Effective mass of a photon in a strong magnetic field

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Abstract

1. In 1971, Adler had calculated the polarization operator in a constant homogenous magnetic field using the proper-time technique developed by Schwinger.The polarization operator on mass shell ($k^2 = 0$, the metric $ab = a^0 b^0 - \mathbf{ab}$ is used) in a strong magnetic field $H \gtrsim H_0 = m^2/e = 4 \cdot 10^{13}$ G (the system of units $\hbar = c = 1$ is used) was investigated well enough in the energy region lower the pair creation threshold. Here we consider the polarization operator for energies less than the third creation threshold of electron and positron on Landau levels. We investigate also the effective mass in the region of large thresold number using the quasiclassical approach. Our analysis is based on the general expression for the contribution of spinor particles to the polarization operator. For the case of pure magnetic field we have in a covariant form the following expression

$$\Pi^{\mu\nu} = -\sum_{i=2,3} \kappa_i \beta_i^\mu \beta_i^\nu,$$
$$\beta_i \beta_j = -\,\delta_{ij}, \quad \beta_i k = 0;$$
$$\beta_2^\mu = (F^* k)^\mu / \sqrt{-(F^* k)^2},$$
$$\beta_3^\mu = (F k)^\mu / \sqrt{-(F^* k)^2},$$
$$\mathrm{Tr} F F^* = 0,$$
$$\mathrm{Tr} F^2 = F^{\mu\nu} F_{\mu\nu} = 2(H^2 - E^2)$$
$$\equiv 2f > 0, (Fk)^\mu = F^{\mu\nu} k_\nu,$$

$$\kappa_i = \frac{\alpha}{\pi} m^2 r$$
$$\times \int_{-1}^{1} dv \int_{0}^{\infty - \mathrm{i}0} f_i(v, x) \exp[\mathrm{i}\psi(v, x)] dx.$$

$$f_2(v, x) = 2\frac{\cos(vx) - \cos x}{\sin^3 x} -$$
$$\frac{\cos(vx)}{\sin x} + v\frac{\cos x \sin(vx)}{\sin^2 x},$$
$$f_3(v, x) = \frac{\cos(vx)}{\sin x} - v\frac{\cos x \sin(vx)}{\sin^2 x}$$
$$-(1 - v^2) \cot x,$$
$$\psi(v, x) = \frac{1}{\mu}(2r\frac{\cos x - \cos(vx)}{\sin x} +$$
$$[r(1 - v^2) - 1]x);$$

$$r = -(F^* k)^2/4m^2 f, \quad \mu^2 = f/H_0^2.$$

The real part of κ_i determines the refractive index n_i of the photon with polarization $e_i = \beta_i$: $n_i = 1 - \frac{\mathrm{Re}\kappa_i}{2\omega^2}$.
At $r > 1$, the proper value of polarization operator κ_i includes the imaginary part which determines the probability per unit length of pair production by photon with the polarization β_i. For $r < 1$, the integration counter over x can be turn to the lower semiaxis ($x \to -\mathrm{i}x$), then the value κ_i becomes real in an explicit form.
2. At $r < 1$, the expression for κ_i takes the form:

$$\kappa_i = \alpha m^2 \frac{r}{\pi} \int_{-1}^{1} dv \int_{0}^{\infty} F_i(v,x) \exp[-\chi(v,x)] dx,$$

$$F_2(v,x) = \frac{1}{\sinh x} (2 \frac{\cosh x - \cosh(vx)}{\sinh^2 x}$$

$- \cosh(vx) + v \sinh(vx) \coth x),$

$F_3(v,x) = \frac{\cosh(vx)}{\sinh x} - v\frac{\cosh x \sinh(vx)}{\sinh^2 x} -$
$(1-v^2)\coth x;$

$\chi(v,x) = \frac{1}{\mu}[2r\frac{\cosh x - \cosh(vx)}{\sinh x} + (rv^2 - r + 1)x].$

For the energy sufficiently close to the thresold $(1-r)/\mu \ll 1$, we add to the integrand for κ_3 and take off the function

$(1-v^2)\exp[-\chi_{00}(v,x)],$
$\chi_{00}(v,x) = \frac{1}{\mu}\left[2r + (rv^2 - r + 1)x\right]$

Integrating over x the deducted part of the integrand, we have

$$\kappa_3^{00} = -\alpha m^2 \frac{\mu}{\pi} \exp\left(-\frac{2r}{\mu}\right) \int\limits_{-1}^{1} dv \frac{r(1-v^2)}{rv^2 - r + 1}.$$

After integration over v, we recover κ_3 in the following well-behaved form:

$$\kappa_3 = \kappa_3^1 + \kappa_3^{00}, \quad \kappa_3^1 = \alpha m^2 \tfrac{r}{\pi} \int\limits_{-1}^{1} dv \int\limits_{0}^{\infty} dx (F_3(v,x)$$
$$\times \exp[-\chi(v,x)] + (1-v^2) \exp[-\chi_{00}(v,x)]),$$

$$\kappa_3^{00} = \alpha m^2 \tfrac{\mu}{\pi} \exp\left(-\tfrac{2r}{\mu}\right) [2 + B(r)] \, ;$$

$$B(r) = \frac{2}{\sqrt{r(1-r)}} \arctan \sqrt{\frac{1-r}{r}} - \frac{\pi}{\sqrt{r(1-r)}} \ .$$

For superstrong fields ($\mu \gg 1$), the value $x \lesssim 1$ contributes in the integral for κ_2 and κ_3^1 and the exponential terms in the integrands can be substitute for unit. As a result, we have for the leading terms of expansion in series of μ:

$$\kappa_2 \simeq -\tfrac{4r}{3\pi} \alpha m^2,$$

$\kappa_3 \simeq \alpha m^2 \frac{\mu}{\pi}(2 + B(r)).$

Near the threshold, when $1 - r \ll 1$, we obtain:

$$\kappa_2 \simeq -\frac{4}{3\pi}\alpha m^2,$$
$$\kappa_3 \simeq -\alpha m^2 \mu \left(\frac{1}{\sqrt{1-r}} - \frac{4}{\pi}\right).$$

In low-energy range ($r \ll 1$), we have:

$$\kappa_2 \simeq -\frac{4r}{3\pi}\alpha m^2, \quad \kappa_3 \simeq -\frac{4r\mu}{3\pi}\alpha m^2.$$

3. We go on to the next energy region, which upper boundary is higher the second threshold r_{10} (but not too close to the third threshold r_{20}). On this threshold, one of the particles is created on the first excited level and another – in the ground state.
In general case

$$r_{lk} = (\varepsilon(l) + \varepsilon(k))^2/4m^2,$$
$$\varepsilon(l) = \sqrt{m^2 + 2eHl} = m\sqrt{1 + 2\mu l}.$$

For $1 < r < r_{10}$, the integration counter over x can be turn to the lower imaginary semiaxis, except the integrand term $-(1 - v^2)\cot x \exp[\mathrm{i}\psi(v, x)]$.
Let's add and take off the function

$$\mathrm{i}(1 - v^2)\exp[\mathrm{i}\psi_{\text{red}}(v, x)],$$
$$\psi_{\text{red}}(v, x) = \frac{1}{\mu}(2\mathrm{i}r + [r(1 - v^2) - 1])x.$$

For the sum of the functions, the integration counter over x can be turn to the lower semiaxis. For the residuary function, the integral over x has the following form

$$\int\limits_0^\infty \exp[\mathrm{i}\psi_{\text{red}}(v, x)]dx = \exp\left(-\frac{2r}{\mu}\right)\frac{\mathrm{i}\mu}{r(1-v^2)-1+\mathrm{i}0} =$$
$$\mu\exp\left(-\frac{2r}{\mu}\right)\left[\mathrm{i}\frac{\mathcal{P}}{r-1-rv^2} + \pi\delta\left(r - 1 - rv^2\right)\right].$$

The operator $\mathcal{P}$ means the principal value integral. Carrying out the integration over v, we have after not complicated calculations

$$-\mathrm{i}r\int\limits_{-1}^{1} dv(1-v^2)\left[\mathrm{i}\frac{\mathcal{P}}{r-1-rv^2}+\pi\delta\left(r-1-rv^2\right)\right]$$
$$=2+B(r),$$
$$B(r)=\frac{2}{\sqrt{r(r-1)}}\ln(\sqrt{r}+\sqrt{r-1})-\frac{\mathrm{i}\pi}{\sqrt{r(r-1)}}.$$

4. The integrals for κ_2 and κ_3^1 have the root divergence at $r=r_{10}$. To bring out these distinctions in an explicit form, let's consider the main asymptotic terms of corresponding integrand at $x\to\infty$:

$$\kappa_i^{10}=\alpha m^2 r\frac{2}{\pi}\int\limits_{-1}^{1} dv\int\limits_{0}^{\infty} d_i(v)\exp[-\chi_{10}(v,x)]dx,$$

$$d_2=v-1,\ d_3=1-v-\frac{2r}{\mu}(1-v^2),$$

$$\chi_{10}(v,x) = \chi_{00}(v,x) + 1 - v =$$
$$\frac{2r}{\mu} + \frac{1}{\mu}\left[(1-v)\mu + rv^2 - r + 1\right]x.$$

After elementary integration over x, one gets

$$\kappa_i^{10} = \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right) \int\limits_{-1}^{1} dv \frac{d_i(v)}{rv^2 - \mu v - r + 1 + \mu}.$$

Performing integration over v, we have:

$$\kappa_2^{10} = \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right)$$
$$\left[\frac{\mu/2r - 1}{\sqrt{h(r)}} A(r) - \frac{1}{2r}\ln(2\mu + 1)\right],$$

$$\kappa_3^{10} = \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right)$$
$$\left(\frac{\mu/2r - 1 - 2/\mu}{\sqrt{h(r)}} A(r) - \frac{1}{2r}\ln(2\mu + 1) + \frac{2}{\mu}\right),$$

$$A(r) = \arctan \frac{r-\mu/2}{\sqrt{h(r)}} + \arctan \frac{r+\mu/2}{\sqrt{h(r)}}$$
$$= \pi - \arctan \frac{\sqrt{h(r)}}{r-\mu/2} - \arctan \frac{\sqrt{h(r)}}{r+\mu/2},$$
$$h(r) = (1+\mu)r - r^2 - \mu^2/4.$$

For $r = r_{10} = (1+\mu+\sqrt{1+2\mu})/2,\ h(r) = 0$ and the values κ_i^{10} diverge at $r = r_{10}$:

$$\kappa_i^{10} \simeq -4\alpha m^2 r \exp\left(-\frac{2r}{\mu}\right) \frac{\beta_i}{\sqrt{h(r)}},$$
$$\beta_2 = \frac{\mu}{2} - \frac{\mu^2}{4r}, \quad \beta_3 = 1 + \frac{\mu}{2} - \frac{\mu^2}{4r}.$$

For higher photon energies $r > r_{10}$, a new channel of pair creation arises, and
$$\kappa_i^{10} = \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right) \int\limits_{-1}^{1} dv d_i(v)[\frac{\mathcal{P}}{rv^2-\mu v-r+1+\mu}$$
$$-\mathrm{i}\pi\delta(rv^2 - \mu v - r + 1 + \mu)];$$

At $r - r_{10} << 1$

$$\kappa_i^{10} \simeq -4\mathrm{i}\alpha m^2 r \exp\left(-\frac{2r}{\mu}\right) \frac{\beta_i}{\sqrt{-h(r)}}.$$

This direct procedure of divergence elimination can be extended further.

5. For strong fields and high energy levels ($\mu \gtrsim 1$, $r \gg \mu$), the main contribution to the integral is given by small values of $x \sim (\mu/r)^{1/3} << 1$. Expanding the entering functions over x, and carrying out the change of variable $x = \mu t$, we get:

$$\kappa_i = \frac{\alpha m^2 \kappa^2}{24\pi} \int_0^1 \alpha_i(v)(1-v^2)dv$$
$$\times \int_0^\infty t \exp[-\mathrm{i}(t + \xi \frac{t^3}{3})]dt;$$
$$\sqrt{\xi} = \frac{\kappa(1-v^2)}{4}, \; \alpha_2 = 3 + v^2, \quad \alpha_3 = 2(3 - v^2),$$
$$\kappa^2 = 4r\mu^2 = -\frac{(Fk)^2}{m^2 H_0^2}.$$

At $\kappa \gg 1$ ($\xi \gg 1$) the small t contributes to the integral ($\xi t^3 \sim 1$), and in the argument of exponent the linear over t term can be omit. The condition $\kappa >> 1$ is identically valid in this case. Carrying out the change of variable:

$\xi t^3/3 = -\mathrm{i}x, \quad t = \exp\left(\frac{-\mathrm{i}\pi}{6}\right)\left(\frac{3x}{\xi}\right)^{1/3},$

one obtains:

$$\kappa_i = \frac{\alpha m^2 \kappa^2}{24\pi} \exp\left(\frac{-\mathrm{i}\pi}{3}\right) \frac{1}{3} \left(\frac{48}{\kappa^2}\right)^{2/3}$$
$$\Gamma\left(\frac{2}{3}\right) \int\limits_0^1 dv \alpha_i(v)(1-v^2)^{-1/3}.$$

After integration over v we have:

$$\kappa_i = \frac{\alpha m^2 (3\kappa)^{2/3}}{7\pi} \frac{\Gamma^3\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} (1 - \mathrm{i}\sqrt{3})\beta_i$$
$$= (0.175 - 0.304\,\mathrm{i})\beta_i \alpha m^2 \kappa^{2/3},$$
$$\beta_2 = 1, \quad \beta_3 = 3/2.$$

6. It follows from the last equation that for $\alpha\kappa^{2/3} > 1$

the photon effective mass becomes larger than the mass of created electron and positron. And so, it seems that it is valid at the photon energy fulfilling the condition $\alpha\kappa^{2/3} \ll 1$. It should be noted that this expression does not depend on the electron mass. At the same time, the first order of the radiation corrections to the electron mass have a form ($\chi = \varepsilon H_\perp/mH_0$, ε is the electron energy):

$$m^2_{\rm rad1} = 2D\ \alpha m^2\chi^{2/3} = 2D\alpha\ \widetilde{\chi}^{2/3},$$
$$D = \frac{7(3)^{1/6}}{27}\Gamma\left(\frac{2}{3}\right)(1-{\rm i}\sqrt{3})$$
$$= (0.422\ - 0.730{\rm i}\,),\quad \widetilde{\chi}^2 = e^2\mathcal{P}F^2\mathcal{P},$$

and does not depend on the mass too. The main term in the second order of the radiaton corrections to the mass has a form

$$m^2_{\rm rad2} = \frac{13\alpha^2m^2\chi}{36\sqrt{3}}[1-{\rm i}\frac{2}{\pi}(\ln\frac{\chi}{2\sqrt{3}} - {\rm C} - \frac{142}{39})]$$
$$= 0.2085\alpha^2m^2\chi[1-0.637{\rm i}(\ln\chi - 5.461)],$$

where C – Euler's constant. This correction includes the additional fuctor $\sim \alpha\kappa^{1/3}$and formally can be larger. But for this value of parameter χ, one can not use the perturbation theory. In this case instead of the Dirac equation, we must use the well-known Schwinger equation

$$[\widehat{\mathcal{P}} - m - M(\mathcal{P},F)]\psi = 0,\ \widehat{\mathcal{P}} = \gamma^\mu \mathcal{P}_\mu \equiv \gamma\mathcal{P},$$
$$\mathcal{P}_\mu = \mathrm{i}\frac{\partial}{\partial x^\mu} - eA_\mu,$$

where M is the mass operator including, generally speaking, the all series of the perturbation theory. At substitution m_{rad1}in place of m into $\alpha\kappa^{1/3}$, we have a value $\sim \sqrt{\alpha}/5$, not depending on any parameter. This value is small and therefore, we can expect that the first order of the perturbation theory contributs mainly into the mass operator M. In this order, the mass operator have a relatively simple form

$$M_1(\mathcal{P},F) \simeq D\alpha\widetilde{\chi}^{-4/3}(e^2\gamma F^2\mathcal{P}).$$

Multiply the Schwinger equation by the operator $\widehat{\mathcal{P}} + m - M$ and take into account that

$$\{\widehat{\mathcal{P}}, e^2\gamma F^2\mathcal{P}\} = 2e^2\mathcal{P}F^2\mathcal{P} = 2\widetilde{\chi}^2,$$

and the term $\propto e^2\gamma F^2\mathcal{P}$ can be omit (that relative value $\sim m^2_{\text{rad1}}/\varepsilon^2$). As a result we have the following squared equation

$$(\widehat{\mathcal{P}}^2 - m^2 - 2D\alpha\ \widetilde{\chi}^{2/3})\psi =$$
$$(\widehat{\mathcal{P}}^2 - m^2 - m^2_{\text{rad1}})\psi = 0$$

All stated above is valid under condition $\varepsilon e H_\perp \gg (m^2 + m^2_{\text{rad1}})^{3/2}$. For $m^2_{\text{rad1}} > m^2$, this condition changes over $1 \gg \alpha^{3/2}$.