asymmetry related to OAM (no confining potential)

Wigner Distributions : Results

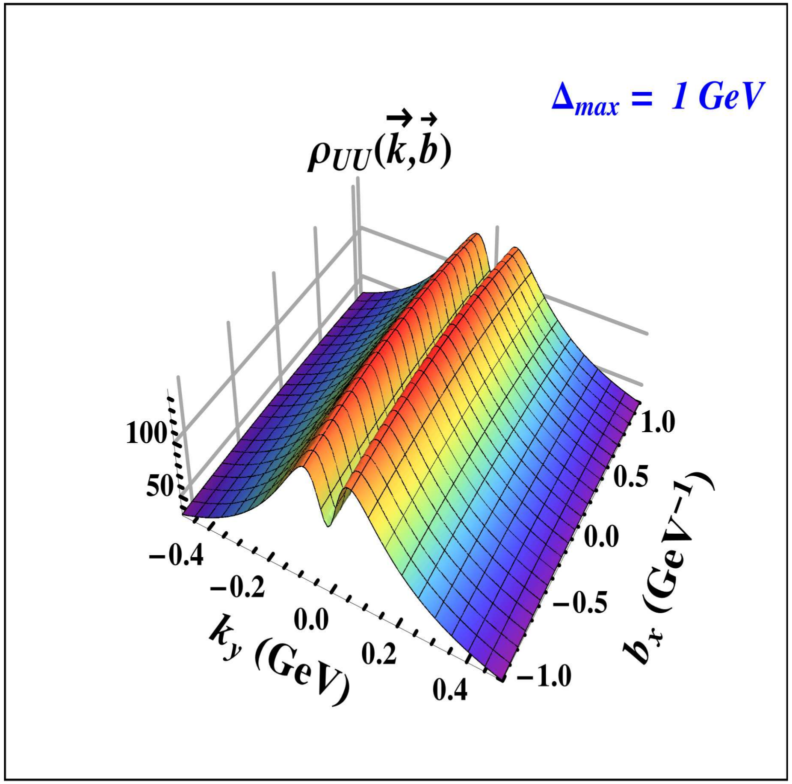


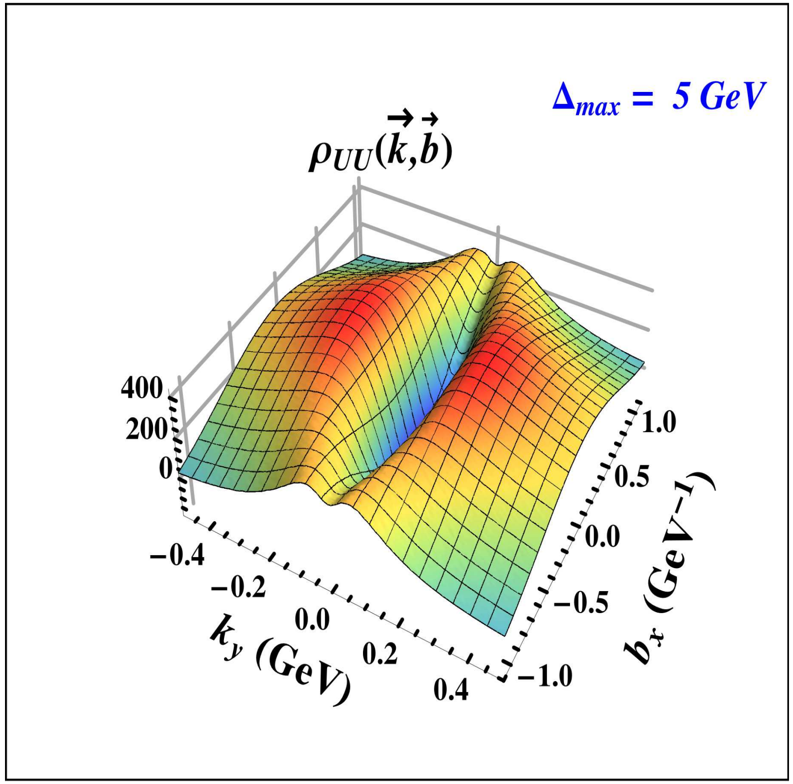


AM, Nair, Ojha, arXiv:1403.6233 [hep-ph]

- For all plots took $m = 0.33$ GeV
- Plots in $k_\perp$ space
- b in y direction

Wigner Distributions : Results

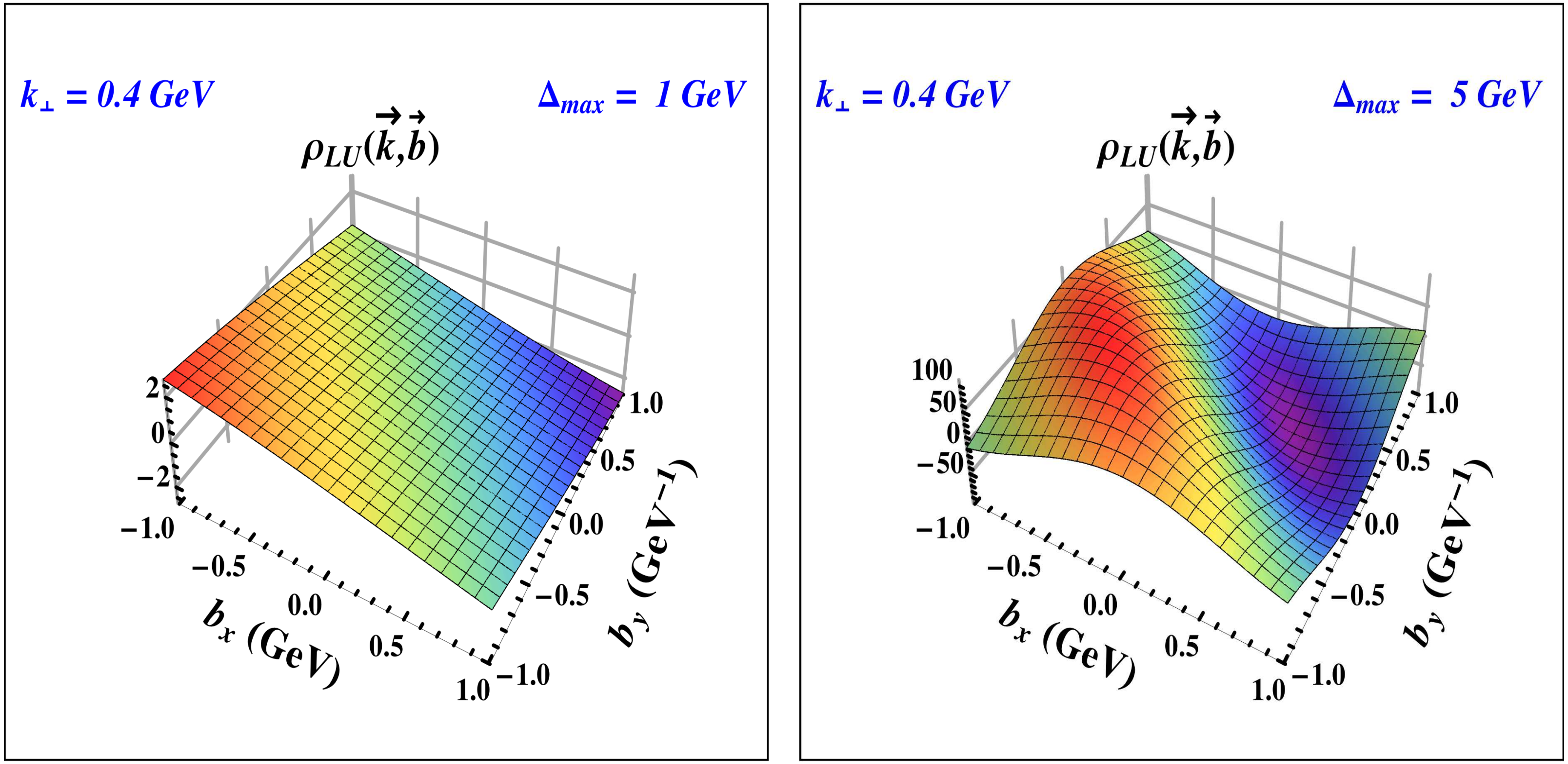




AM, Nair, Ojha, arXiv:1403.6233 [hep-ph]

- Plots in mixed space : probabilistic interpretation
- Minima at $b_x = 0$ and $k_y = 0$: minima is observed for all b_x values for $k_y = 0$
- Probability of finding a quark with fixed k_y and b_x first increases away from $k_y = 0$ and then decreases

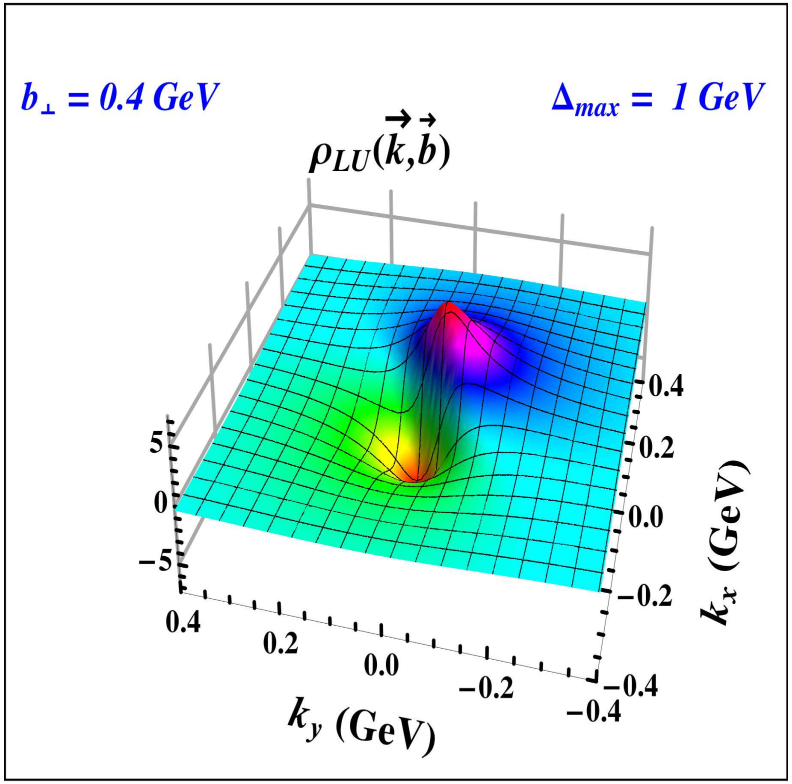
Wigner Distributions : Results

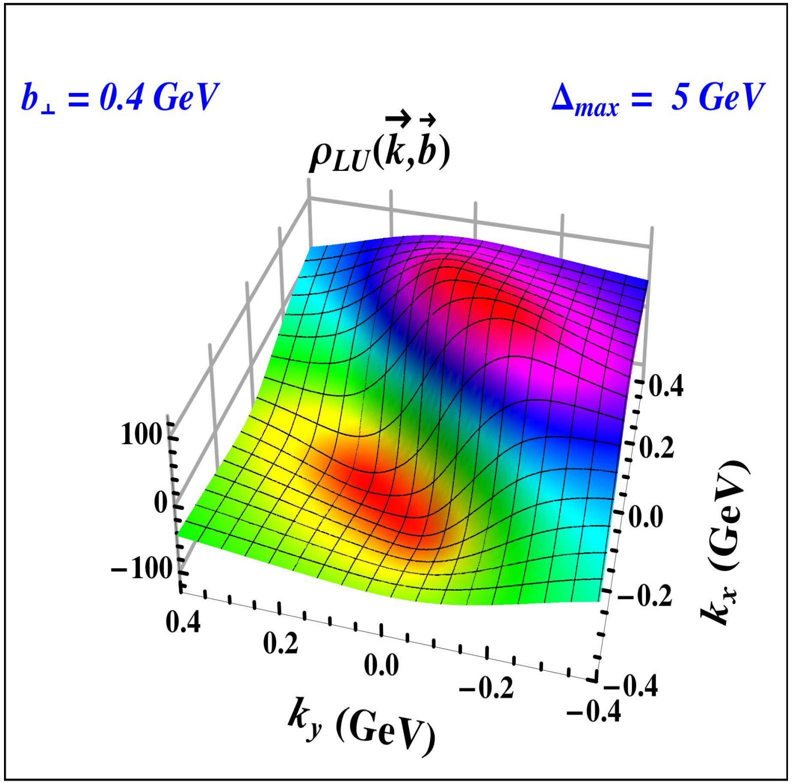


AM, Nair, Ojha, arXiv:1403.6233 [hep-ph]

- Distortion of the Wigner distribution of unpolarized quarks due to the longitudinal polarization of the dressed quark.
- ρ_{LU} is the same as ρ_{UL} in this model
- $k_\perp$ in y direction; dipole structure similar to other models

Wigner Distributions : Results

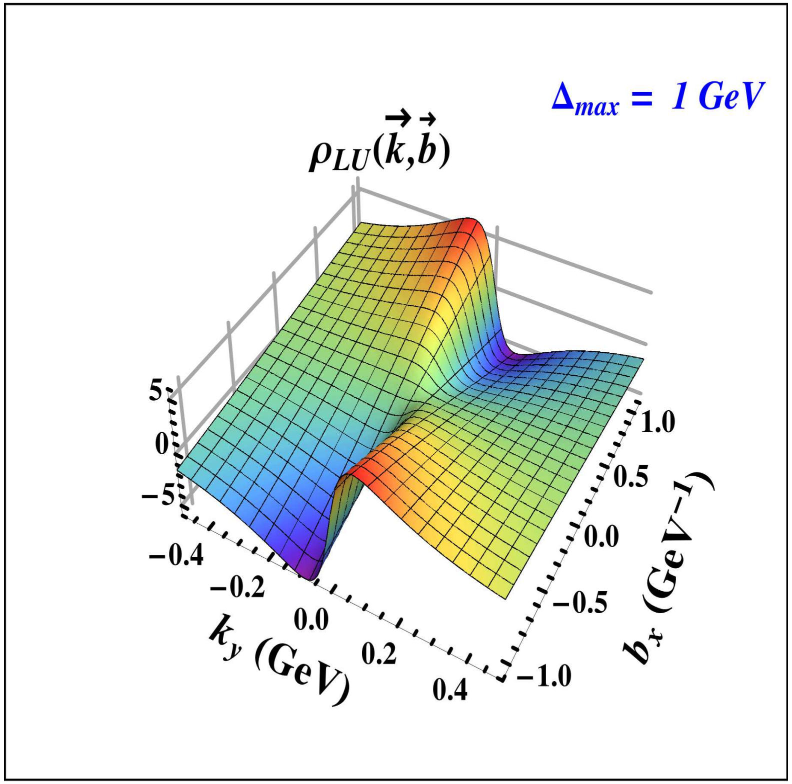


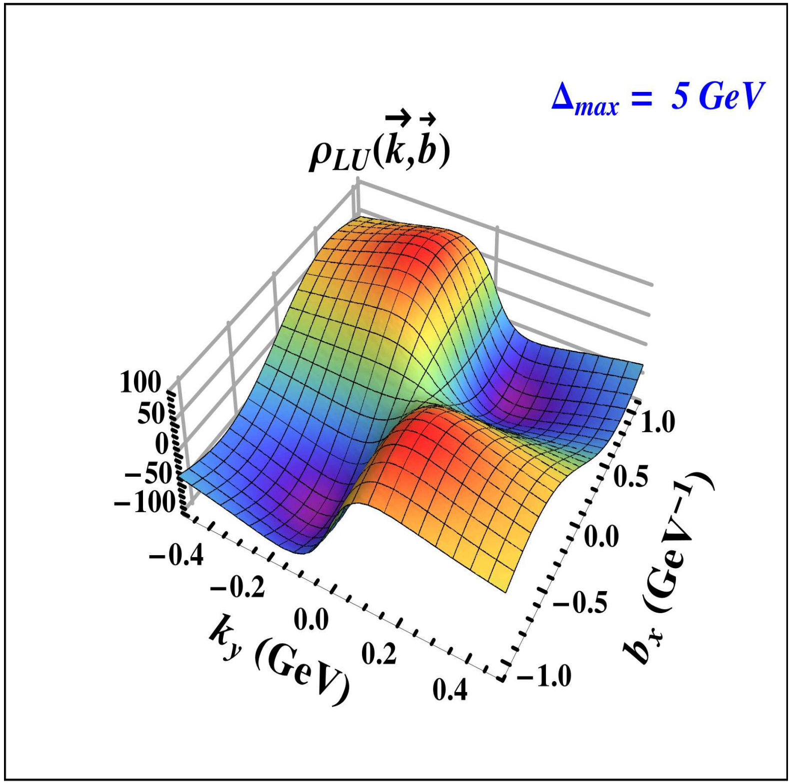


AM, Nair, Ojha, arXiv:1403.6233 [hep-ph]

- b in y direction
- Dipole structure

Wigner Distributions : Results





AM, Nair, Ojha, arXiv:1403.6233 [hep-ph]

- Quadrupole structure in the mixed space like other models
- Peaks increase in magnitude with increasing $\Delta_{\perp}^{max}$

Orbital Angular Momentum of Quarks

- EMC experiment (1989) showed that the quark intrinsic spin contribution was much less than expected
- Recent polarized beam experiments suggest that the gluon polarization contribution to the total spin of the proton is also very small
- A substantial part of the spin of the nucleon comes from quark and gluon OAM
- Issue of gauge invariance and experimental measurability of the OAM contribution complicates the issue of a full understanding of such contributions
- Theoretically there exist mainly two definitions of OAM : one obtained from the sum rules of GPDs and the other, canonical OAM distribution in the light cone gauge
- These two different distributions are projections of Wigner distributions with different choice of gauge links and they are related by a gauge dependent potential term

Hatta, PLB (2012), Lorce, PLB (2013), Burkardt, PRD (2013).

Orbital Angular Momentum of Quarks

Quark-quark correlator defining the Wigner distributions can be parameterized in terms of generalized transverse momentum dependent parton distributions (GTMDs)

$$W^{[\gamma^+]}_{\lambda,\lambda'} = \frac{1}{2M}\bar{u}(p',\lambda')\left[F_{1,1} - \frac{i\sigma^{i+}k_{i\perp}}{P^+}F_{1,2} - \frac{i\sigma^{i+}\Delta_{i\perp}}{P^+}F_{1,3} + \frac{i\sigma^{ij}k_{i\perp}\Delta_{j\perp}}{M^2}F_{1,4}\right]u(p,\lambda);$$

$$\begin{aligned} W^{[\gamma^+\gamma_5]}_{\lambda,\lambda'} &= \frac{\bar{u}(p',\lambda')}{2M}\left[\frac{-i\epsilon^{ij}_{\perp}k_{i\perp}\Delta_{j\perp}}{M^2}G_{1,1} - \frac{i\sigma^{i+}\gamma_5 k_{i\perp}}{P^+}G_{1,2}\right. \\ &\left. -\frac{i\sigma^{i+}\gamma_5\Delta_{i\perp}}{P^+}G_{1,3} + i\sigma^{+-}\gamma_5 G_{1,4}\right]u(p,\lambda). \end{aligned}$$

For the dressed quark, the GTMDs can be calculated analytically

Confirm in our model calculation that the GTMDs F_{14} and G_{11} exist and are non-zero; conclusion agrees with

Kanazawa, Lorce, Metz, Pasquini, Schlegel, arXiv:1403.5226[hep-ph]

Quark GTMDs

$$F_{11} = -\frac{N\left[4k_\perp^2(1+x^2) + (x-1)^2(4m^2(x-1)^2 - (1+x^2)\Delta_\perp^2)\right]}{D(q_\perp)D(q'_\perp)2x^2(x-1)^3};$$

$$F_{12} = \frac{2Nm^2\Delta_\perp^2}{D(q_\perp)D(q'_\perp)x(k_y\Delta_x - k_x\Delta_y)};$$

$$\begin{aligned} F_{13} \quad &= \quad \frac{N}{D(q_\perp)D(q'_\perp)4x(k_y\Delta_x - k_x\Delta_y)}\Bigg[8m^2(k_\perp\Delta_\perp) \\ &- \frac{(k_y\Delta_x - k_x\Delta_y)(4k_\perp^2(1+x^2) + (x-1)^2(4m^2(x-1)^2 - (1+x^2)\Delta_\perp^2))}{x(x-1)^3}\Bigg]; \end{aligned}$$

$$F_{14} = \frac{2Nm^2(1+x)}{D(q_\perp)D(q'_\perp)x^2(1-x)}.$$

Quark GTMDs

$$G_{11} = -\frac{2Nm^2(1+x)}{D(q_\perp)D(q'_\perp)x^2(x-1)};$$

$$G_{12} = \frac{-N}{D(q_\perp)D(q'_\perp)x(x-1)}\left[4m^2\frac{k_\perp.\Delta_\perp}{(k_y\Delta_x - k_x\Delta_y)} - \frac{(1+x)\Delta_\perp^2}{x}\right];$$

$$G_{13} = \frac{N\left[(1+x)\left(\Delta_y^2 - \Delta_x^2 + \Delta_x\Delta_y(k_y^2 - k_x^2)\right) + 4xm^2k_\perp^2\right]}{D(q_\perp)D(q'_\perp)x^2(x-1)(k_y\Delta_x - k_x\Delta_y)};$$

$$G_{14} = \frac{N\left[-4k_\perp^2(1+x^2) + (x-1)^2\left(4m^2(x-1)^2 - (1+x^2)\Delta_\perp^2\right)\right]}{D(q_\perp)D(q'_\perp)2x^2(x-1)^3};$$

Quark Orbital Angular Momentum

$$L_z^q = \frac{1}{2}\int dx\{x[H^q(x,0,0) + E^q(x,0,0)] - \tilde{H}^q(x,0,0)\}$$

X. Ji, PRL (1997)

GPDs in the above equation are defined at $\xi = 0$ or when the momentum transfer is purely in the transverse direction

The GPDs in the above expression are related to the GTMDs by the following relations :

$$H(x,0,t) = \int d^2k_\perp F_{11}$$

$$E(x,0,t) = \int d^2k_\perp \Big[- F_{11} + 2\Big(\frac{k_\perp.\Delta_\perp}{\Delta_\perp^2} F_{12} + F_{13}\Big)\Big]$$

$$\tilde{H}(x,0,t) = \int d^2k_\perp G_{14}$$

Meissner, Metz, Schlegel, JHEP (2009)

Quark Orbital Angular Momentum

Using the GTMDs calculated we have the following final expression for the kinetic orbital angular momentum of quarks in the dressed quark model:

$$L_z^q = \frac{N}{2} \int dx \Big\{ - f(x) I_1 + 4m^2 (1-x)^2 I_2 \Big\};$$

$$I_1 = \int \frac{d^2 k_\perp}{m^2(1-x)^2 + (k_\perp)^2} = \pi log \left[\frac{Q^2 + m^2(1-x)^2}{\mu^2 + m^2(1-x)^2} \right];$$

$$I_2 = \int \frac{d^2 k_\perp}{\Big(m^2(1-x)^2 + (k_\perp)^2 \Big)^2} = \frac{\pi}{(m^2(1-x)^2)}; f(x) = 2(1+x^2)$$

Q and μ are the upper and lower limits of the integration respectively, Q is the large scale

μ can be safely taken to be zero provided the quark mass is non-zero

Quark Orbital Angular Momentum

$F_{1,4}$ is not reducible to any GPDs or transverse-momentum dependent parton distributions (TMDs) in any limit: related to the canonical OAM as

$$l_z^q = -\int dx d^2 k_\perp \frac{k_\perp^2}{m^2} F_{14}.$$

Canonical quark OAM in the dressed quark model becomes

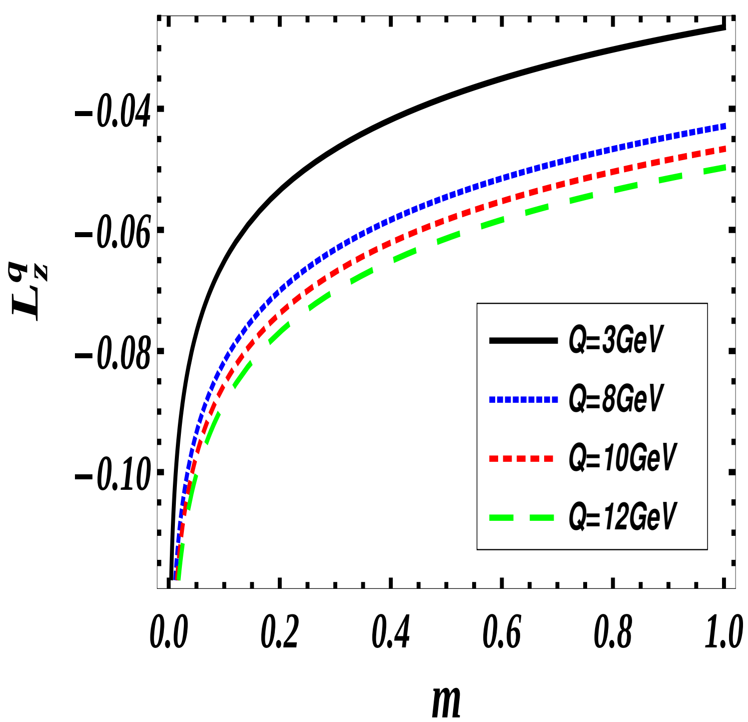
$$l_z^q = -2N \int dx (1 - x^2) \Big[I_1 - m^2 (x-1)^2 I_2 \Big].$$

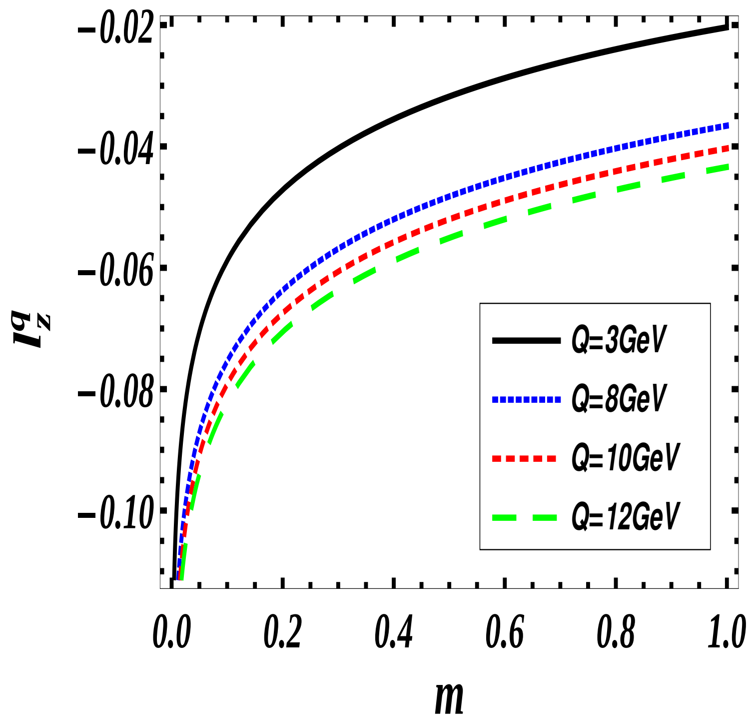
Agrees with

Harindranath and Kundu (1999); Hikmat and Burkardt(2012),
Kanazawa, Lorce, Metz, Pasquini, Schlegel, arXiv:1403.5226[hep-ph]

Spin-orbit correlation is given by the same expression in this model
Does not depend on the gauge link

Orbital Angular Momentum : Results





AM, Nair, Ojha, arXiv:1403.6233 [hep-ph]

Both OAM decrease in magnitude with increasing mass of the quark

The two OAM differs in this model

In this model gluon intrinsic helicity contribution to the helicity sum rule cancels the contribution from the quark and gluon OAM and the helicity sum rule is satisfied

Harindranath and Kundu, PRD, (1999)

Contribution from the single particle sector (at $x = 1$) is important

Summary and Conclusions

- We calculated the Wigner distributions for a quark state dressed with a gluon using the overlap representation in terms of the LFWFs

- This is a simple composite relativistic spin-1/2 system which has a gluonic degree of freedom : used light-front wave functions

- Although the Wigner distributions in quantum mechanics are not measurable and do not have probabilistic interpretation, after integrating out some of the variables a probabilistic interpretation is possible to obtain

- We calculated the Wigner distributions both for unpolarized and longitudinally polarized target state and showed the correlations in transverse momentum and position space

- We also calculated the kinetic quark OAM using the GPD sum rule and the canonical OAM in light-front gauge and showed that these are different in magnitude; does not depend on the gauge link

- Further work would involve calculating the Wigner distributions for the gluons and also including transverse polarization of the target and the quark