Unknown Preconditions and Abnormal Features of Cherenkov Radiation and X-ray (γ-ray) Laser Amplification in Realistic Media

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The main problem at creation of X-ray Cherenkov generator is connected with satisfaction of threshold conditions

$$\beta n(\omega)\cos\theta \equiv \beta\sqrt{\varepsilon_{eff}(\omega)}\cos\theta = 1;\ \beta = v/c$$

According standard point of view effective characteristics (susceptibility $\chi_{eff}(\omega)$ and the permittivity $\varepsilon_{eff}(\omega)$) must be determined by averaging the local characteristics within the spatial region that exceeds the average size $\bar{d}$ of the medium's atomic (molecular) inhomogeneities

$$\varepsilon_{eff}(\omega) \equiv <\varepsilon(\vec{r},\omega)>_{local} \equiv$$

$$1 + <\chi(\vec{r},\omega)>_{local} \equiv 1 + \chi_{eff}(\omega)$$

(4π is inserted in $\chi(r,\omega)$).

This **semi-intuitive** method corresponds to a transition to macroscopic electrodynamics.

In this approach the wave equation for X-ray waves is written **(in fact, postulated**) in the form of the equation

$$\Delta\vec{E}(\vec{r},\omega)+(\omega/c)^2\varepsilon_{eff}(\omega)\vec{E}(\vec{r},\omega)=0,$$

determining the evolution of the locally averaged field

$$\vec{E}(\vec{r},\omega)\equiv<\vec{E}(\vec{r},\omega)>_{local}$$

in a medium with the locally averaged characteristics

$$\varepsilon_{eff}(\omega)=<\varepsilon(\vec{r},\omega)>\equiv 1+\chi_{eff}(\omega)=1+<\chi(\vec{r},\omega)>$$

In this case, the electron susceptibility corresponds to the “plasma” approximation

$$\chi_{eff}(\omega)\equiv<\chi(\vec{r},\omega)>=-4\pi<n_e(\vec{r})>e^2/m\omega^2\equiv-<\omega_p^2>/\omega^2$$

In the case of resonant nuclei, we have $\chi_{eff}(\omega)=\dfrac{4\pi<n_n(r)>c\sigma_0\gamma}{\omega_0^2-\omega^2-i\gamma\omega}$;

$$\sigma_0=\frac{\lambda^2}{2\pi}\frac{2J_2+1}{2J_1+1}\frac{f}{1+\alpha}-cross\,section\,of\,resonat\;scattering$$

The simple analysis has shows that this method of introduction of $\chi_{eff}(\omega)$ is incorrect and leads to significant errors!

In what follows, we show that the heuristic assumption

$$\chi_{eff}(\omega) \equiv < \chi(\vec{r}, \omega) >$$

which is used in analysis of nondiffracted beam propagation, is incorrect in the X- and γ-ray ranges, and the actual value of $\chi_{eff}(\omega)$ depends not only on $\chi(\vec{r}, \omega)$ but also on $grad\ \chi(\vec{r}, \omega) \equiv \nabla \chi(\vec{r}, \omega)$ which changes the condition of Cherenkov radiation

Wave equation for X-Ray radiation in real (actual) media

From the system of Maxwell equations for nonmagnetic medium

$$\nabla \times \vec{H}(\vec{r},\omega) = -i\frac{\omega}{c}\vec{D}(\vec{r},\omega),\ \nabla \times \vec{E}(\vec{r},\omega) = i\frac{\omega}{c}\vec{H}(\vec{r},\omega)$$

in the case of harmonic fields

$$\vec{E} = \vec{E}(\vec{r},\omega)e^{-i\omega t},\ \vec{H} = \vec{H}(\vec{r},\omega)e^{-i\omega t},\ \vec{D} = \varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)e^{-i\omega t}$$

follows the equation for the electric field

$$\nabla \times \nabla \times \vec{E}(\vec{r},\omega) \equiv \nabla\nabla\vec{E}(\vec{r},\omega) - \Delta\vec{E}(\vec{r},\omega) = (\omega / c)^2 \varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$$

This equation can be transformed using the relation

$$\nabla\vec{E}(\vec{r},\omega) = -\{\nabla\varepsilon(\vec{r},\omega) / \varepsilon(\vec{r},\omega)\}\vec{E}(\vec{r},\omega)$$

following from the “material” equation of the system of Maxwell equations:

$$\nabla\vec{D}(\vec{r},\omega) \equiv \nabla\{\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)\} = 4\pi\rho_q = 0$$

in the absence of uncompensated electric charges in the medium.

After substituting $\nabla\vec{E}(\vec{r},\omega)$ we obtain the modified **wave equation** in real (not model) medium with the real **nonuniform distribution atoms and molecules**

$$\Delta\vec{E}(\vec{r},\omega)+(\omega/c)^2\tilde{\varepsilon}(\vec{r},\omega)\vec{E}(\vec{r},\omega)=0;$$

Here

$$\tilde{\varepsilon}(\vec{r},\omega)=\varepsilon(\vec{r},\omega)+\frac{c^2}{\omega^2}\left\{\nabla\left(\frac{\nabla\varepsilon(\vec{r},\omega)}{\varepsilon(\vec{r},\omega)}\right)-\left(\frac{\nabla\varepsilon(\vec{r},\omega)}{\varepsilon(\vec{r},\omega)}\right)^2\right\}\equiv$$

$$\varepsilon(\vec{r},\omega)+\frac{c^2}{\omega^2}\left\{\frac{\Delta\varepsilon(\vec{r},\omega)}{\varepsilon(\vec{r},\omega)}-2\left(\frac{\nabla\varepsilon(\vec{r},\omega)}{\varepsilon(\vec{r},\omega)}\right)^2\right\}$$

- effective local dielectric permittivity of nonuniform medium.

[V.I.Vysotskii, M.V.Vysotskyy **Conditions for Generation of X-Ray Cherenkov Radiation during Motion of Charges in Realistic Media.** *Journal of Surface Investigation. X-ray, Synchrotron and Neutron Techniques, 2013, Vol. 7, No. 1, 51*]

In the one-dimensional case, the solution

$$E(\vec{r},\omega) = A(\vec{r},\omega)\exp\{iS(\vec{r},\omega)\}$$

of modified wave equation

$$\frac{d^2E(x,\omega)}{dx^2} + \tilde{k}^2(x,\omega)E(x,\omega) = 0,\ \tilde{k}^2(x,\omega) \equiv \frac{\omega^2}{c^2}\tilde{\varepsilon}(x,\omega)$$

under the condition $|d^2A(x)/dx^2| << |\tilde{k}^2(x)A(x)|$

has the simple form

$$E(x) = E_0\sqrt{\tilde{k}(0)/\tilde{k}(x)}\exp\left\{\pm i\int_0^x \tilde{k}(x)dx\right\} \equiv$$

$$E_0\sqrt{\tilde{k}(0)/\tilde{k}(x)}\exp\left\{\pm i < \tilde{k}(x) > x\right\};\ < \tilde{k}(x) > =$$

$$\frac{\omega}{c}\left\langle\left[\varepsilon(x,\omega) + \frac{c^2}{\omega^2}\frac{1}{\varepsilon(x,\omega)}\frac{d^2\varepsilon(x,\omega)}{dx^2} - 2\frac{c^2}{\omega^2}\left(\frac{1}{\varepsilon(x,\omega)}\frac{d\varepsilon(x,\omega)}{dx}\right)^2\right]^{1/2}\right\rangle$$

Taking into account that in UV and X-ray and gamma-ray ranges

$$\tilde{\chi}(x,\omega)=\tilde{\varepsilon}(x,\omega)-1,\ |\tilde{\chi}(x,\omega)|<<1$$

we find $E(x)\approx$

$$E_0\exp\left\{\pm i\frac{\omega}{c}\left[1+\frac{\bar{\chi}(\omega)}{2}+\frac{c^2}{2\omega^2}\left(\left\langle\frac{d^2\chi(x,\omega)}{dx^2}\right\rangle-\left\langle\chi(x)\frac{d^2\chi(x,\omega)}{dx^2}\right\rangle-2\left\langle\left(\frac{d\chi(x,\omega)}{dx}\right)^2\right\rangle\right)\right]\right.$$

In one-dimensional spatially periodic medium with the period L for which

$$\chi(x,\omega)=\chi(x\pm L,\omega) \quad \text{and} \quad d\chi(x,\omega)/dx\,|_x=d\chi(x,\omega)/dx\,|_{x\pm L}$$

we have the **final expressions for electric field and effective dielectric susceptibility in crystall**

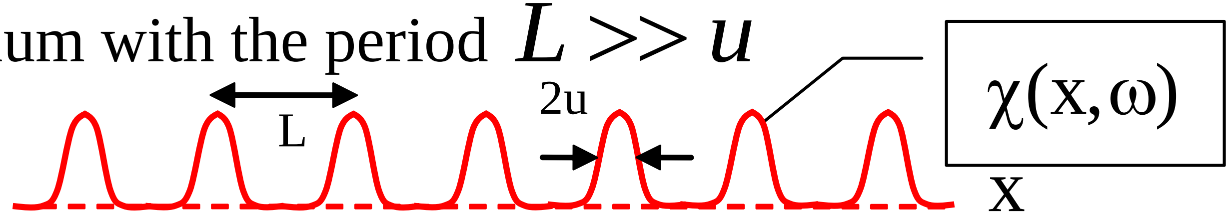
$$E(x)=E_0\exp\left\{\pm i\frac{\omega}{c}\left(1+\frac{\chi_{eff}(\omega)}{2}\right)x\right\};$$

$$\chi_{eff}(\omega)=<\chi(x,\omega)>-2\frac{c^2}{\omega^2}<\left(\frac{d\chi(x,\omega)}{dx}\right)^2>$$

Let's consider the model case of the one-dimensional susceptibility:

$$\chi(x,\omega)=\sum_n\left\{\bar{\chi}(\omega)L/\sqrt{\pi u^2}\right\}\exp\left[-(x-nL)^2/u^2\right] \qquad (*)$$

in a medium with the period $L >> u$



Here, the quantity u is the **rms amplitude of the thermal oscillations** of atoms forming the crystal plane.

The corresponding coefficients in (*) are determined by taking into account that the condition $<\chi(x,\omega)>=\bar{\chi}(\omega)$ is satisfied.

In such a one-dimensional medium

$$\chi_{eff}(\omega)=\bar{\chi}-\frac{c^2}{\omega^2}\bar{\chi}^2\frac{1}{L}\int_{-\infty}^{\infty}\frac{4x^2L^2}{\pi u^6}e^{-2x^2/u^2}dx=$$

$$=\bar{\chi}-\bar{\chi}^2\frac{\lambda^2 L}{4\sqrt{2}\pi^{5/2}u^3}\equiv\bar{\chi}-\bar{\chi}^2K,\ K\approx 0.01(\lambda^2 L/u^3)$$

It follows from these results that in the case of a spatially nonuniform distribution of electrons, the effective susceptibility of the crystal decreases

$$\chi_{eff}(\omega) = \bar{\chi} - \bar{\chi}^2 K,\ K \approx 0.01(\lambda^2 L / u^3)$$

and, accordingly, the threshold energy

$$E_{tresh} = \gamma_{tresh} mc^2 = mc^2 / \sqrt{1 - \beta_{tresh}^2} \approx mc^2 / \sqrt{\bar{\chi}(\omega)},$$

$$\beta_{tresh} \bar{n}(\omega) = 1, \quad \beta_{tresh} = 1 / \sqrt{1 - \bar{\chi}(\omega)}$$

of fast charged particles, which is required to generate Cherenkov radiation, increases by the quantity

$$\delta E_{tresh} \approx mc^2 K \sqrt{\bar{\chi}(\omega)} / 2,$$

$$\delta E_{tresh} / E_{tresh} \approx K \bar{\chi}(\omega) / 2$$

For this model, in the case of actual parameters of the crystal and X-ray radiation (

$$\lambda \approx 10^{-8}\, cm,\ u \approx 2.10^{-9}\, cm$$

$$L \approx 2.10^{-8}\, cm,\ \bar{\chi} \approx 10^{-5} - 10^{-6}$$

the condition

$$| d^2 A(x) / dx^2 | << | \tilde{k}^2 (x) A(x) |$$

takes the form

$$\left| \frac{d}{dx} \frac{1}{\tilde{k}(x)} \right| \approx \frac{\lambda \bar{\chi} L}{2\pi^{3/2} u^2} \approx 10^{-4} - 10^{-5} << 1$$

This criterion can be valid not only in the case of $\lambda < u$, but also for $\lambda > u$ (e.g. for Mossbauer nuclei), if the entire set of parameters satisfies the condition

$$\frac{\lambda \bar{\chi} L}{2\pi^{3/2} u^2} << 1$$

For Mossbauer Fe^{57} isotope with $E_\gamma = 14.4\, keV$, $\lambda \approx 0.7 * 10^{-8}\, cm$ and $\bar{\chi}_n \approx 10^{-4}$, $u \approx 10^{-9}\, cm$ we have

$$\left| \frac{d}{dx} \frac{1}{\tilde{k}(x)} \right| \approx \frac{\lambda \bar{\chi} L}{2\pi^{3/2} u^2} \approx 10^{-2} << 1$$

In 3-D system (e.g. crystal) we have

$$\Delta\vec{E} + \frac{\tilde{\varepsilon}(\vec{r})\omega^2}{c^2}\vec{E} = 0;\ \tilde{\varepsilon}(\vec{r}) = \varepsilon(\vec{r})\left\{1 + \frac{c^2}{\omega^2}\left[\nabla\left(\frac{\nabla\varepsilon}{\varepsilon}\right) - \left(\frac{\nabla\varepsilon}{\varepsilon}\right)^2\right]\right\}$$

If

$$\chi(r) \equiv \varepsilon(\vec{r}) - 1 = \bar{\chi}V\frac{1}{(\pi u)^{3/2}}e^{-r^2/u^2} \rightarrow \bar{\chi}V\frac{1}{(\pi u)^{3/2}}\sum_{n,l,m}e^{-(\vec{r}-\vec{r}_{nlm})^2/u^2}$$

that

$$\varepsilon_{eff} = \left\{1 + \bar{\chi} - \bar{\chi}^2\frac{\lambda^2 L^3}{64\sqrt{2}\pi^{5/2}u^5}\right\} = 1 + \bar{\chi} - \bar{\chi}^2 K_3, K_3 = 5.10^{-4}\frac{\lambda^2 L^3}{u^5}$$

$$At\ \lambda \approx 10^{-8}\,cm,\ u \approx 10^{-9}\,cm,\ L \approx 2.10^{-8}\,cm,\ \bar{\chi} \approx 10^{-4}$$

$$(e.g.,\ \ Fe^{57}\ Mossbauer\ isotope)\ we\ \ have:$$

$$K_3 \approx 400; \qquad \delta E_{tresh}/E_{tresh} \approx K_3\bar{\chi}(\omega)/2 \approx 0.02$$

In the opposite case (if the quantities λ and $\bar{\chi} \sim \lambda^2 < n_e >$ are increased to $\lambda \approx 10^{-6}\, cm$ and $\bar{\chi} \approx 10^{-1} - 10^{-3}$, which corresponds to the VUV range for light atoms, this criterion is invalid and

$$\left| \frac{d}{dx} \frac{1}{\tilde{k}(x)} \right| \approx 10^1 - 10^3 >> 1$$

This result shows that solution

$$\chi_{eff}(\omega) = < \chi(x,\omega) > -2\frac{c^2}{\omega^2} < \left(\frac{d\chi(x,\omega)}{dx} \right)^2 > = \bar{\chi} - \bar{\chi}^2 \frac{\lambda^2 L}{4\sqrt{2}\pi^{5/2} u^3}$$

are inapplicable in “ordinary” optics in the case of the propagation in crystals of radiation with wavelengths that are longer than X-ray radiation and **do not change previously known optical laws**!

Features of X-ray laser amplification in realistic (nonuniform, inhomogeneous) media

The presence of absorption ($\bar{\chi}'' \neq 0$) lead to modification of the expression for χ_{eff} :

$$\chi_{eff} = \bar{\chi} - \bar{\chi}^2 K = \chi'_{eff} + i\chi''_{eff};$$

$$\chi'_{eff} = \bar{\chi}' - (\bar{\chi}')^2 K + (\bar{\chi}'')^2 K, \quad \chi''_{eff} = \bar{\chi}''(1 - 2\bar{\chi}' K)$$

Two important results follows from this result.

The presence of absorption in a spatially inhomogeneous medium:

a) increases the real part of the effective susceptibility χ'_{eff} (which assists in the generation of Cherenkov radiation);

b) Changes resonant absorption (or amplification) by **decreasing** $|\chi''_{eff}|$ in the range of frequencies $\omega < \omega_0$ with $\bar{\chi}' > 0$ and by **increasing** it for $\bar{\chi}' < 0$ (at $\omega > \omega_0$).

The presence of the spatial structure of the medium leads to change in condition of X-Ray (γ-Ray) laser generation.

The "standard" coefficient of laser amplification in inverted system of resonant nuclei are the following [V.I. Vysotskii, R.N. Kuz'min. *Gamma-Lasers*, Moscow, 1989.]

$$G_{\textbf{homog}} = \frac{\omega}{c}\bar{\chi}'' = \frac{\omega}{c}\,\mathrm{Im}\left\{\frac{<\Delta n(r)> c\sigma_0\gamma}{\omega_0^2 - \omega^2 - i\gamma\omega}\right\} = \frac{<\Delta n(r)>\sigma_0\gamma^2}{(\omega_0-\omega)^2+\gamma^2};$$

$$<\Delta n_n> = <n_n^{(e)}> - <n_n^{(g)}>$$ - average value of the inverse population on resonant transition;

$$\sigma_0 = \frac{\lambda^2}{2\pi}\frac{2J_2+1}{2J_1+1}\frac{f}{1+\alpha}$$ - cross-section of resonant transition.

Coefficient of laser amplification in inverted system of resonant nuclei for inhomogeneous medium:

$$G_{\mathit{in}\,\mathrm{hom}\,\mathit{og}} = \frac{\omega}{c}\chi''_{\mathit{eff}} \equiv \frac{\omega}{c}\bar{\chi}''(1-2\bar{\chi}'K) \approx$$

$$\frac{<\Delta n(r)>\sigma_0\gamma^2}{(\omega_0-\omega)^2+\gamma^2}\left\{1+K\left(\frac{<\Delta n_n>\sigma_0 c\gamma(\omega-\omega_0)}{\omega[4(\omega_0-\omega)^2+\gamma^2/4]} + \frac{8\pi <n_e(\vec{r})> e^2}{m\omega^2}\right)\right\}$$

Conclusions

Presented results demonstrate the essential influence of inhomogeneous (atomic and electronic) structure of material media in which fast charged particles travel to effective susceptibility and permittivity in the X-ray range and to condition of Cherenkov radiation, as compared with cases of model homogeneous media with the same average concentration of electrons.

It is shown that the function of the spatial distribution of electrons and nuclei in a target affects the conditions for generating laser radiation in the X- and γ-ray ranges (on the problem of X- and γ-ray lasers).