PDFs from Lattice QCD

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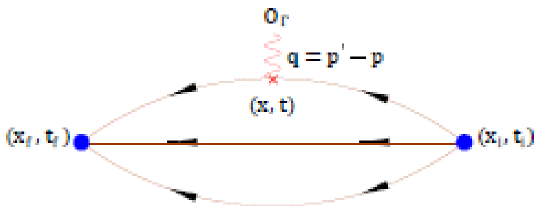
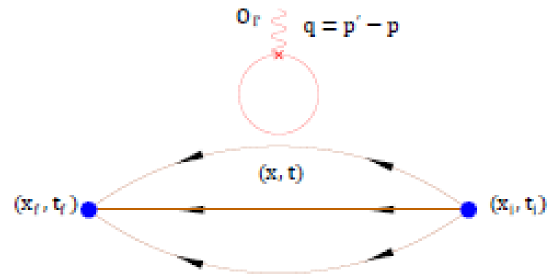
To access the nucleon structure in lattice QCD, there are basically two classes of diagrams to be computed





Given an operator, the corresponding correlation funcion is calculated

$$C_3^{\alpha'\alpha}(x', y, x) = \frac{\int \mathcal{D}U\, \mathcal{D}(\bar{\psi}\psi)\ \exp\left(-S[U, \bar{\psi}, \psi]\right) N_{\alpha'}(x')\mathcal{O}(y)\bar{N}_\alpha(x)}{\int \mathcal{D}U\, \mathcal{D}(\bar{\psi}\psi)\ \exp\left(-S[U, \bar{\psi}, \psi]\right),}$$

We work with twisted mass fermions

$$S_F[\chi, \overline{\chi}, U] = a^4 \sum_x \overline{\chi}(x) \left[D_W + m_0 + i\mu_q \gamma_5 \tau^3 \right] \chi(x)$$

with is an O(a) improved action at maximum twist

Frezzotti et al., JHEP 0108 (2001) 058; Frezzotti and Rossi, JHEP 0408(2004) 007.

Outline

- Quark distributions
- Quasi-quark distributions
- Relation between quark and quasi-quark distributions
- Calculation of the matrix elements of the quasi distributions
- Results

Light cone quark distributions

QCD

↓

OPE

↓

Matrix element of local operators

↓

Inverse Mellin transform

↓

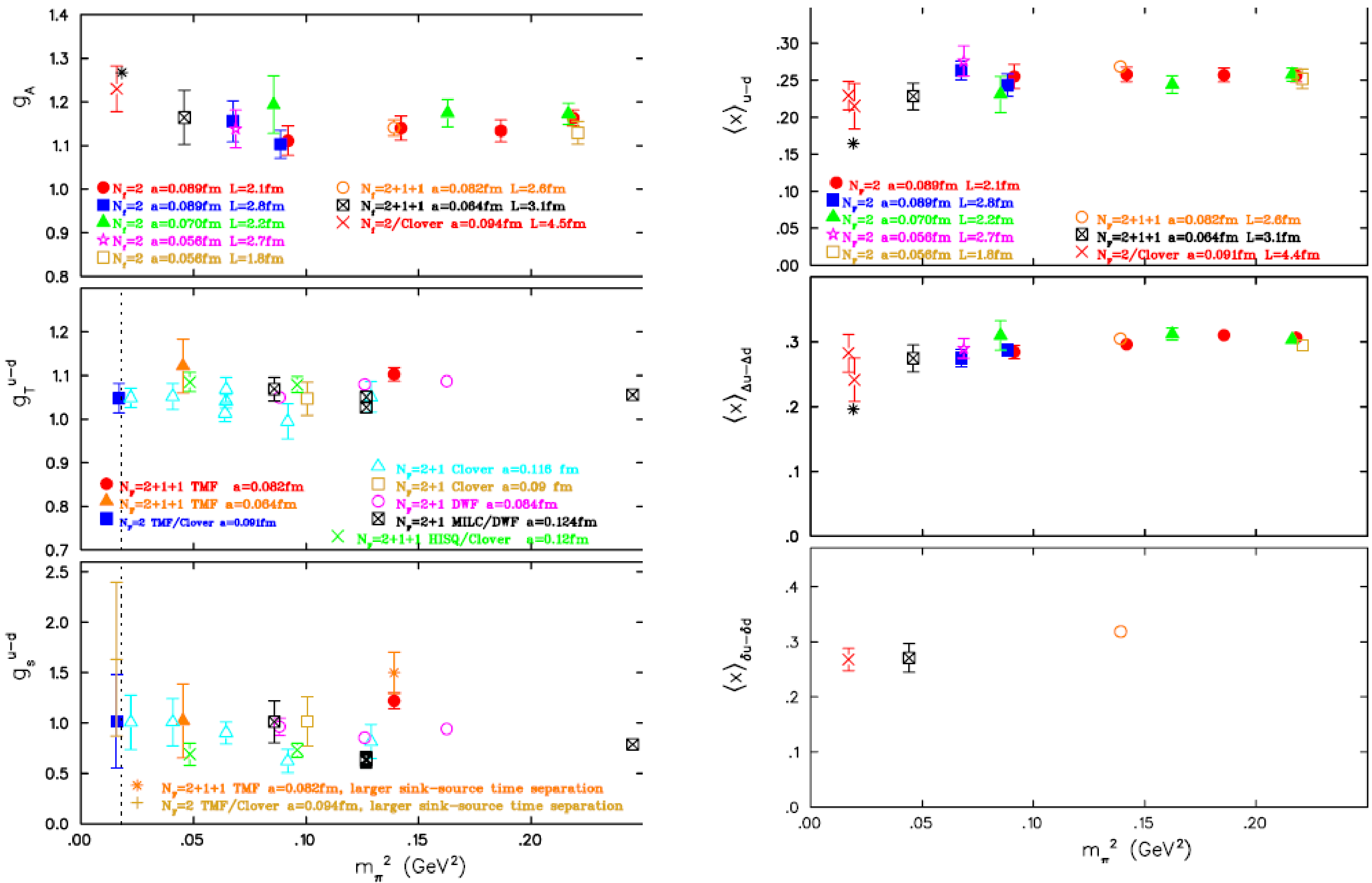
$$q(x) = \frac{1}{2\pi}\int d\xi^- e^{-ixp^+\xi^-}\langle P|\bar{\psi}(\xi^-)\Gamma\mathcal{L}(\xi^-,0)\psi(0)|P\rangle$$

$$= e^{-ig\int_0^{\xi^-} d\eta^- A^+(\eta^-)}$$

- Light cone correlations in the nucleon rest frame
- Equivalent to distributions in the IMF
- Light cone dominated: $\xi^2 = t^2 - z^2 \sim 0$
- Not calculable on Euclidian Lattice as $t^2 + z^2 \sim 0$

However, the moments are calculable: $a_n = \int_0^1 dx\, x^{n-1} q(x) = \frac{1}{(p^+)^n} \left\langle P \middle| \bar{\psi}(0) \left(i \overleftrightarrow{D}^+\right)^n \psi(0) \middle| P \right\rangle$

- If a sufficient number of moments are calculated, one can reconstruct the x dependence of the distributions
- Hard to simulate high order derivatives on the lattice
- Nevertheless, the first few moments as well as charges can and have been calculated



See also the talk by S. Collins

Quasi Distributions

X. Ji, PRL 110, 262002 (2013)

In general: $\langle P|O^{\mu_1\cdots\mu_n}|P\rangle = 2a_n(P^{\mu_1}\cdots P^{\mu_n} - \ ...)$

Taking: $\mu_1 = \mu_2 = \cdots = \mu_z = z \ \Rightarrow \langle P|O^{z\cdots z}|P\rangle = 2\,\tilde{a}_n\,(P^z)^n\left(1 + \mathcal{O}\left(\frac{M^2}{(P^z)^2}\right)\right)$

Inverting for a twist-2 operator,

$$\tilde{q}(x,P^z) = \frac{1}{2\pi}\int dze^{-ixp^z z}\langle P|\bar{\psi}(z)\Gamma\mathcal{L}(z,0)\psi(0)|P\rangle + \mathcal{O}\left(\frac{M^2}{(P^z)^2}\right)$$

- Nucleon moving with finite momentum in the z direction
- Pure spatial correlation
- Can be simulated in a lattice

What are these quasi-distributions? Do they have a partonic interpretation?

The light cone distributions:

$$x = \frac{k^+}{P^+}$$

$$0 \leq x \leq 1$$

Distributions can be defined in an infinite momentum frame: $p^z, p^+ \to \infty$

Quasi distributions:

P^z large but finite

Some constituents can be moving backward or even with momentum greater than P^z

$x < 0$ or $x > 1$ is possible

Usual partonic interpretation is lost

But they can be related to each other!

Matching Condition

- Relating finite to infinite momentum
- Axial gauge $A^Z = 0$
- UV divergence regulate with $|k_T| \leq \Lambda \sim \dfrac{1}{a}$
- Renormalization scale μ

$$\tilde{q}_{NS}(x, \Lambda, P^z) = \int \frac{dy}{y}\, Z\left(\frac{x}{y}, \frac{\Lambda}{P^z}, \frac{\mu}{P^z}\right) q_{NS}(y, \mu) + \mathcal{O}\left(\frac{M^2}{P^{z2}}\right)$$

No partonic interpretation

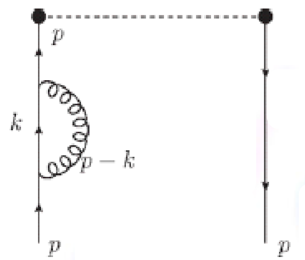
Partonic interpretation

How to calculate Z?

Calculating Z

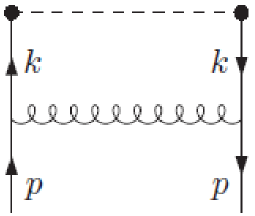
$$Z(y) = Z^0\left(y, \frac{\mu}{Pz}\right) + Z^{(1)}\left(y, \frac{\mu}{Pz}\right) + \cdots$$

$$q(x, \Lambda) = \left(1 + Z_F^{(1)}(\Lambda)\right)\delta(x-1) + q^{(1)}(x, \Lambda) + \cdots$$



Self-energy correction

Vertex correction



$$\tilde{q}(x, \Lambda, \mathrm{P^z}) = \left(1 + \tilde{Z}_F^{(1)}(\Lambda, \mathrm{P^z})\right)\delta(x-1) + \tilde{q}^{(1)}(x, \Lambda, \mathrm{P^z})$$

$$\mathcal{O}\,(\alpha_s{}^0): \qquad \delta(1-x) = \int_0^1 \frac{dy}{y}\, Z^{(0)}\left(\frac{x}{y}\right)\delta(1-y)$$

$$Z^{(0)} = \delta(1-x)$$

$$\mathcal{O}\,(\alpha_s{}^1): \qquad \left(1+\tilde{Z}_F^{(1)}\right)\delta(x-1) + \tilde{q}^{(1)}(x) =$$

$$= \int \frac{dy}{y}\,\delta\left(\frac{x}{y}-1\right)\left[\left(1+Z_F^{(1)}\right)\delta(y-1) + q^{(1)}(y)\right]$$

$$+ \int \frac{dy}{y} Z^{(1)}\left(\frac{x}{y}\right)\delta(y-1)$$

$$\Longrightarrow \boxed{Z^{(1)}\left(\frac{x}{y},\frac{\Lambda}{P^z},\frac{\mu}{P^z}\right) = \tilde{q}^{(1)}(x,\Lambda,P^z) - q^{(1)}(x,\Lambda) + \underbrace{(\tilde{Z}_F^{(1)} - Z_F^{(1)})}_{\delta Z^{(1)}}\delta(x-1)}$$

Results

P^z fixed, $\Lambda \to \infty$

$$\tilde{q}^{(1)}(x,\Lambda,P^z) = \frac{\alpha_S C_F}{2\pi}\begin{cases} \frac{1+x^2}{1-x}\ln\frac{x}{x-1} + 1 + \frac{\Lambda}{(1-x)^2P^z}, & x>1, \\ \frac{1+x^2}{1-x}\ln\frac{(P^z)^2}{m^2} + \frac{1+x^2}{1-x}\ln\frac{4x}{1-x} - \frac{4x}{1-x} + 1 + \frac{\Lambda}{(1-x)^2P^z}, & 0<x<1, \\ \frac{1+x^2}{1-x}\ln\frac{x-1}{x} - 1 + \frac{\Lambda}{(1-x)^2P^z}, & x<0, \end{cases}$$

$$\tilde{Z}_F^{(1)}(\Lambda,P^z) = \frac{\alpha_S C_F}{2\pi}\int dy \begin{cases} -\frac{1+y^2}{1-y}\ln\frac{y}{y-1} - 1 - \frac{\Lambda}{(1-y)^2P^z}, & y>1, \\ -\frac{1+y^2}{1-y}\ln\frac{(P^z)^2}{m^2} - \frac{1+y^2}{1-y}\ln\frac{4y}{1-y} + \frac{4y^2}{1-y} + 1 - \frac{\Lambda}{(1-y)^2P^z}, & 0<y<1, \\ -\frac{1+y^2}{1-y}\ln\frac{y-1}{y} + 1 - \frac{\Lambda}{(1-y)^2P^z}, & y<0. \end{cases}$$

Λ fixed, $P^z \to \infty$

$$q^{(1)}(x,\Lambda) = \frac{\alpha_S C_F}{2\pi}\begin{cases} 0, & x>1 \quad \text{or} \quad x<0, \\ \frac{1+x^2}{1-x}\ln\frac{\Lambda^2}{m^2} - \frac{1+x^2}{1-x}\ln(1-x)^2 - \frac{2x}{1-x}, & 0<x<1, \end{cases}$$

$$Z_F^{(1)}(\Lambda) = \frac{\alpha_S C_F}{2\pi}\int dy \begin{cases} 0, & y>1 \quad \text{or} \quad y<0, \\ -\frac{1+y^2}{1-y}\ln\frac{\Lambda^2}{m^2} + \frac{1+y^2}{1-y}\ln(1-y)^2 + \frac{2y}{1-y}, & 0<y<1, \end{cases}$$

X. Xiong et al., Phys. Ver. D90,014051 (2014)

And

$$Z^{(1)}(\xi)/C_F = \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{\xi}{\xi-1}+1+\frac{1}{(1-\xi)^2}\frac{\Lambda}{P^z}, \qquad \boxed{\xi > 1}$$

$$Z^{(1)}(\xi)/C_F = \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{(P^z)^2}{\mu^2}+\left(\frac{1+\xi^2}{1-\xi}\right)\ln[4\xi(1-\xi)] - \frac{2\xi}{1-\xi}+1+\frac{\Lambda}{(1-\xi)^2 P^z}, \qquad \boxed{0 < \xi < 1}$$

$$Z^{(1)}(\xi)/C_F = \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{\xi-1}{\xi}-1+\frac{\Lambda}{(1-\xi)^2 P^z} \qquad \boxed{\xi < 0}$$

$$\delta Z^{(1)} = \frac{\alpha_S C_F}{2\pi}\int dy \begin{cases} -\frac{1+y^2}{1-y}\ln\frac{y}{y-1}-1-\frac{\Lambda}{(1-y)^2P^z}, & y>1, \\ -\frac{1+y^2}{1-y}\ln\frac{(P^z)^2}{\mu^2}-\frac{1+y^2}{1-y}\ln[4y(1-y)]+\frac{2y(2y-1)}{1-y}+1-\frac{\Lambda}{(1-y)^2P^z}, & 0<y<1, \\ -\frac{1+y^2}{1-y}\ln\frac{y-1}{y}+1-\frac{\Lambda}{(1-y)^2P^z}, & y<0, \end{cases} \qquad \boxed{\xi = 1}$$

All divergences cancel, with the exception of a UV log divergence

$$\tilde{q}(x) = \int_{-1}^{+1}\frac{dy}{|y|}Z\left(\frac{x}{y}\right)q(y)$$

- It should be there
- It appears as a low x cut

Calculating the quark distibution

$$\tilde{q}(x,\Lambda,P^z) = q(x,\mu) + q(x,\mu)\delta Z^{(1)}\left(\frac{\Lambda}{P^z},\frac{\mu}{P^z}\right) + \frac{\alpha_s}{2\pi}\int_{-1}^{+1}\frac{dy}{|y|}Z^{(1)}\left(\frac{x}{y},\frac{\Lambda}{P^z},\frac{\mu}{P^z}\right)q(y,\mu)$$

From pQCD

From the lattice

Desired quantity

Calculation of the matrix elements in a lattice

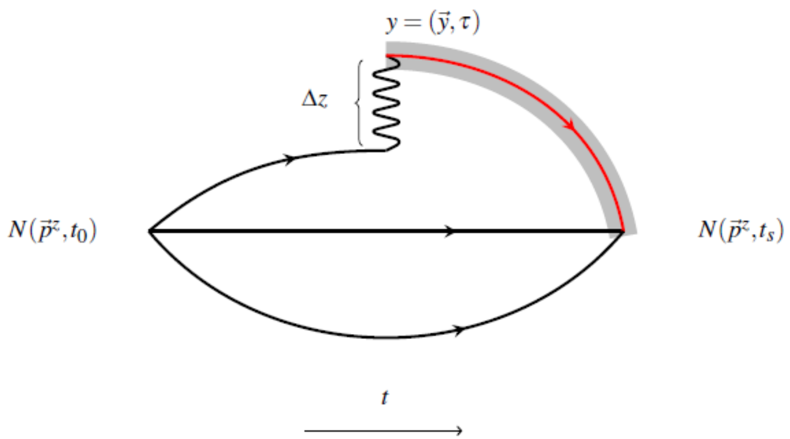
C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos,
K. Hadjiyiannakou, K. Jansen, FS, C. Wiese, Lattice 2014 , arXiv:1411.0891

We want: $h(p^z,\Delta z) = \langle N(p^z)\big|\bar{\psi}(\Delta z)\gamma^z\mathcal{L}(\Delta z,0)\psi(0)\big|N(p^z)\rangle$

Let : $C^{3pt}(t,\tau,0) = \langle \Gamma_{\alpha\beta} N_\alpha(\vec{p}^z,t)\mathcal{O}(\tau)\bar{N}_\beta(\vec{p}^z,0)\rangle$

$$N_\alpha(\vec{p}^z,t) = \sum_{\vec{x}} e^{i\vec{p}^z.\vec{x}} \varepsilon^{abc}\, u^a{}_\alpha(x)\left(d^{b^T}(x)\mathcal{C}\gamma_5 u^c(x)\right)$$

$$\mathcal{O}\,(\Delta z,\tau,Q^2=0) = \sum_{\vec{y}} \bar{\psi}(y+\ \Delta z)\gamma^z\mathcal{L}(y+\Delta z,y)\psi(y)$$



All to all propagators needed

Stochastic method is used

Flavour structure: u - d

Extraction of the matrix elements

$$\frac{C^{3pt}(t,\tau,0;\vec{p}^z)}{C^{2pt}(t,0;\vec{p}^z)} \xrightarrow{0\ll\tau\ll t} \frac{-ip^z}{E}\, h(p^z,\Delta z)$$

$8a, 10a$ Source – sink separation

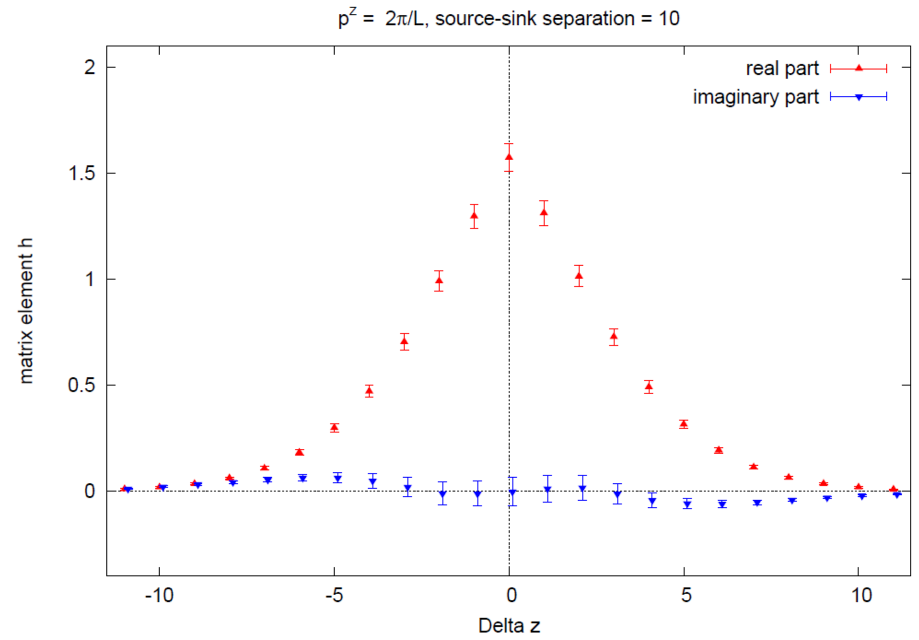
$32^3 \times 64$ Lattice

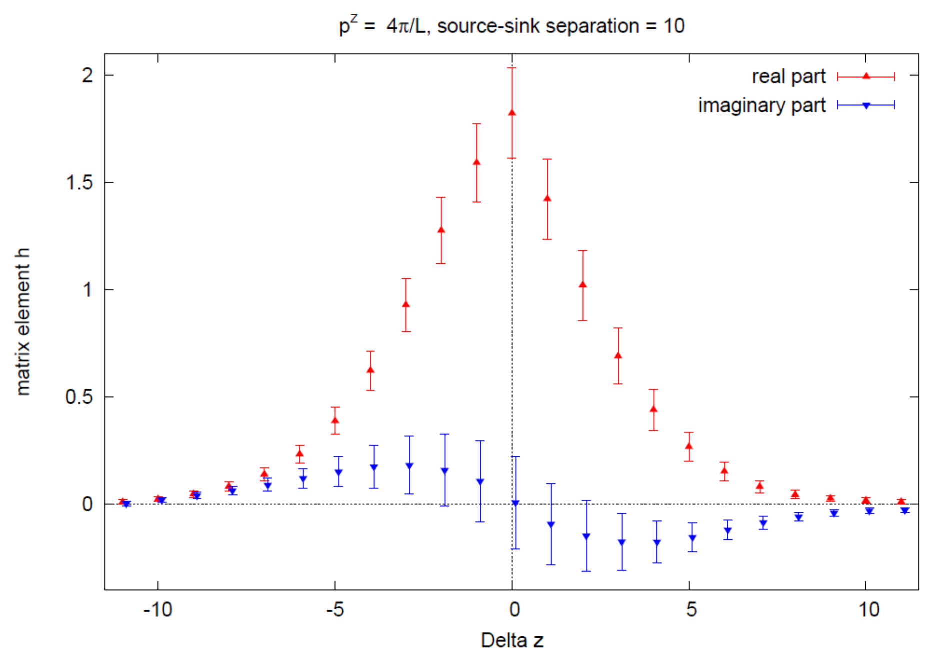
$\beta = \frac{6}{g_0^2} = 1.95$ $a \approx 0.078fm$ $N_f = 2+1+1$

Maximally twisted mass ensemble: $a\mu_q = 0.0055 \Rightarrow m_{PS} \approx 390\ MeV$

$$P^z = \frac{2\pi}{L}, 2\frac{2\pi}{L}, \cdots$$

1000 gauge configurations





What is the minimum Bjorken x ?

$$\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Spatial extent
of the lattice

Largest momentum $|\vec{p}| = \frac{\pi}{a}$

Smallest momentum $|\vec{p}| = \frac{2\pi}{L}$

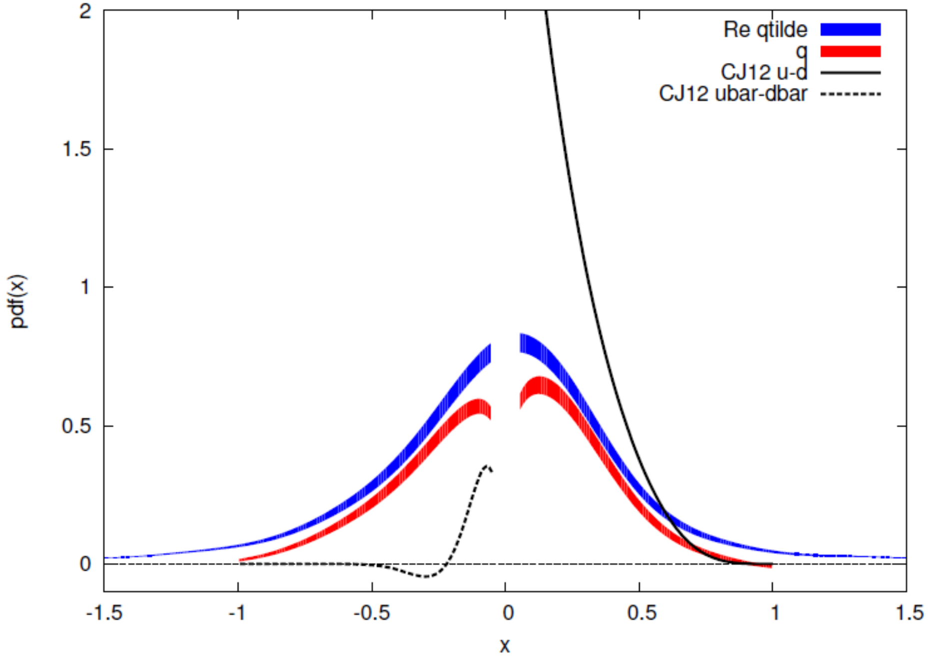
The injected momentum can be distributed at most between a/L lattice points

$$x = \frac{k^z}{P^z} \rightarrow x_{min} \sim \frac{1}{L/a}$$

Present approach is valid at intermediate and large x: cut imposed by the Lattice size

Results for

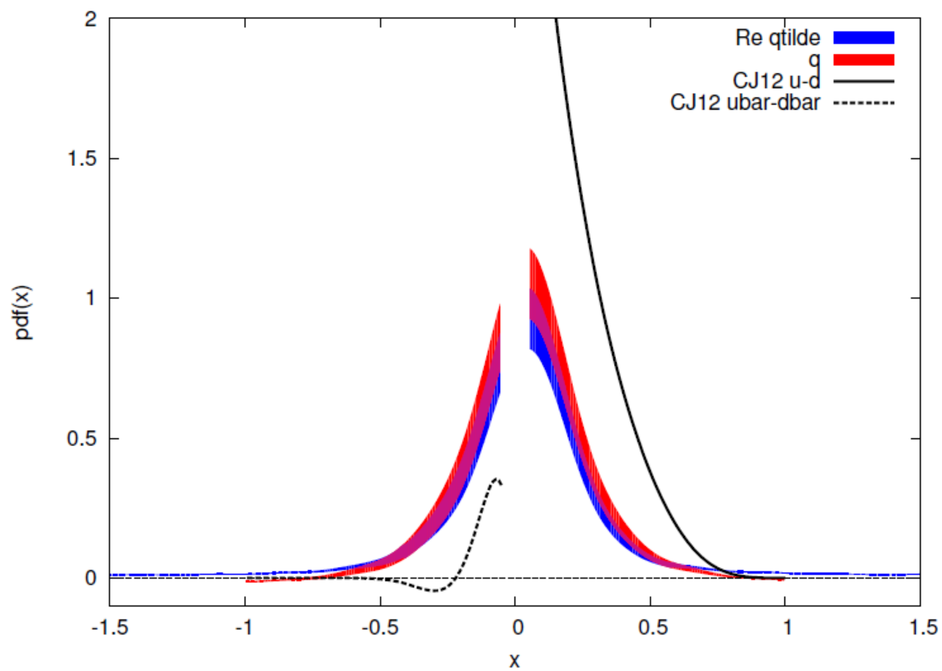
$$u(x) - d(x)$$



$$\boxed{aP^z = \frac{2\pi}{L}}$$

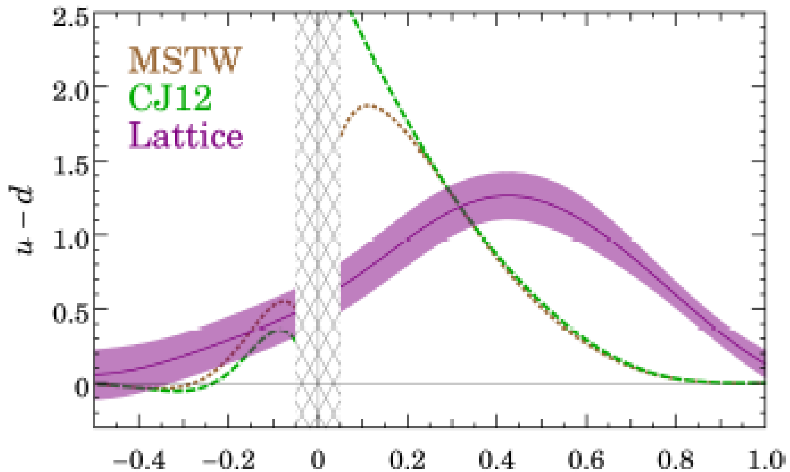
Next steps:

- Nucleon mass corrections
- Renormalization
- Physical pion mass



$$\boxed{aP^z = 2\frac{2\pi}{L}}$$

Only other result



Huey-Wen Lin et al. arXiv:1402.1462

$24^3 \times 48$

$a \approx 0.12 fm \qquad N_f = 2+1+1$

$m_{PS} \approx 310\ MeV$

Uses highly improved staggered quarks and HYP smearing

Nucleon mass corrections

$$\tilde{a}_n(P_Z) = \frac{1}{2P_z^n}\langle P_z|\bar{\psi}(0)\gamma^z(i\overleftarrow{D}^z)^{n-1}\psi(0)|P_z\rangle$$

$$\langle P_z|O^{z\ldots z}|P_z\rangle = 2\tilde{a}_{2k}^{(0)}(P_z^2)^k\sum_{j=0}^{k}\mu^j\begin{pmatrix} 2k-j \\ j \end{pmatrix} \qquad \mu = \frac{M^2}{4P_z^2}$$

$$\tilde{a}_n^{(0)} = \int_{-\infty}^{+\infty} x^{n-1}\tilde{q}^{(0)}(x,P_z)dx \qquad \tilde{a}_n = \int_{-\infty}^{+\infty} x^{n-1}\tilde{q}(x,P_z)dx$$

$$\boxed{\tilde{q}(x,P_z) = \sum_{j=0}^{n_{max}}\frac{(-1)^j}{j!}\mu^j x^{1-j}\frac{d^j}{dx^j}\tilde{q}^{(0)}(x,P_z)x^{j-1}}$$

It is being implemented

Summary & Outline

- First attempts of a direct QCD calculation of quark distributions;
- Valuable information on the large x region;
- Include the nucleon mass corrections to arbitrary order;
- Renormalization;
- Higher order correction;
- Go to up 30000 gauge configurations – errors go down by a factor of 5;
- Compute at the physical mass – smaller number of configurations available at the moment;
- Go to the continuum;
- Polarized sector, transverse distributions, singlet combinations, etc
- Much to be done!