Generalized Bessel Functions and Their Use In Electromagnetic Processes

Dedicated to

P. L. OTTAVIANI

1941-2014

friend and mentor

G. DATTOLI

ENEA FRASCATI

A) Why Bothering with Bessel functions in these Days?

B) Why Inventing new families of special functions?

A) Why not?

B) Why not?

A) What is a Bessel Function?

$$\frac{1}{1+x} = \sum_{r=0}^{\infty} (-x)^r \quad \equiv \quad \text{Rational (hyperbolic)}$$

$$e^{-x} = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \quad \equiv \quad \text{Transcendental (exponential)}$$

$$\frac{1}{1+x^2} = \sum_{r=0}^{\infty} (-x)^r \quad \equiv \quad \text{Rational (Lorentzian)}$$

$$e^{-x^2} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{r!} \equiv \quad \text{Transcendetal (Gaussian)}$$

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{r!\,r!} \equiv \text{hygher transcendental (Bessel)}$$

$$J_{0,0}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!\,r!\,r!} \equiv \quad (\text{hygher})^2 \text{ transcendental (Humbert)}$$

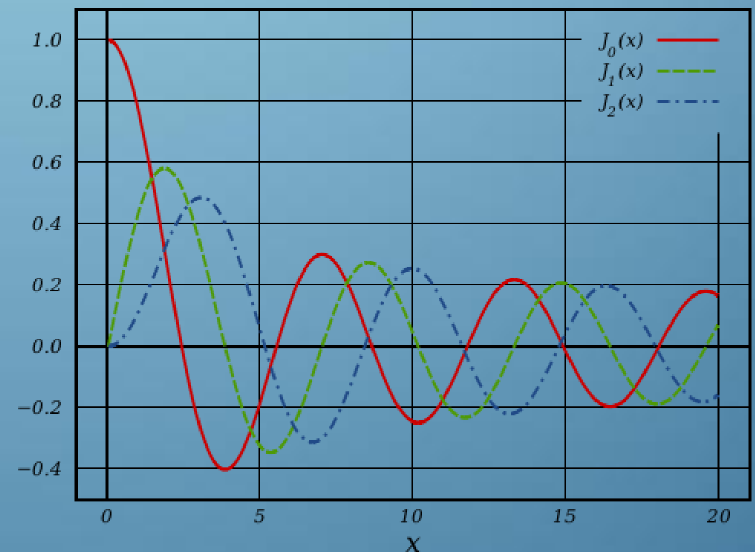
....

Higher transcendental Functions

- Bessel Functions (cylindrical of first kind) higher order transcendental even though some recent point of view tend to reduce them to ordinary Gaussians $J_0(x) = e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0$
- According to the orthodox prescriptions they are analytic functions expressible through the series

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{r!\,r!},$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!}$$



Bessel Functions in a Nut shell-1

- Generating Function
- Recurrences

$$\sum_{n=-\infty}^{+\infty} t^n J_n(x) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} \to \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{xt}{2}\right)^r \sum_{s=0}^{\infty}\frac{1}{s!}\left(\frac{-x}{2t}\right)^s \to \text{Cauchy Product} \to$$

$$r-s=n \to \sum_{n=-\infty}^{+\infty} t^n \left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r}\right)$$

$$2\frac{d}{dx}J_n(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$\frac{2n}{x}J_n = J_{n-1}(x) + J_{n+1}(x)$$

Bessel functions in a nut shell-II

- Shift Operators and differential equation

$$\hat{E}_{-} = \frac{d}{dx} + \frac{\hat{n}}{x},$$

$$\hat{E}_{+} = -\frac{d}{dx} + \frac{\hat{n}}{x}$$

$$\hat{E}_{\pm} J_n(x) = J_{n\pm1} \Rightarrow \hat{E}_{+}\hat{E}_{-} J_n(x) = J_n(x) \Rightarrow$$

$$\Rightarrow x^2 y'' + 2\,x\,y' + (x^2 - n^2)y = 0$$

Bessel Functions in a Nut-Shell III

- The Jacobi Anger Generating function and Integral representation

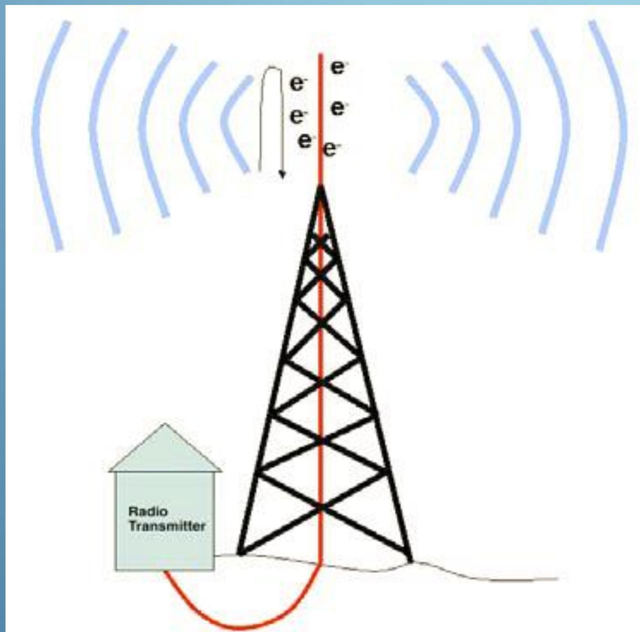
$$\sum_{n=-\infty}^{+\infty} t^n J_n(x) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} \rightarrow t = e^{i\vartheta} \Rightarrow$$

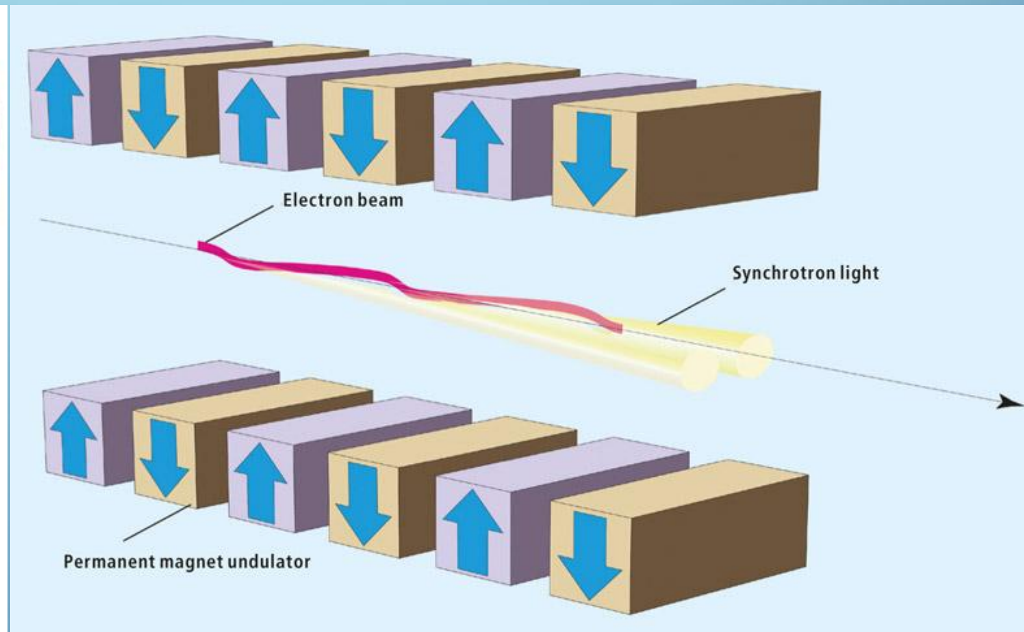
$$\sum_{n=-\infty}^{+\infty} e^{in\vartheta} J_n(x) = e^{ix\sin(\vartheta)} \Rightarrow$$

$$\Rightarrow J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\vartheta - x\cdot\sin(\vartheta))\, d\vartheta$$

Bessel Functions and radiation emission by oscillating charges

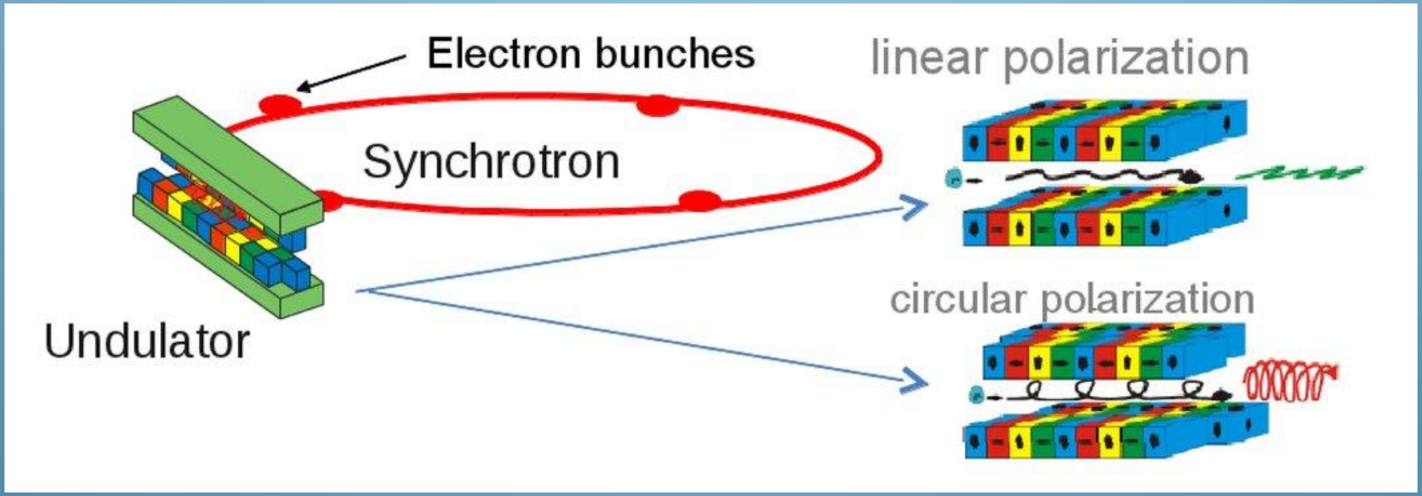
- *A simple radiating device: an antenna, an undulator…*



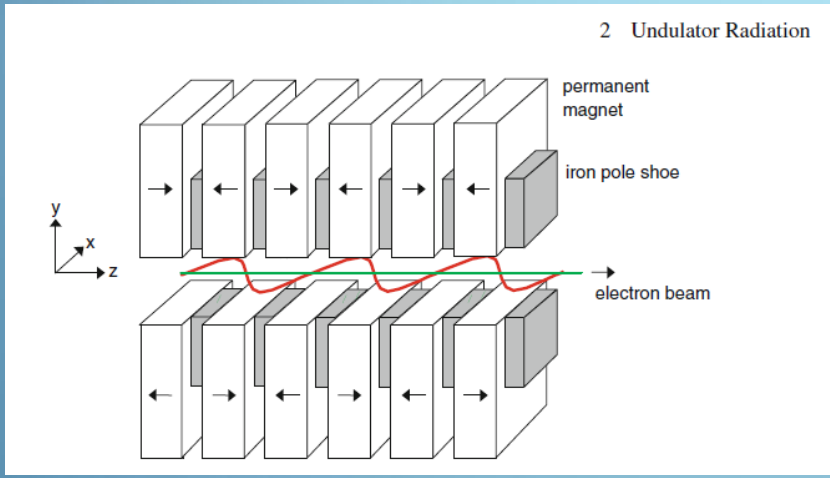


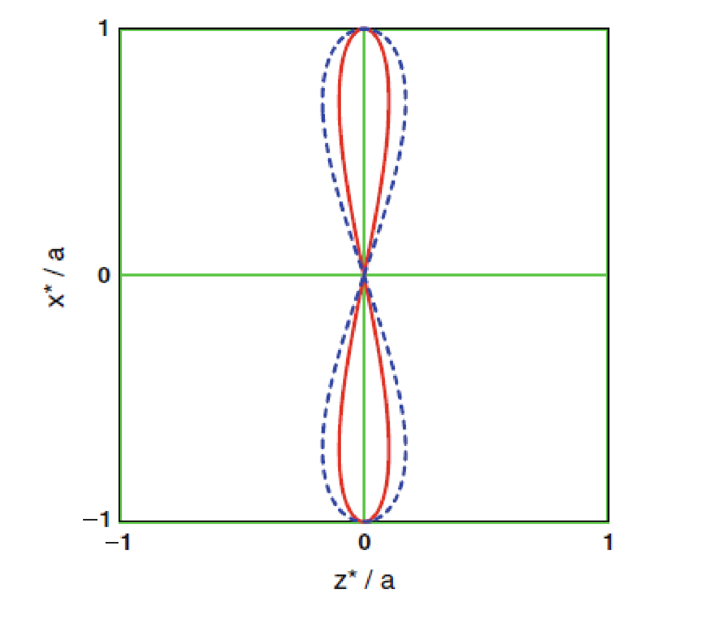
Undulator Geometry

- Linear
- Circular



Electron dynamics inside the undulator
Lorenz equation of motion (Linear)





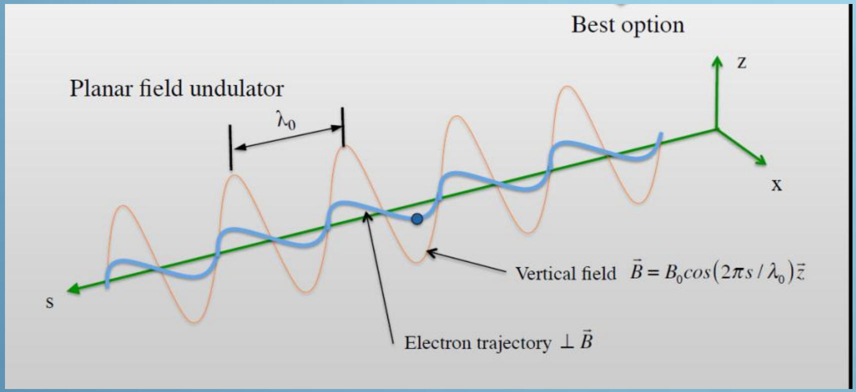
$$\vec{B} = -B_0 \sin(k_u z)\vec{e}_y, k_u = \frac{2\pi}{\lambda_u},$$

$$\gamma m_e \frac{d}{dt}\vec{v} = -e\vec{v} \times \vec{B},$$

$$x(t) = \frac{K}{\gamma k_u}\sin(\omega_u t), z(t) = \bar{v}_z t - \frac{K^2}{8\gamma^2 k_u}\sin(2\omega_u t)$$

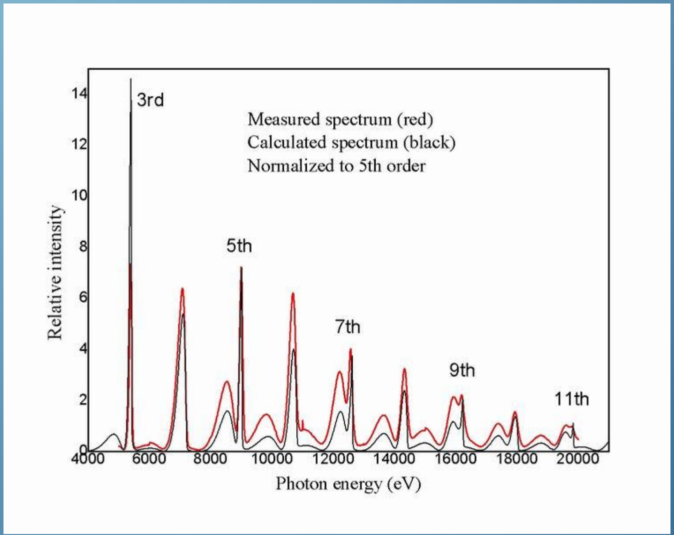
Undulator Radiation Spectral details: Radiation Integral

$$\vec{r} \equiv (\frac{K}{\gamma k_u}\sin(\omega_u t), 0, z(t) = \bar{v}_z t - \frac{K^2}{8\gamma^2 k_u}\sin(2\omega_u t))$$



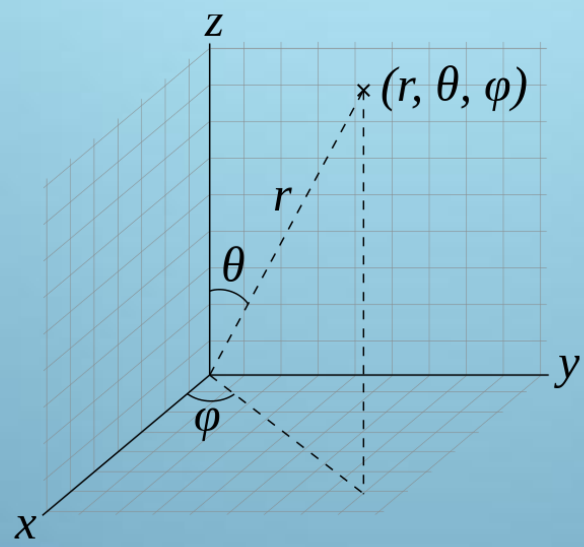
$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c}\left|\omega\int_{-\infty}^{+\infty} dt[\vec{n}\times[\vec{n}\times\vec{\beta}]]e^{i\omega(t-\frac{\vec{n}\cdot\vec{r}}{c})}\right|^2,$$

$$\lambda_n = \frac{\lambda_1}{n}, \lambda_1 = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$



Bessel functions enter the Game

- Jacobi Anger Expansion and radiation Integral



$$e^{i\omega(t-\frac{\vec{n}\cdot\vec{r}}{c})} \rightarrow \vec{n}\cdot\vec{r} = \sin(\vartheta)\cos(\varphi)\cdot\frac{K}{\gamma k_u}\sin(\omega_u t) + \cos(\vartheta)\left[\bar{v}_z t - \frac{K^2}{8\gamma^2 k_u}\sin(2\omega_u t)\right] \rightarrow$$

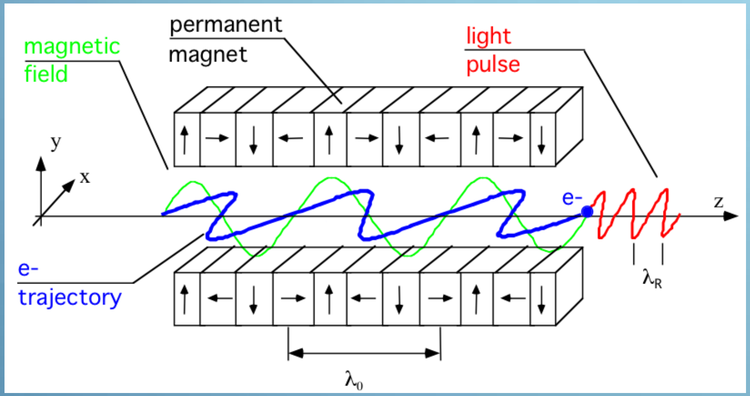
$$\rightarrow$$

$$K << 1 \Rightarrow e^{iA\sin(\omega_u t)} = \sum_{n=-\infty}^{+\infty} e^{in\omega_u t} J_n(A)$$

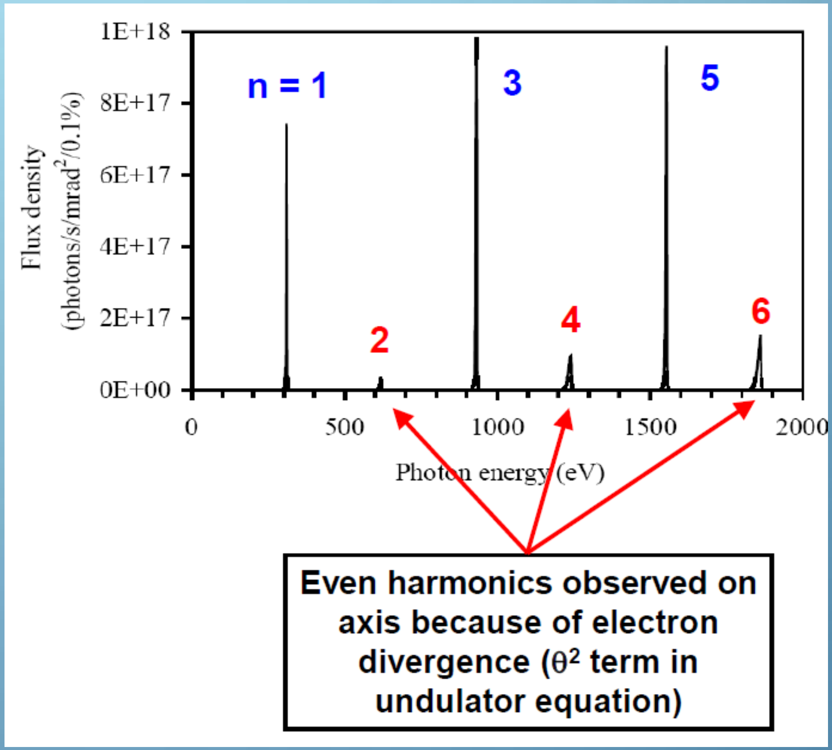
$$A = \omega\sin(\vartheta)\cos(\varphi)\frac{K}{\gamma k_u}$$

On axis radiation

- ## Even Harmonics on axis



$$\frac{d^2 I}{d\omega d\Omega}\Big|_{\vartheta=0} \propto \sum_{n=odd} f_n \left[J_{\frac{n-1}{2}}(n\xi) - J_{\frac{n+1}{2}}(n\xi) \right]^2$$



What if the dypole approximation does not hold?

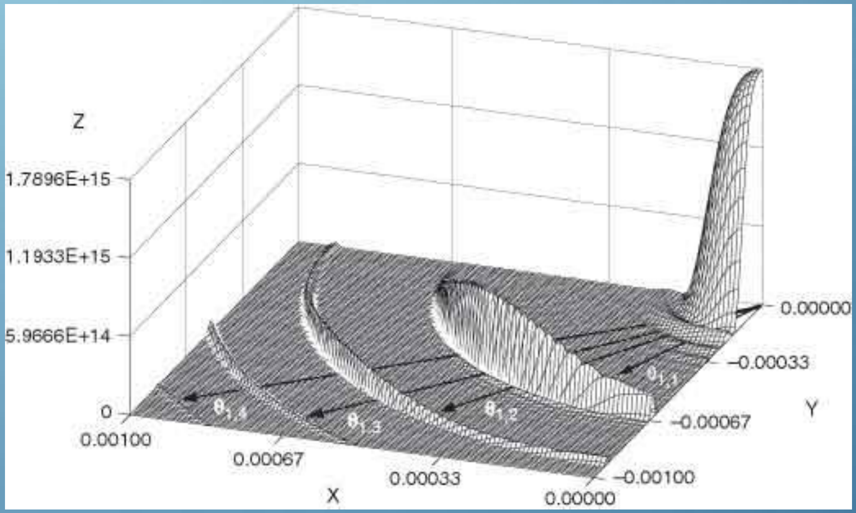
- **Generalized Bessel Functions: The «new» tool**

$$K \geq 1$$

$$e^{i\omega\left(t-\frac{\vec{n}\cdot\vec{r}}{c}\right)} \rightarrow e^{i A \sin(\omega_u t)+i B \sin(2\omega_u t)} = ???$$

$$A = \omega \sin(\vartheta)\cos(\varphi)\frac{K}{\gamma k_u},$$

$$B = -\omega \cos(\vartheta)\frac{K}{8\gamma^2 k_u}$$



Two variable Bessel

- *A Step Further*

$$e^{i x \sin(\varphi) + i y \sin(2\varphi)} = \sum_{n=-\infty}^{+\infty} e^{i n \varphi} J_n(x, y),$$

$$t \to e^{i\varphi}$$

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right) + y\left(t^2 - \frac{1}{t^2}\right)} = \sum_{n=-\infty}^{+\infty} e^{i n \varphi} J_n(x, y)$$

$$J_n(x, y) = ???$$

Theory of TVBF in « a nut shell» I

- Recurrences

$$\partial_x J_n(x,y) = \frac{1}{2}\left(J_{n-1}(x,y) - J_{n+1}(x,y)\right),$$

$$\partial_y J_n(x,y) = \frac{1}{2}\left(J_{n-2}(x,y) - J_{n+2}(x,y)\right),$$

$$2n\, J_n(x,y) = x\left[J_{n-1}(x,y) + J_{n+1}(x,y)\right] + y\left[J_{n-2}(x,y) + J_{n+2}(x,y)\right]$$

- Shift operators

$$\hat{E}_- = \frac{n + \frac{y}{2}}{x} + \frac{y}{x}\partial_x^2 + \frac{1}{x}\partial_x = \frac{1}{x}\hat{N} + \frac{1}{x}\partial_x,$$

$$\hat{E}_+ = \frac{n + \frac{y}{2}}{x} + \frac{y}{x}\partial_x^2 - \frac{1}{x}\partial_x = \frac{1}{x}\hat{N} - \frac{1}{x}\partial_x,$$

$$\hat{N} = \hat{n} + y\left(\frac{1}{2} + \partial_x^2\right)$$

Theory of TVBF in « a nut shell» II

- Differential equation

$$\hat{E}_+\hat{E}_- J_n(x,y) = J_n(x,y) \Rightarrow$$

$$\left[\left(\frac{n-1+\frac{y}{2}}{x}+\frac{y}{x}\partial_x^2\right)-\frac{1}{x}\partial_x\right]\left[\left(\frac{n+\frac{y}{2}}{x}+\frac{y}{x}\partial_x^2\right)+\frac{1}{x}\partial_x\right]J_n(x,y) = J_n(x,y)$$

- Series expansion

$$J_n(x,y) = \sum_{l=-\infty}^{+\infty} J_{n-2l}(x)\, J_l(y)$$

- Integral representation

$$J_n(x,y) = \frac{1}{\pi}\int_0^{\pi}\cos(n\vartheta - x\sin(\vartheta) - y\sin(2\vartheta))\, d\vartheta$$

Radiation Integral and TVBF

- Expansion without the dipole approximation

$$e^{i\omega(t-\frac{\vec{n}\cdot\vec{r}}{c})} = \sum_{n=-\infty}^{+\infty} e^{in\omega_u t} J_n(A,B)$$

- So What???
- Have we really made any progress?
- Or we have just expressed in «well educated» terms largely well known stuff?

Critical points

- *If we limit the discussion to the undulator picture we have in mind*
- *«an oscillating field orthogonal to the electron trajectory»*
- *No doubt that our description is just helpful to*
  - ***«fill the vacuum with nothing!!!»***

Going Further...

- Why not inventing something even more tricky!!!
- Use the Generating Function as starting point

$$e^{i\,x\sin(\varphi)+i\,y\sin(m\varphi)} = \sum_{n=-\infty}^{+\infty} e^{i\,n\varphi\,(m)} J_n(x,y),$$

$$t \to e^{i\varphi}$$

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)+y\left(t^m-\frac{1}{t^m}\right)} = \sum_{n=-\infty}^{+\infty} t^n \left[{}^{(m)}J_n(x,y)\right]$$

$${}^{(m)}J_n(x,y) = \sum_{l=-\infty}^{+\infty} J_{n-m\,l}(x)\,J_l(y)$$

...and further!!!

- Modular Character of Generalized Bessel Functions

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^m-\frac{1}{t^m}\right)+\frac{z}{2}\left(t^3-\frac{1}{t^3}\right)+\frac{u}{2}\left(t^{3m}-\frac{1}{t^{3m}}\right)} = \sum_{n=-\infty}^{+\infty} t^n \left[{}^{(m,3)}J_n(x,y;z,u)\right],$$

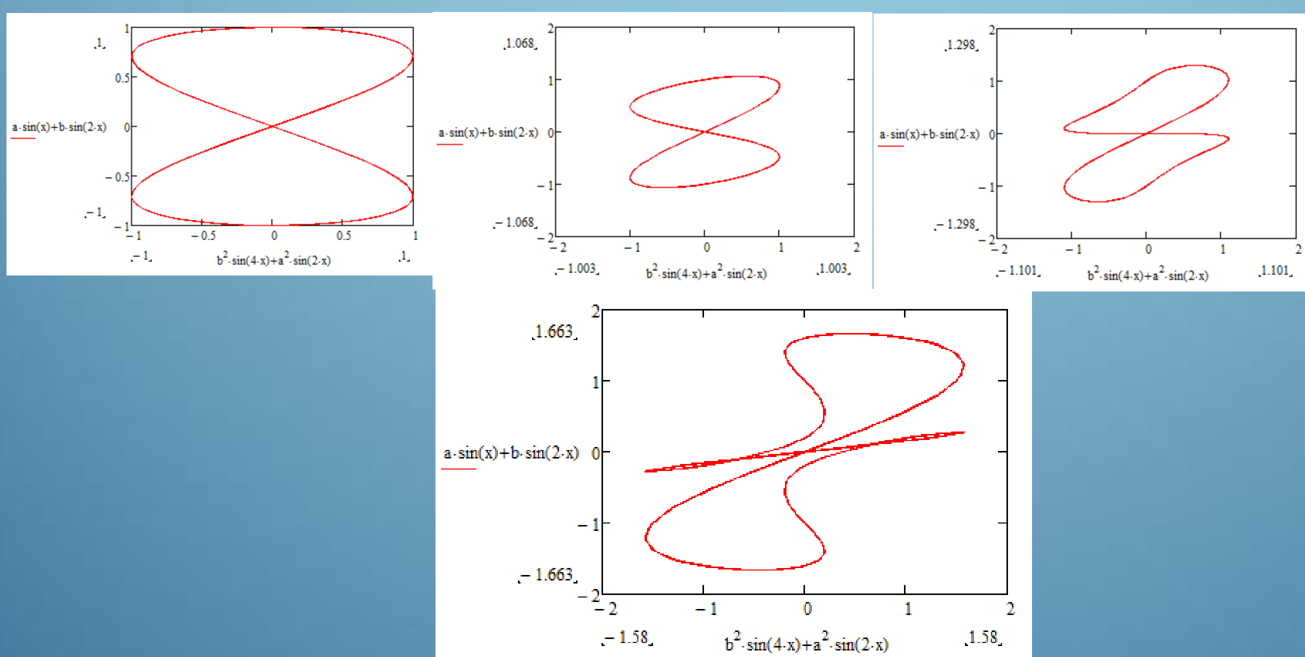
$$e^{i\,x\sin(\varphi)+i\,y\sin(m\varphi)+i\,z\sin(3\varphi)+i\,u\sin(3m\varphi)} = \sum_{n=-\infty}^{+\infty} e^{i\,n\varphi}\left[{}^{(m,3)}J_n(x,y;z,u)\right],$$

$$J_n(x,y;z,u) = \sum_{l=-\infty}^{+\infty} {}^{(m)}J_{n-3l}(x,y)\,{}^{(m)}J_l(u,v)$$

May have they any relevance to radiation problems???

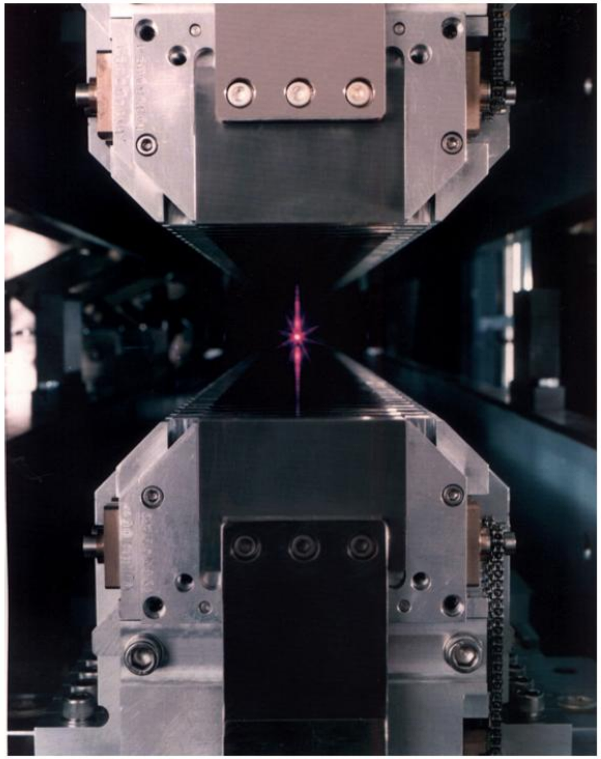
- Yes, For special type of undulators

$$\vec{r} \equiv (\frac{K_1}{\gamma k_u}\sin(\omega_u t) + \frac{K_h}{\gamma k_u}\sin(h\omega_u t), 0, z(t) = \bar{v}_z t + ...o(K^2))$$



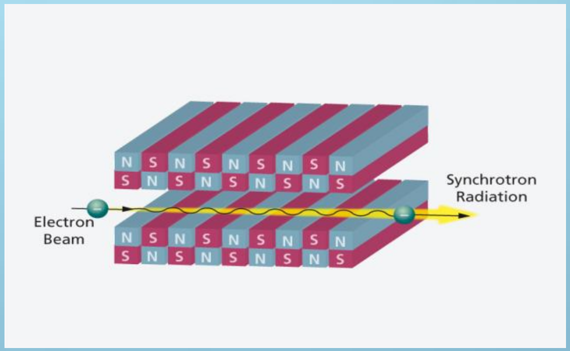
So What???

- Undulators Are complicated devices with not so nice on axis fields, which, even though often forgotten, **satisfy Maxwell equations**

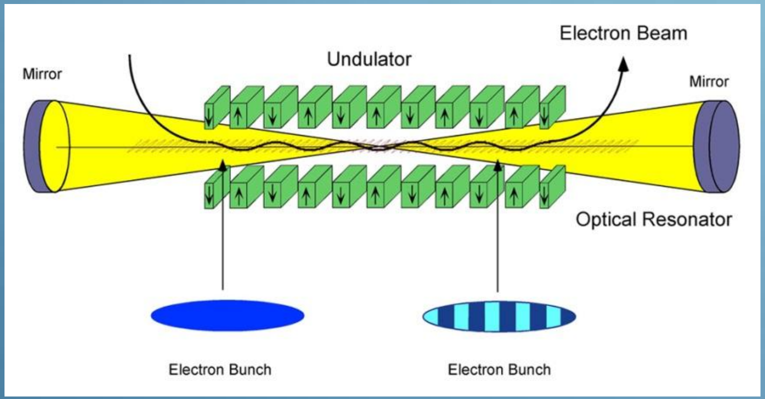


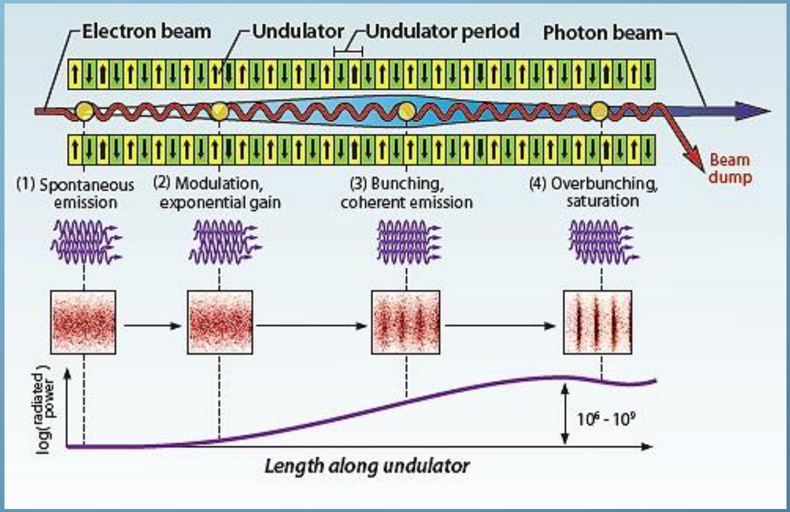
ALS U5 undulator, beamline 7.0, N = 89, λ_u = 50 mm

Free Electron Lasers (FEL)

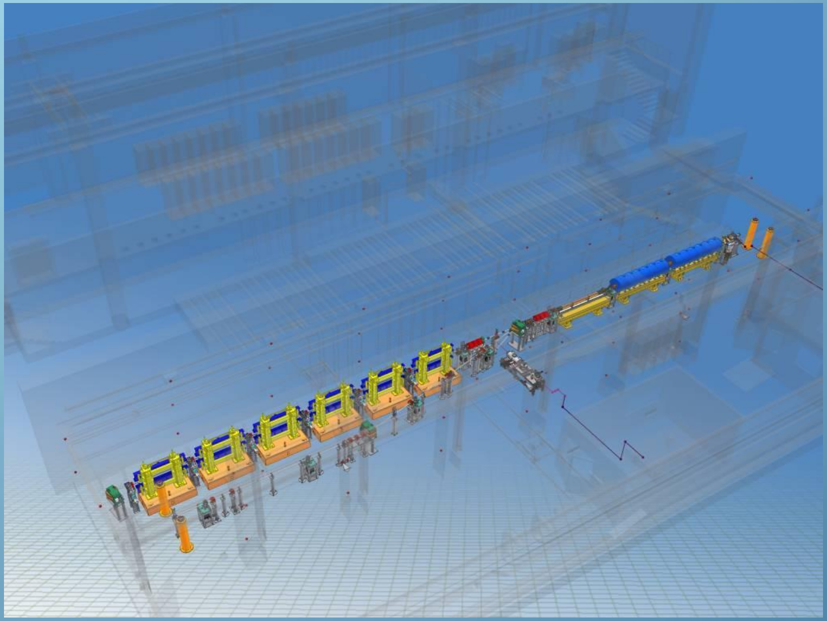
- *From Bremmstrahlung*  *to FEL*

- *Oscillators & SASE Amplifiers*





Layout of LCLS and of SPARC Free electron laser facilities



Bessel Functions and FEL gain

-

$$g_0 = 4\pi \frac{|J|}{I_0}\left(\frac{N}{\gamma}\right)^3 \left(\lambda_u K^* f_b\right)^2$$

$$f_b = \begin{cases} 1 & helical \\ J_0(\xi) - J_1(\xi) & linear \end{cases}, \xi = \frac{1}{4}\frac{K^2}{1+\frac{K^2}{2}}$$

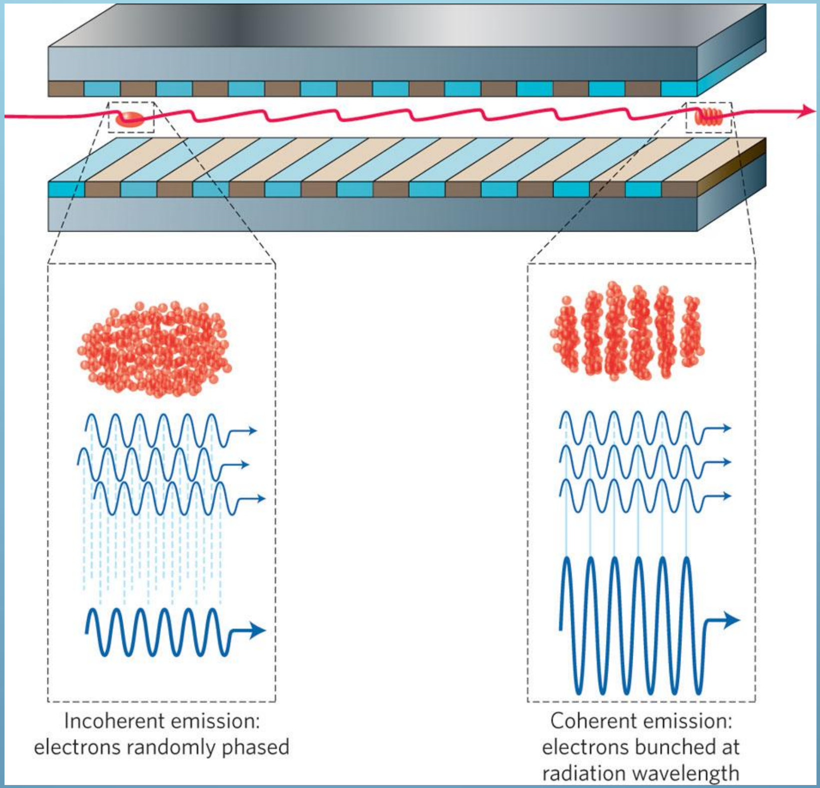
$$g_n \propto f_{b,n}^2,$$

$$f_{b,n} = J_{\frac{n-1}{2}}(\xi) - J_{\frac{n+1}{2}}(\xi)$$

-

FEL Harmonic Generation

- bunching



Bi-Harmonic Undulators

T. Tanaka, H. Kitamura (NIM 1995)

(G. D., P. L. Ottaviani, NIM (2001))

- Multi-field: Bi-harmonic (poly-harmonic)

$$\vec{B} \equiv (0, B_1 \sin(k_u z) + B_h \sin(h\, k_u z), 0),$$
$$\vec{B} \equiv (\vec{B}_1 \sin(k_u z), B_h \sin(h\, k_u z), 0)$$

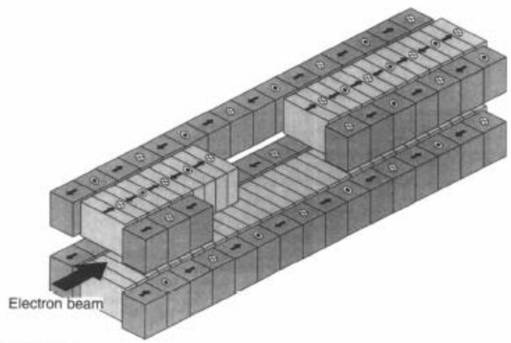


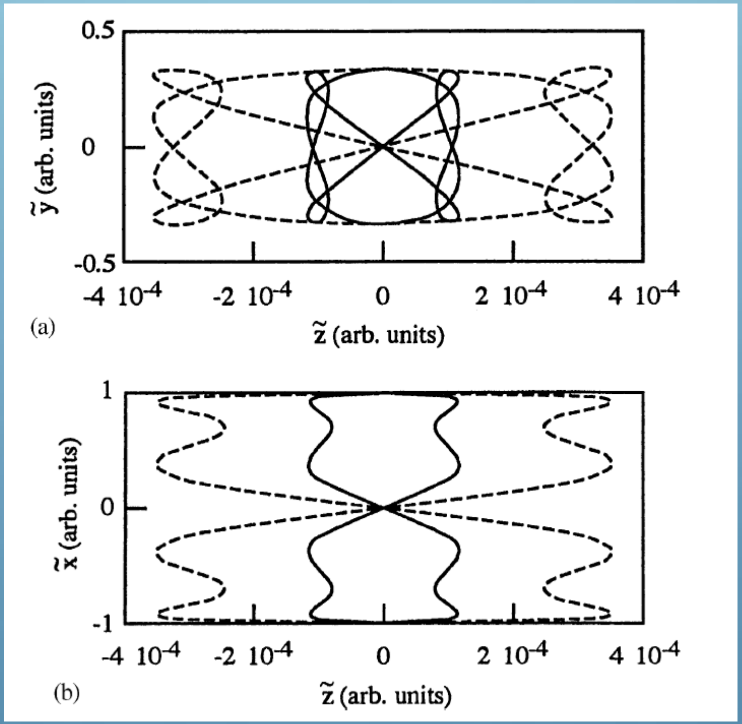
Figure 1
Schematic illustration of the figure-8 undulator. The period length of the vertical undulator is twice as long as that of the horizontal undulator.

- Why have they been proposed?
- To produce Coherent radiation operating at orthogonal polarizations with different harmonics

Biharmonic, 1st & 3rd

- *On axis undulator Field*

$$\vec{B} \equiv (\vec{B}_3 \sin(3k_u z), B_1 \sin(k_u z), 0)$$



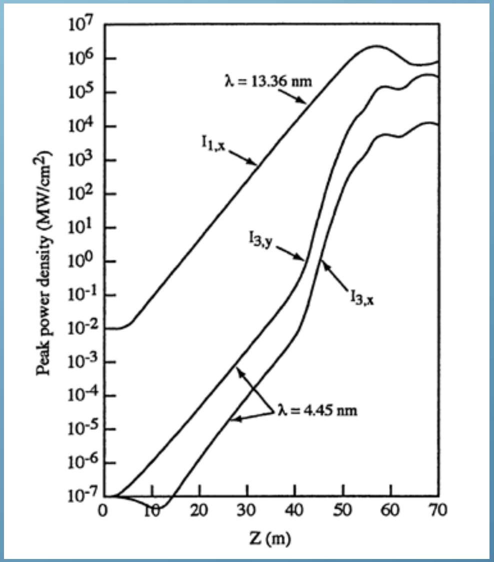
Biharmonic Fel gain Bessel factor

- The Bessel Factor Reflects the more complicated structure of the undulator itself

$$F_x^{(m)} = \left(\frac{4m\xi}{K}\right) \times \left[{}^{(3)}J_{(m-1)/2}\left(m\xi, \frac{m}{3}\xi\right) - {}^{(3)}J_{(m+1)/2}\left(m\xi, \frac{m}{3}\xi\right) \right]$$

$$F_y^{(m)} = \left(\frac{4m\xi}{K}\right) \times \left[{}^{(3)}J_{(m-3)/2}\left(m\xi, \frac{m}{3}\xi\right) - {}^{(3)}J_{(m+3)/2}\left(m\xi, \frac{m}{3}\xi\right) \right]$$

$$\xi = \frac{1}{4}\frac{K^2}{1+K^2} \qquad (24)$$



…an error diagnostic tool

- *Undulators are not so naive devices, the on axis field is not just a mathematical device.* Significant distortion are introduced by the magnetization errors….*and can e.g. modelled through an expansion in* ***Fourier series*** *of the on axis field (G. D., OTTAVIANI & PAGNUTTI, 2004)*

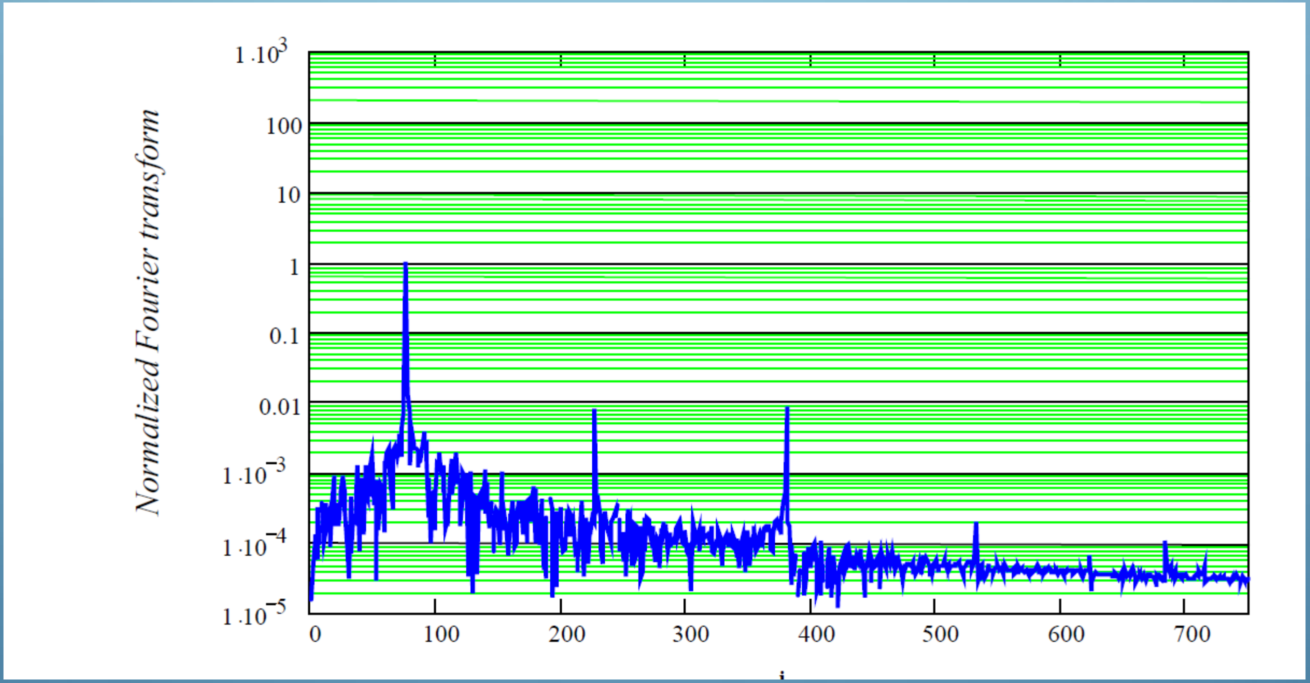
$$\vec{B} \equiv (\vec{B}_{h,x} \sin(hk_u z + \phi_h), B_m \sin(m k_u z + \phi_m), 0)$$

- ***Even though apparetly intrigued the method is helpful in understanding the spectra and thus the error magnetization contributions.***

Measured on axis field (G.D., E. SABIA (2007)

- Trigonometric fit

$$B_y = B_0 \left\{ \sin\left(\frac{2\pi}{\lambda_u} z\right) + 10^{-2} \left[\sin\left(\frac{6\pi}{\lambda_u} z\right) + \sin\left(\frac{10\pi}{\lambda_u} z\right) \right] \right\}$$



E-Beam diagnostic tool

- *Inclusion of betatron motion*
- *G. D., G. Voykov and M. Carpanese (1995)*
- *G. D. and G. K. Voykov (1996)*

PRL **109**, 194801 (2012) PHYSICAL REVIEW LETTERS week ending 9 NOVEMBER 2012

Observation of Picometer Vertical Emittance with a Vertical Undulator

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(Received 11 July 2012; published 8 November 2012)

Using a vertical undulator, picometer vertical electron beam emittances have been observed at the Australian Synchrotron storage ring. An APPLE-II type undulator was phased to produce a horizontal magnetic field, which creates a synchrotron radiation field that is very sensitive to the vertical electron beam emittance. The measured ratios of undulator spectral peak heights are evaluated by fitting to simulations of the apparatus. With this apparatus immediately available at most existing electron and positron storage rings, we find this to be an appropriate and novel vertical emittance diagnostic.

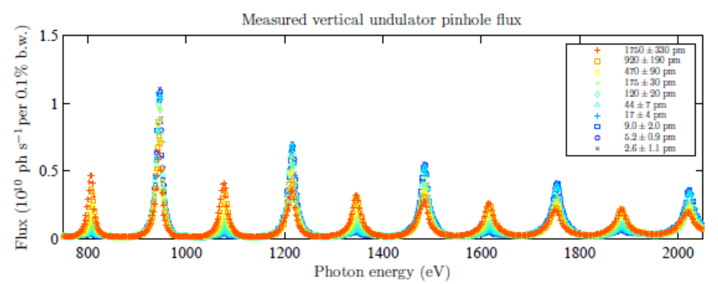


Figure 8.1: Measured undulator spectra for vertical emittances calibrated with LOCO [199], from minimum in blue up to maximum in red. Shown are undulator harmonics 6 – 15. [K.P. Wootton, et al., *Phys. Rev. Lett.*, **109** (19), 194801 (2012). © 2012 American Physical Society.]

Is it possible to do more?

- Probably yes!!!
- An example is provided by undulator operating with undulators having different relatively prime periods.
- The Bessel functions we have studied so far are no more useful and the problem of deciphering the undulator spectroscopy becomes extremely complicated.
- A very helpful tool is provided by a class of BF characterized by many variable and many indices

….

- *On axis field (Iracane and Bamas: FEL operating with narrow spectrum and large extraction efficiency PRL 1991) Ciocci, G. D., Giannessi, Voykov (PRE) (1993)*

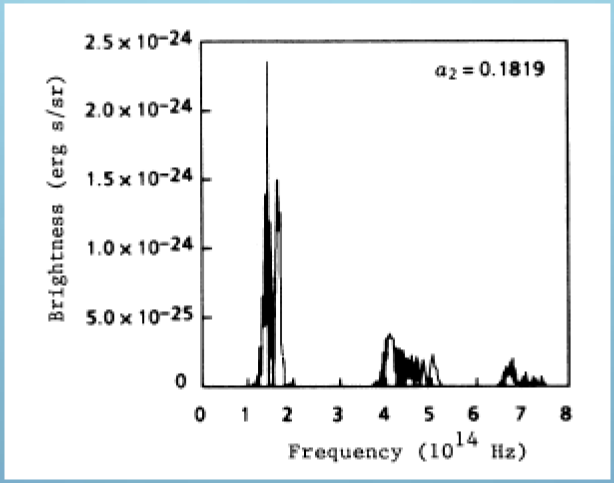
$$\vec{B} \equiv (B_1 \sin(k_{u,1} z) + B_2 \sin(k_{u,2} z), 0)$$

- *What do we expect?*
- *A fairly intrigued oscillations but, limiting to the dypole axxproximations*

$$e^{i\,x \sin(\varphi) + i\,y \sin(\vartheta) + i\,z \sin(\vartheta - \varphi)} = \sum_{m,n=-\infty}^{+\infty} e^{i\,m\,\varphi} e^{i\,n\,\vartheta} J_{m,n}(x, y; z)$$

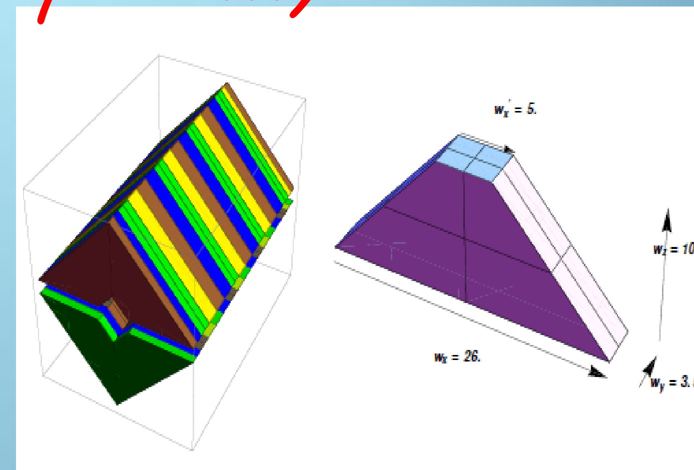
Typical spectra

- Ciocci, G. D., Giannessi and Voykov (PRE) 1993



Few Experimental results

(We are not only painting equations!!!)



- DELTA like undulator

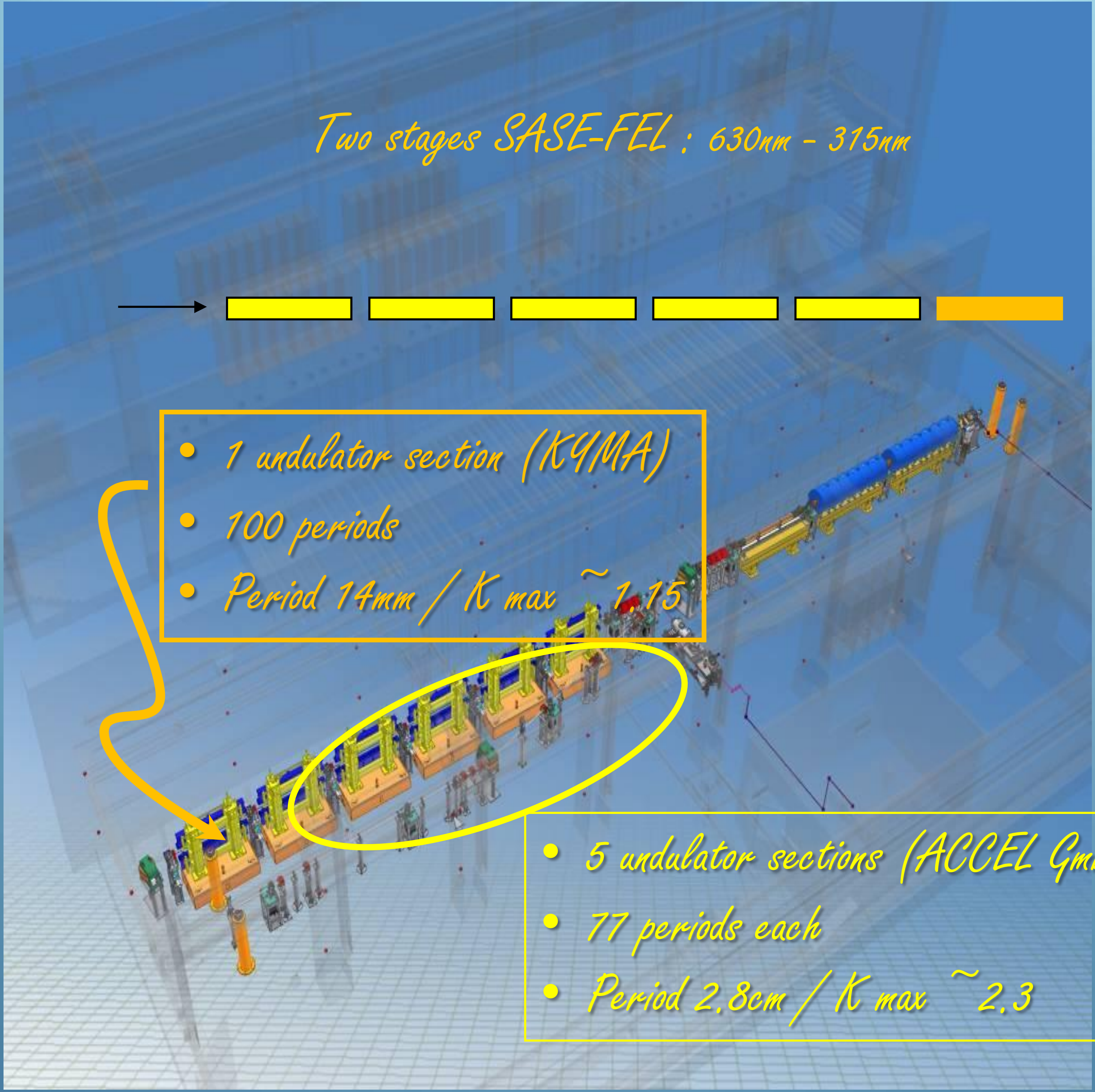
  ℓ_u = 14.0mm, gap g = 5mm, Br = 1.22T.

Undulator tested in two stage SASE-FEL:

630nm to 315 nm

KYMA undulator

Two stages SASE-FEL : 630nm - 315nm



- 1 undulator section (KYMA)
- 100 periods
- Period 14mm / K max ~1.15

- 5 undulator sections (ACCEL Gmbh)
- 77 periods each
- Period 2.8cm / K max ~2.3

Radial emittance

$e_x = 2.23783$ mm mrad

$b_x = 64.9801$ m

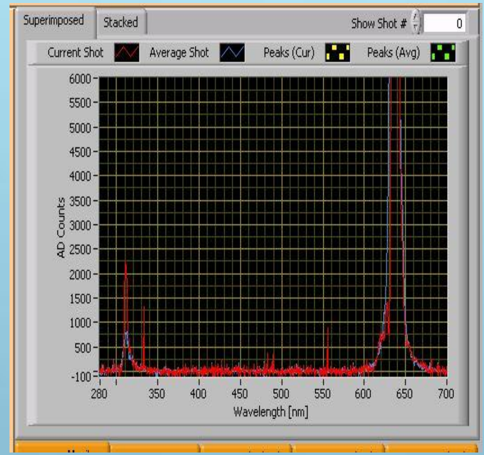
$a_x = 0.039368$

Vertical emittance

$e_y = 1.28451$ mm mrad

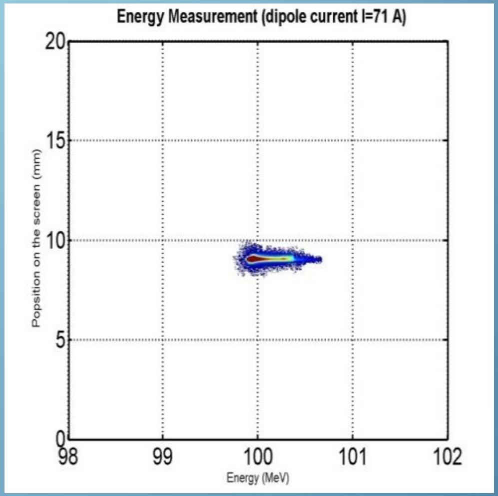
$b_y = 40.5836$ m

$a_y = -1.19299$

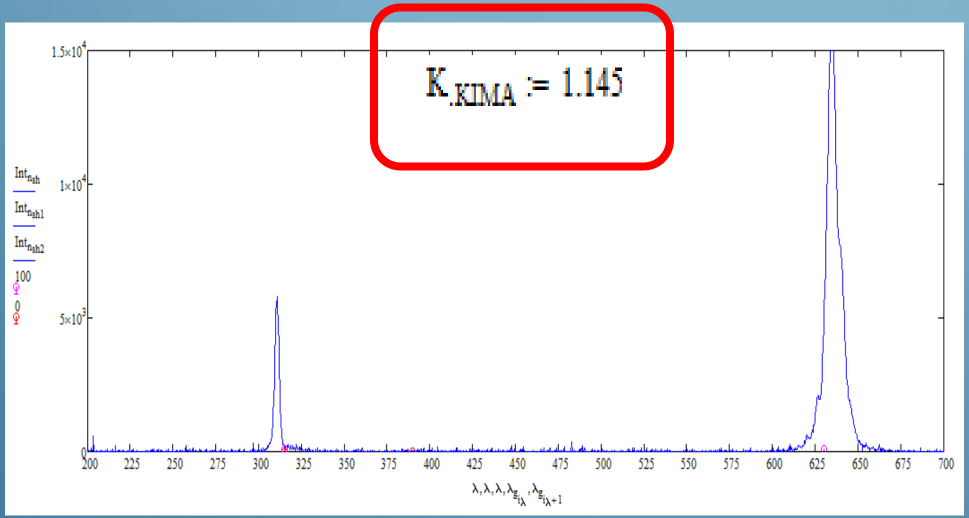


MeanEnergy(MeV)=100.087101

DE(MeV)=0.176304 → $s_e=7.5x10^{-4}$



Correct e.beam energy=98 MeV



Charge 125 pC

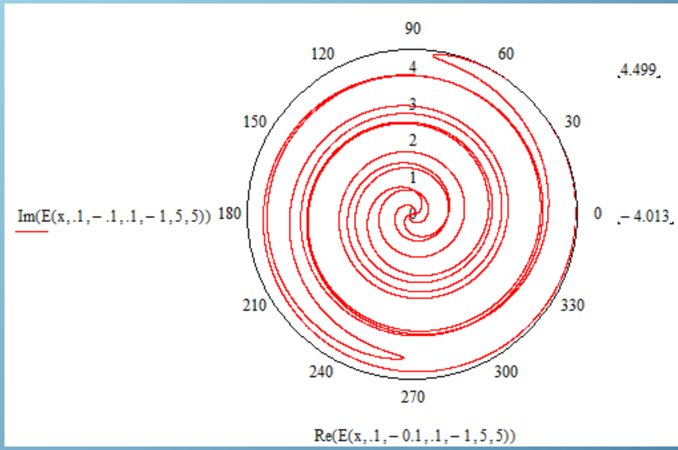
Bunch length ≈ 200 fs

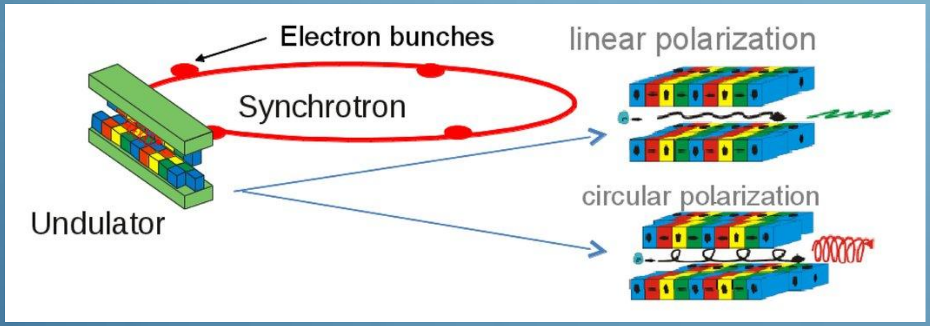
⇩

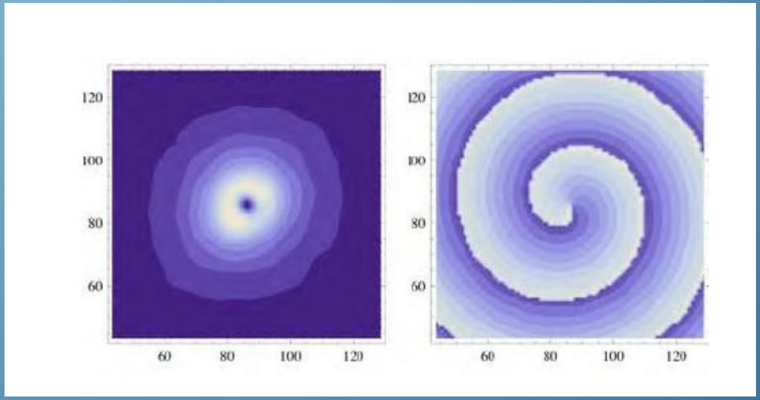
Peak current ≈ 250 A

Optical Vortex propagation and electron undulator interaction

- Towards the optical beam shaping with electrons mode converter
- Helical-Linear Undulator
- Linear V-H Undulator







What is an undulator when considered from the point of view of the electrons?

- If we sit with the electrons the undulator is moving towards us and will appear as an intense electromagnetic wave!!!
- Therefore all we discussed so far can be extended to electron (charged particle) wave interaction.

$$\lambda \cong \frac{\lambda^*}{4\gamma^2}\left(1+\frac{K^2}{2}\right)$$

$$K \approx 0.85 \cdot 10^{-5} \lambda^*[m]\sqrt{\left(I\left[\frac{W}{m^2}\right]\right)}, K \cong 1 \Rightarrow \lambda^*[m] \cong 10^{-5},$$

$$I\left[\frac{W}{m^2}\right] \cong 10^{20}$$

1

Numbers!!!!

- Just quoting from
- D. Bauer «Theory of Intense Laser Matter Interaction» June 22 (2012) dieter.bauer@mpi-bd.mpg.de

$$10^{20}\,\frac{W}{m^2}$$

- «Is peanuts for current laser technology an order of magnitude larger is routnely achieved in laboratories around the world. $10^{26}\,\frac{W}{m^2}$ Seems to be the maximum achieved so far»

Why are these number interesting?

- 1) Free Electron lasers operating with wave undulators
- 2) New Physical regimes and break down of ordinary QM perturbative treatment ($>10^{16}\frac{\mathrm{W}}{\mathrm{m}^2}$)
- 3) New effects in non linear QED
- 4) Ionization effects...

Relativistic Dynamics of a Charged Particle in an electromagnetic wave

- *Fields*

$$\vec{A}(r,t) = \vec{A}_0 \sigma(\eta)\Pi(\eta), \eta = \omega t - \vec{k}\cdot\vec{r},$$

$$\vec{A}\cdot\vec{k} = 0,$$

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A}, \qquad \vec{B} = \vec{\nabla}\times\vec{A},$$

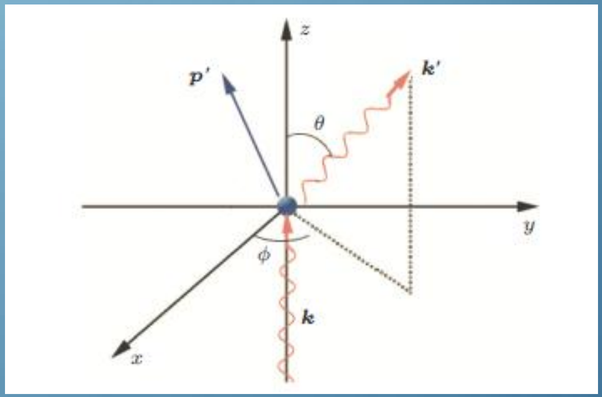
$$\Pi(\eta) \equiv \text{Slow Varying Amplitude}$$

- *Electron wave interaction*

-

$$H = c^2\sqrt{m^2c^2 - (\vec{P} - q\vec{A})^2},$$

$$\vec{P} \equiv \text{Canonical Momentum}$$



...Hamilton Jacobi theory...

- Oscillation center frame ($\vec{A} = A_0 \vec{e}_y \sin(\eta)$ linear polarization)

$$kx = \frac{q^2 \bar{A}_0^{\,2}}{8\left(1 + \frac{q^2 \bar{A}_0^{\,2}}{2}\right)} \sin(2\eta),$$

$$k\,y = \frac{q\bar{A}_0}{\sqrt{1 + \frac{q^2 \bar{A}_0^{\,2}}{2}}} \cos(\eta),$$

$$z = 0, \quad \bar{A}_0 = \frac{A}{m\,c}, K_r = \frac{q\,A}{m\,c}$$

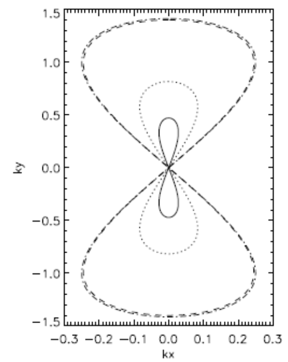


Figure 2.2: Figure-eight dynamics of a charged particle in an electromagnetic wave, as seen in the oscillation center frame. The trajectory in the xy-plane, where x is the propagation direction of the laser pulse, and y is the polarization direction, is shown for $a = 0.5$ (solid), $a = 1$ (dotted), $a = 10$ (dashed), and $a = 100$ (dashed-dotted). As a increases, the amplitudes $k\hat{x}$ and $k\hat{y}$ approach the calculated values $1/4$ and $\sqrt{2}$, respectively.

Multiphoton Compton Scattering
Laser Dressed QED

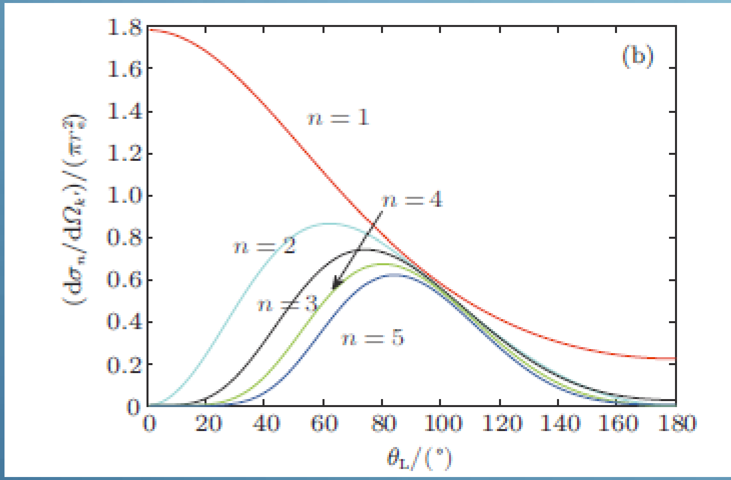
- Just few equations!!!

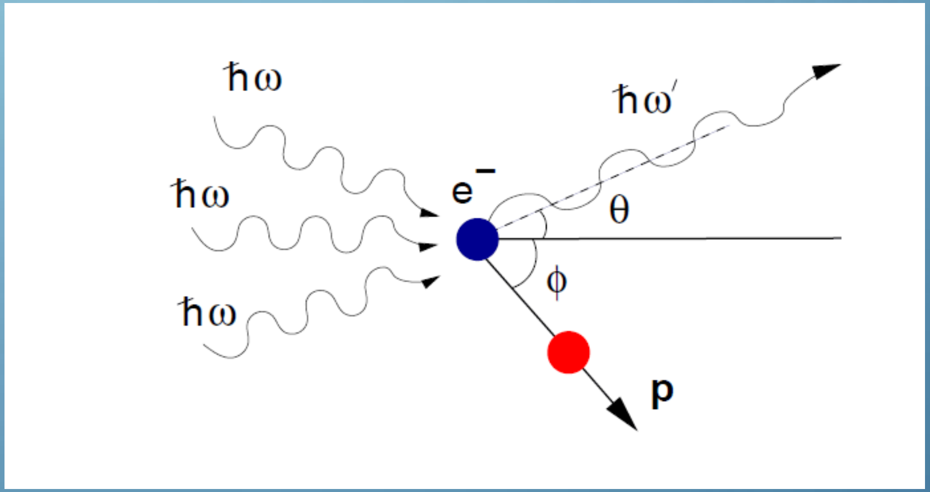
$$\left(-\partial^2 - 2ieA\cdot\partial + e^2A^2 - m^2 - ie\not{k}\frac{\mathrm{d}\not{A}}{\mathrm{d}\varphi}\right)\psi_{\mathrm{p}}(x) = 0$$

$$\psi_{\mathrm{p}}(x) = \mathrm{e}^{-\mathrm{i}p\cdot x}F(\varphi)$$

$$2\mathrm{i}k\cdot p\frac{\partial F}{\partial\varphi} = \left(2ep\cdot A - e^2A^2 + \mathrm{i}e\not{k}\frac{\partial\not{A}}{\partial\varphi}\right)F.$$

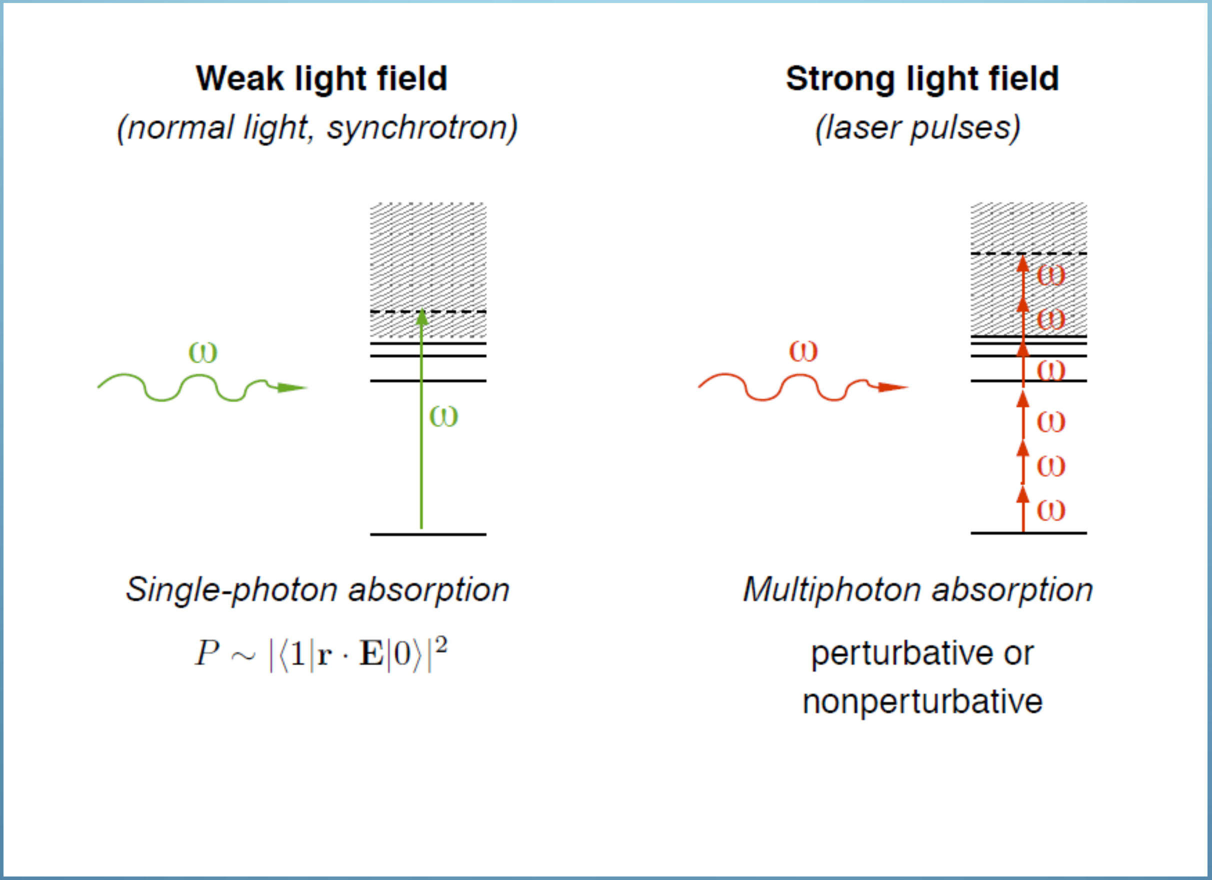
$$F = \exp\left[-\mathrm{i}\int_{-\infty}^{k\cdot x}\left(\frac{ep\cdot A}{pk} - \frac{e^2A^2}{2p\cdot k}\right)\mathrm{d}\varphi + \frac{e\not{k}\not{A}}{2p\cdot k}\right] \times u(p,s),$$





....What's more???

- GBF and photoionization...



...what is the link with undulator?

- Mmmhhh!!!

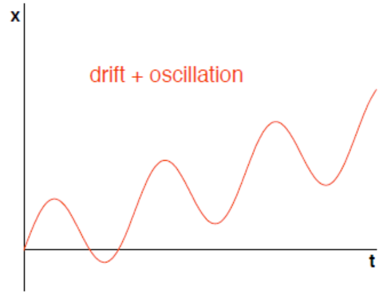
Classical electron in a monochromatic laser field

Equation of motion: $\boxed{\ddot{\mathbf{r}}(t) = -\mathbf{E}_0 \sin(\omega t)}$

(using dipole approximation; $\mathbf{E}_0 \sin(\omega t)$ = electric field, linearly polarized)

Velocity: $\dot{\mathbf{r}}(t) = \mathbf{v}_{\text{drift}} + \frac{\mathbf{E}_0}{\omega} \cos(\omega t)$

Position: $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_{\text{drift}} t + \frac{\mathbf{E}_0}{\omega^2} \sin(\omega t)$



Oscillation amplitude: $\alpha = \mathbf{E}_0/\omega^2$

Kinetic energy: $T(t) = \frac{v_{\text{drift}}^2}{2} + \mathbf{v}_{\text{drift}} \cdot \frac{\mathbf{E}_0}{\omega} \cos(\omega t) + \frac{E_0^2}{2\omega^2} \cos^2(\omega t)$

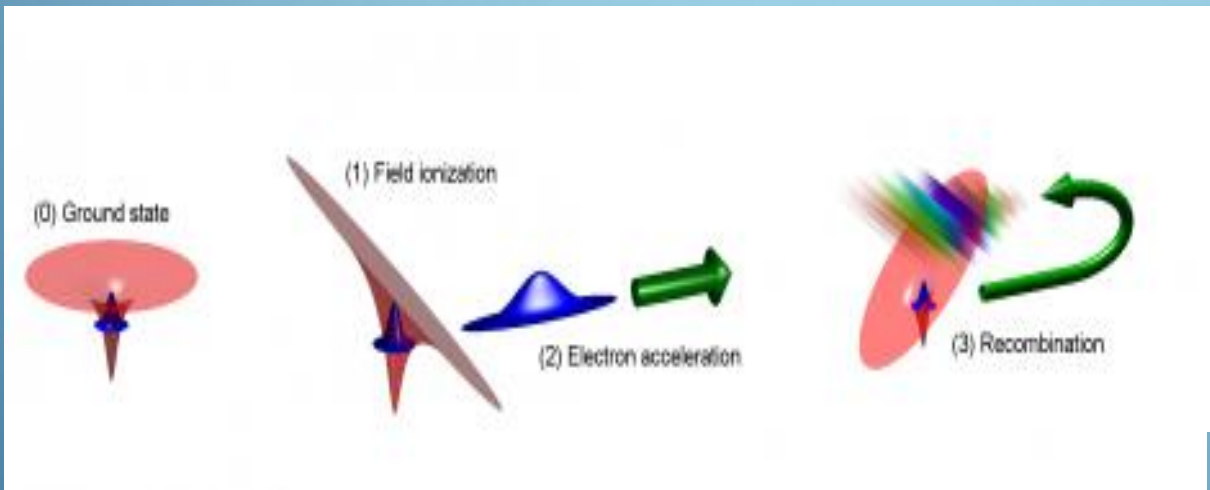
Average kinetic energy: $\boxed{\bar{T} = \frac{v_{\text{drift}}^2}{2} + \frac{E_0^2}{4\omega^2}}$

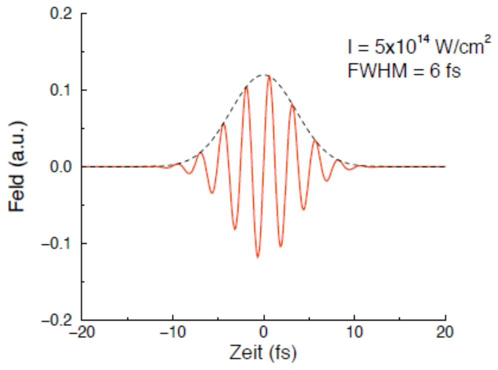
$\rightarrow$ Define **ponderomotive potential**: $\boxed{U_{\text{p}} = \frac{E_0^2}{4\omega^2}}$

If field amplitude is position dependent, there will be a ponderomotive force $\mathbf{F}_{\text{p}} = -\nabla U_{\text{p}}(\mathbf{r})$.

Combination of tunnel effect and laser pulse field

- Harmonic generation





What a lovely Plot!!!!

Dedicated to all the ladies in this Room