

Non-perturbative effects for the BFKL equation

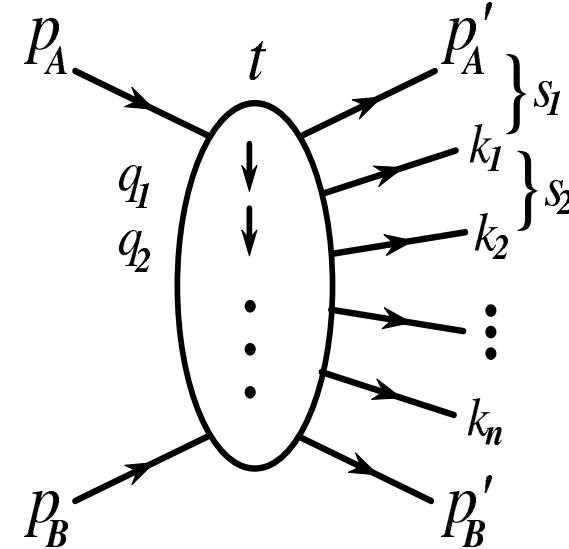
L. N. Lipatov

Petersburg Nuclear Physics Institute, Russia

Content

1. Gluon reggeization in the multi-Regge kinematics
2. BFKL equation in Higgs model and in QCD
3. Next-to-leading corrections and running coupling
4. Spectrum of Pomerons in QCD
5. Non-perturbative phase at the Higgs model
6. Pomeron at the thermostat and anti-Meisner effect
7. Compactification of ρ -space as a confinement model
8. Non-Fredholm properties of the kernel and continuous spectrum
9. Discussion

1 Amplitudes in multi-Regge kinematics



$$M_{2 \rightarrow 2+n}^{FKL} = 2s g \frac{s_1^{\omega_1}}{q_1^2 + m^2} g T_{c_2 c_1}^{d_1} C_1^\mu e_\mu^1 \frac{s_2^{\omega_2}}{q_2^2 + m^2} \dots g T_{c_{n+1} c_n}^{d_n} C_n^\sigma e_\sigma^n \frac{s_{n+1}^{\omega_{n+1}}}{q_{n+1}^2 + m^2} g ,$$

$$C_1 = -q_1^\perp - q_2^\perp - p_A \left(\frac{q_1^2 + m^2}{p_A k_1} - \frac{p_B k_1}{p_A p_B} \right) + p_B \left(\frac{q_2^2 + m^2}{p_B k_1} - \frac{p_A k_1}{p_A p_B} \right) ,$$

$$\omega_r = j_r - 1 = -\frac{g^2 N_c}{16\pi^3} \int \frac{d^2 k (q_r^2 + m^2)}{(k^2 + m^2)((q_r - k)^2 + m^2)}$$

2 Fadin-Kuraev-Lipatov equation (1975)

Total cross-section in the Higgs model at high energies in LLA

$$\sigma_t \sim s^\Delta, \quad \Delta = -\frac{\alpha_s N_c}{2\pi} E_0 = \frac{4\alpha_s N_c}{\pi} \ln 2$$

FKL equation for the Pomeron wave function at $q = t = 0$

$$Ef(r) = Hf(r), \quad H = T(p) + V(r), \quad r = |x|$$

Kinetic energy related to two Regge trajectories

$$T(p) = \frac{2(|p|^2 + m^2)}{|p|\sqrt{|p|^2 + 4m^2}} \ln \frac{\sqrt{|p|^2 + 4m^2} + |p|}{\sqrt{|p|^2 + 4m^2} - |p|}, \quad |p|^2 = -\frac{1}{r} \partial r \partial$$

Potential energy

$$V(r) = -4 K_0(r m) + \frac{N_c^2 + 1}{N_c^2} \hat{P}, \quad \hat{P} \phi(p) = \frac{m^2}{|p|^2 + m^2} \int \frac{d^2 p'}{\pi} \frac{\phi(p')}{|p'|^2 + m^2}$$

Semiclassical solution (Levin, Lipatov, Siddikov (2014))

3 BFKL equation in LLA (1978)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E \Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12} \Psi(\vec{\rho}_1, \vec{\rho}_2) : \Delta = -\frac{\alpha_s N_c}{2\pi} E_0 = \frac{4\alpha_s N_c}{\pi} \ln 2$$

Holomorphic separability (L.)

$$H_{12} = h_{12} + h_{12}^*, \quad [h_{12}, h_{12}^*] = 0, \quad E = \epsilon + \epsilon^*, \quad \epsilon = \psi(m) + \psi(1-m) - 2\psi(1)$$

Holomorphic Hamiltonian

$$h_{12} = \frac{1}{p_1} (\ln \rho_{12}) p_1 + \frac{1}{p_2} (\ln \rho_{12}) p_2 + \ln(p_1 p_2) - 2\psi(1)$$

Möbius-invariant solution (L.)

$$\Psi = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}} \right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*} \right)^{\tilde{m}}, \quad m = i\nu + \frac{1+n}{2}, \quad \tilde{m} = i\nu + \frac{1-n}{2}$$

4 Phase of the mixed representation

Pomeron wave function in the mixed representation

$$\Psi(\vec{\rho}, \vec{q}) = \int d^2 R e^{i\vec{q}\vec{R}} \Psi(\vec{\rho}_1, \vec{\rho}_2), \quad \vec{\rho}_1 = \vec{R} + \frac{\vec{\rho}}{2}, \quad \vec{\rho}_2 = \vec{R} - \frac{\vec{\rho}}{2}$$

Asymptotic behavior of Ψ at small ρ for $n = 0$

$$\lim_{\rho \rightarrow 0} \Psi(\vec{\rho}, \vec{q}) \sim \rho^{1+2i\nu} + e^{i\delta} \rho^{1-2i\nu}$$

Phase δ at a finite temperature and the Higgs model

$$e^{i\delta_{m,\tilde{m}}^0(\vec{Q})} = (-1)^n \left(\frac{|Q|}{4} \right)^{-4i\nu} \left(\frac{Q}{Q^*} \right)^n \frac{\Gamma(m + \frac{1}{2})}{\Gamma(\frac{3}{2} - m)} \frac{\Gamma(\tilde{m} + \frac{1}{2})}{\Gamma(\frac{3}{2} - \tilde{m})}$$

$$e^{i\delta_{m,\tilde{m}}^T(\vec{Q})} = |Q|^{4i\nu} (-1)^n \left(\frac{Q^*}{Q} \right)^n \frac{\Gamma(1 - m + iQ)}{\Gamma(m + iQ)} \frac{\Gamma(1 - \tilde{m} - iQ^*)}{\Gamma(\tilde{m} - iQ^*)}.$$

$$\delta_H = 1.865 \nu$$

5 Pomeron in next-to-leading order

Eigenvalue of BFKL kernel at QCD in NLO (F., L. (1998))

$$\omega = 4 \hat{a} \chi(n, \gamma) + 4 \hat{a}^2 \Delta(n, \gamma), \quad \hat{a} = g^2 N_c / (16\pi^2), \quad \gamma = 1/2 + i\nu$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n, \gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}, \quad M = \gamma + \frac{|n|}{2},$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2), \quad \beta'(z) = \frac{1}{4} \left[\Psi' \left(\frac{z+1}{2} \right) - \Psi' \left(\frac{z}{2} \right) \right]$$

Maximal transcendentality (2002) and integrability (1997)

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M) \left(\Psi(1) - \Psi(M) \right),$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M} \left(\Psi'(k+1) - \frac{\Psi(k+1) - \Psi(1)}{k+M} \right)$$

6 Pomeron and graviton in N=4 SUSY

Eigenvalue of the BFKL kernel in a diffusion approximation

$$j = 2 - \Delta - \Delta \nu^2, \quad \gamma = 1 + \frac{j-2}{2} + i\nu$$

AdS/CFT relation for the graviton Regge trajectory

$$j = 2 + \frac{\alpha'}{2} t, \quad t = E^2/R^2, \quad \alpha' = \frac{R^2}{2} \Delta$$

Large coupling expansion of Δ (KLOV, BPST, KL)

$$\Delta = 2\lambda^{-1/2} + \lambda^{-1} - 1/4 \lambda^{-3/2} - 2(1 + 3\zeta_3)\lambda^{-2} + \dots, \quad \lambda = \frac{\alpha N_c}{2\pi}$$

Exact expression for the slope of γ at $j = 2$ (KLOV, V., Basso)

$$\gamma'(2) = -\frac{\lambda}{24} + \frac{1}{2} \frac{\lambda^2}{24^2} - \frac{2}{5} \frac{\lambda^3}{24^2} + \frac{7}{20} \frac{\lambda^4}{24^4} - \frac{11}{35} \frac{\lambda^5}{24^5} + \dots = -\frac{\sqrt{\lambda}}{4} \frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}$$

7 Equation with running coupling

Simplified BFKL equation in QCD at $q = 0$

$$\omega f_\omega(t) = \frac{1}{ct} \chi(\hat{\nu}) f_\omega(t), \quad t = \ln \frac{k^2}{\Lambda_{QCD}^2}, \quad \hat{\nu} = -i \frac{\partial}{\partial t}, \quad c = \frac{11}{12} - \frac{n_f}{18}$$

Pomeron wave functions falling at large t

$$f_\omega(t) = \int_{-\infty}^{\infty} d\nu e^{it\nu} g_\omega(\nu), \quad g_\omega(\nu) = e^{-\frac{2i\nu}{c\omega} \psi(1)} \left(\frac{\Gamma(1/2 + i\nu)}{\Gamma(1/2 - i\nu)} \right)^{\frac{1}{c\omega}}$$

Equation for saddle points $\nu = \pm \tilde{\nu}_\omega(t)$

$$\omega = \frac{1}{ct} \chi(\tilde{\nu}_\omega(t)), \quad \chi(\nu) = 2\psi(1) - \psi(1/2 + i\nu) - \psi(1/2 - i\nu)$$

Oscillations of eigenfunctions at small t

$$f_\omega \sim \cos \delta_\omega^p(t), \quad \delta_\omega^p(t) = \frac{\pi}{4} + t \tilde{\nu}_\omega(t) - \frac{2\psi(1)}{c\omega} \tilde{\nu}_\omega(t) + \frac{\Im}{c\omega} \ln \frac{\Gamma(1/2 + i\tilde{\nu}_\omega(t))}{\Gamma(1/2 - i\tilde{\nu}_\omega(t))}$$

8 Matching to non-perturbative physics

Green function of the BFKL equation

$$G_0^\omega(t, t') = -\frac{it'}{\omega} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} e^{it\nu} g_\omega(\nu) \int_{-\infty}^{\infty} \frac{d\nu'}{2} \epsilon(\nu - \nu') e^{it'\nu'} g_\omega(\nu')$$

Integral operator of evolution in rapidity Y

$$G_0^Y(t, t') = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} e^{Y\omega} G_0^\omega(t, t'), \quad \lim_{Y \rightarrow 0} G_0^Y(t, t') = \delta(t - t')$$

Oscillations of G_0 at small t'

$$G_0^\omega(t, t') \approx \frac{t'}{\omega} f_\omega(t) (2\pi(\chi'(-\tilde{\nu}_\omega))^{-1/2} \sin \delta_\omega^p(t'))$$

Green function with a non-perturbative phase

$$G^\omega(t, t') = G_0^\omega(t, t') + \frac{t' \cot \phi_\omega}{4\pi\omega} f_\omega(t) f_\omega(t'), \quad \delta_\omega^{np}(t) = \phi_\omega + \delta_\omega^p(t)$$

9 Spectrum of Pomerons in QCD

Dispersion representation for the Green function

$$G^\omega(t, t') = \sum_{n=1}^{\infty} \frac{c_n(t)}{\omega - \omega_n}, \quad \phi(\omega_n) \approx \frac{a}{\omega_n} - \eta = \pi n$$

Spectrum of Pomerons and physics BSM (KLR)

$$\omega_n \approx \frac{0.5}{1 + 0.95n}, \quad \bar{k}_n \approx \Lambda_{QCD} e^{4n}, \quad \bar{k}_3 \approx 10 TeV$$

DGLAP equation for the Pomeron wave function

$$-i \frac{d}{dt} f_\omega(t) = \tilde{\nu}_\omega(t) f_\omega(t), \quad \chi(\tilde{\nu}_\omega(t)) = c\omega t$$

BFKL singularity and the "turning" point t_c

$$\tilde{\nu}_\omega(t) \sim \sqrt{\omega - \frac{4\alpha_c(t)N_c}{\pi} \ln 2}, \quad \frac{4\alpha_c(t_c)N_c}{\pi} \ln 2 = \omega$$

10 Phase ϕ_ω in the Higgs model

Pomeron wave function at the Higgs model (LLS)

$$\lim_{k^2 \rightarrow \infty} f(\nu, \frac{k^2}{m^2}) \approx \sin \left(\nu \ln \frac{6.456 k^2}{m^2} \right)$$

Running coupling and phase δ_ω^p at Higgs scale

$$\alpha(t_m) = \frac{\pi}{3c \ln t_m}, \quad \delta_{\omega,m}^p = \delta_\omega^p(t_m), \quad t_m = \ln \frac{m^2}{6.456 \Lambda_s^2}$$

Matching of perturbative and "non-perturbative" phases

$$\phi_\omega = \frac{\pi}{2} - \delta_{\omega,m}^p = \pi n$$

Quantization of Pomeron intercepts

$$\omega_n \approx \frac{3.68}{11 - \frac{2}{3} n_f} \frac{1}{n + \frac{1}{4}}$$

11 Pomeron at a thermostat

Gluon coordinates and momenta at a non-zero temperature

$$\rho_r = x_r + iy_r, \quad 0 < y_r < 1/T_t; \quad p_r = p_r^x + ip_r^y, \quad p_r^y = 2\pi T_t k, \quad k = 0, \pm 1, \pm 2, \dots$$

BFKL Hamiltonian at a thermostat with $T_t \neq 0$ (de Vega, Lipatov)

$$h = \sum_{s=1,2} \left(\psi \left(1 + i \frac{p_s}{2\pi T_t} \right) + \psi \left(1 - i \frac{p_s}{2\pi T_t} \right) - 2\psi(1) + \frac{2}{p_s} \ln(2 \sinh(\pi T_t \rho_{12})) p_s \right)$$

Conformal transformation to the zero temperature and integrability

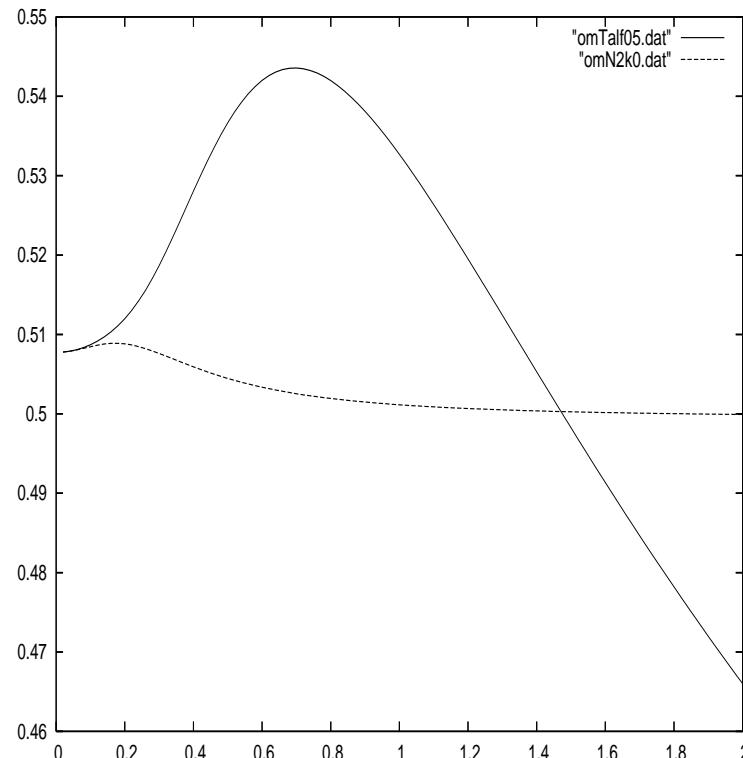
$$\rho_r = \frac{1}{2\pi T_t} \ln \rho'_s$$

Running α_s and Pomeron trajectories for $T_t \neq 0$

$$\omega_n(\vec{q}) = \alpha_s(q) \chi(\nu_n), \quad q = q_x + 2\pi Ni, \quad \delta_\omega^{np}(q, T) - \delta_\omega^p(q) = \pi n$$

12 T-dependence of Regge trajectories

Leading trajectories $\omega(\vec{q}, T)$ at $\alpha_s(q) = 0.5$ for $N = 0$ and $N=2$
(H. de Vega, L. Lipatov (2013))



Gluon attraction for increasing T and anti-Meissner effect

Confinement can be imitated by a cylinder-type topology (a bag)

13 BFKL Pomeron in a rectangle bag

BFKL hamiltonian on a torus

$$H = T + V, \quad T = \ln |\tilde{p}_1|^2 + \ln |\tilde{p}_2|^2, \quad V = v + v^*$$

Regge trajectory on the torus

$$\ln |\tilde{p}|^2 = \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \frac{T_x^{-1} T_y^{-1} \pi^{-1}}{\frac{n_x^2}{T_x^2} + \frac{n_y^2}{T_y^2}} \frac{p_x^2 + p_y^2}{(p_x - 2\pi \frac{n_x}{T_x})^2 + (p_y - 2\pi \frac{n_y}{T_y})^2}$$

Potential energy and Green function

$$v = \frac{1}{p_1 p_2^*} G(\vec{\rho}_{12}) p_1 p_2^*, \quad G(\vec{\rho}) = \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \frac{T_x^{-1} T_y^{-1} (2\pi)^{-1}}{\frac{n_x^2}{T_x^2} + \frac{n_y^2}{T_y^2}} e^{2\pi i (\frac{x n_x}{T_x} + \frac{y n_y}{T_y})}$$

Equation for the Pomeron trajectories

$$\omega_n(\vec{q}) = \alpha_s(q) \chi(\nu_n), \quad \delta_{\omega}^{np}(q, T_x, T_y) - \delta_{\omega, q}^p = \pi n$$

14 Conformal transformations

Pomeron function on the impact parameter plane

$$\Psi = \left(\frac{\rho'_{12}}{\rho'_{10}\rho'_{20}} \right)^m \left(\frac{\rho'^*_{{12}}}{\rho'^*_{{10}}\rho'^*_{{20}}} \right)^{\tilde{m}}$$

Transformation to a rectangle $|\Re\rho| < a, 0 < \Im\rho < b$

$$\rho' = sn \left(\frac{\rho K}{a}; k \right) = sn \left(\frac{\rho K'}{ib}; k' \right), \quad k^2 + k'^2 = 1$$

Inverse transformation

$$\rho = \frac{a}{K} \int_0^{\rho'} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad K = \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

Pomeron trajectory quantization

$$\omega_n(\vec{q}) = \alpha_s(q) \chi(\nu_n), \quad \delta_\omega^{np}(q, a, b) - \delta_{\omega,q}^p = \pi n$$

15 Non-Fredholm properties of the BFKL kernel

Divergency of the Fredholm integral for the BFKL kernel

$$\int d^2 p d^2 p' |K(p^2, p'^2, (p - p')^2)|^2 = \infty$$

Asymptotic freedom at large p and confinement at small p

$$\lim_{p \sim p' \rightarrow \infty} |K(p^2, p'^2, (p - p')^2)|^2 \sim f(p/p') \frac{1}{|p|^2 |p'|^2} \frac{1}{\ln^2(\max(p^2, p'^2))}$$

Divergency of K at $p - p' = 0$ and a continuous spectrum at $\omega < 0$

$$|K(p^2, p'^2, (p - p')^2)|^2 \sim (\omega(p^2) \delta^2(p - p'))^2$$

Semiclassical prediction for the BFKL spectrum in $N = 4$ SUSY

$$\lim_{|m| \rightarrow \infty} \omega^{(0)}(n, \nu) = -\gamma_K(a) \ln |m|$$

16 Discussion

1. Production amplitudes in multi-Regge kinematics
2. BKP equation in the Higgs model and QCD
3. Running coupling and infrared boundary conditions
4. Spectrum of Pomerons in QCD and confinement
5. Pomeron at the thermostat and the anti-Meisner effect
6. Compactification of the impact parameter space
7. Fredholm integral divergency and BFKL eigenvalues at large ν