Scattering Amplitudes

new perspectives
on Feynman Integral Calculus

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Motivation

- Identify a unique Mathematical framework for any Multi-Loop Amplitude
- Simplify the calculations in High-Energy Physics
- Computing the uncomputable
- Discover hidden properties of Quantum Field Theories

Path

- Scattering Amplitudes in QFT
- Unitarity and Analyticity
- Poles and *Residues*
- Amplitudes Decomposition
- Unitarity-based methods and Cauchy's Residue Theorem
- Multiloop *Integrand Reduction* and principles of Algebraic Geometry
- Application: H+3jets and HtTj production at NLO
- Application: beyond one-loop
- Differential Equations for Feynman Integrals: Magnus Exponential
- Conclusions

Origins

1. What is the major discovery of the mankind?

2. What is the major invention of the mankind?

3. How do human beings acquire knowledge?

Origins

1. What is the major discovery of the mankind?
   The Fire
2. What is the major invention of the mankind?
   The Wheel
3. How do human beings acquire knowledge?
   By successive approximation

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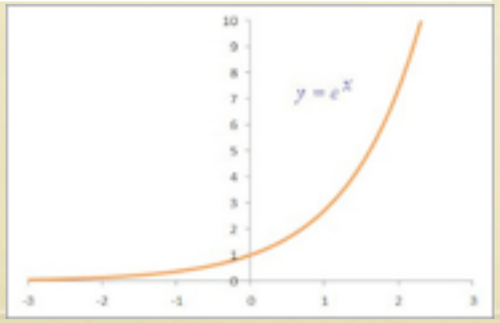
What Particle Physics has to do with that?

Origins

- Focusing energy in one point

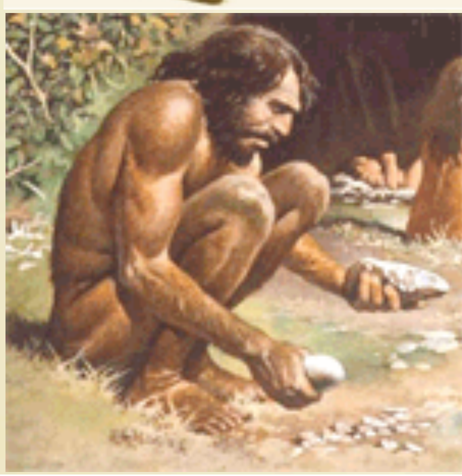
- Energy from collisions

- usefulness of circular shapes



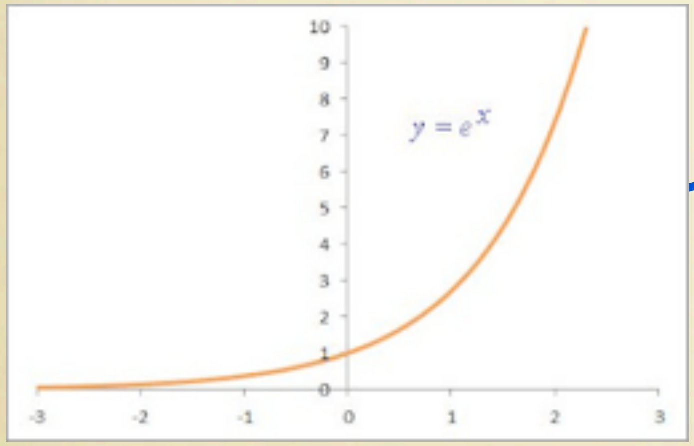
- Exponential function

Origins



Particle Physics...

...in practice

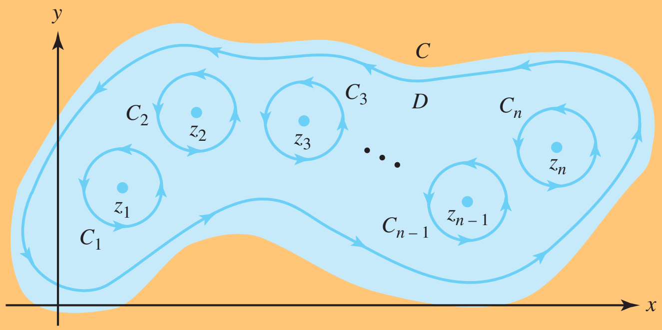


Particle Physics...

$$e^{ix} = \left(1 + \frac{-x^2}{2!} + \frac{x^4}{4!} + \frac{-x^6}{6!} + \frac{x^8}{8!} + \cdots\right) + i\left(x + \frac{-x^3}{3!} + \frac{x^5}{5!} + \frac{-x^7}{7!} + \frac{x^9}{9!} \cdots\right)$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad + \quad i \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= \cos(x) + i \sin(x)$$

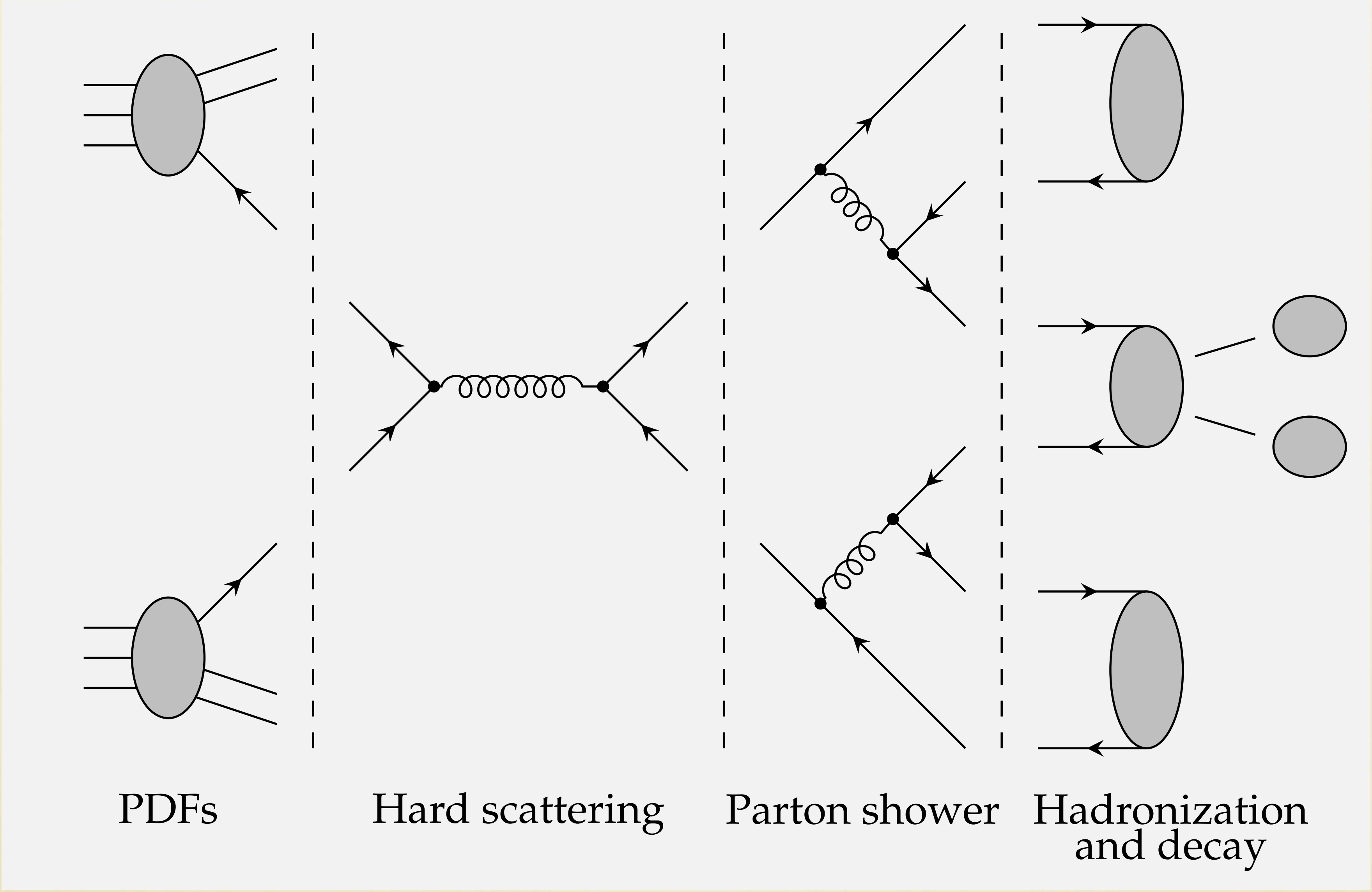


...in theory

Perturbation Theory

- Goal :: Discovery = Caos - Known
- Tool :: Factorization Hypothesis => Observables = Non-Perturbative x Perturbative
- Perturbative Approach
  - organize the knowledge in successive approximations
  - delaying our ignorance to higher-orders

Anatomy of the Scattering Process



Scattering Matrix

$$S_{fi} = \lim_{\substack{t_1 \to +\infty \\ t_2 \to -\infty}} \langle \phi_f | U(t_1, t_2) | \phi_i \rangle = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p_f - p_i) M_{fi}.$$

The transition rate for a transition from the initial state i to the final state f per unit time is $w_{fi} = \frac{|S_{fi}|^2}{T}$

total scattering cross section $\sigma(a+b \to 1+2+...+n)$

$$\sigma = \frac{\#\text{transitions per unit of time}}{\#\text{incoming particles per surface per time}} = \frac{w_{fi}}{\text{flux}}$$

$$\text{flux} = \frac{\#\text{particles}}{\text{volume}} \cdot |\text{relative velocity}|$$

The cross section is given by

$$\sigma = \frac{1}{2s} \int \mathrm{d}\phi_n \left(p_1, \ldots, p_n; Q\right) \frac{1}{S} \sum_{\text{spin}} |M_n|^2 \,,$$

where Q is the total incoming momentum, $(s = Q^2)$ and

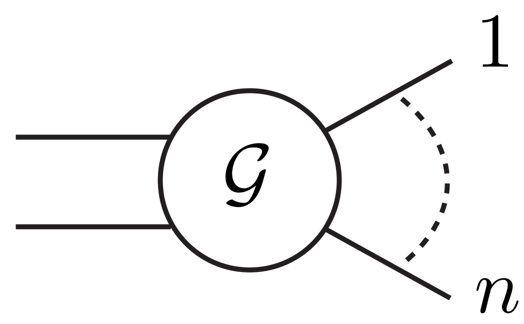
$$\mathrm{d}\phi_n = (2\pi)^4 \delta^d \left(Q^\mu - \sum_{j=1}^{n} p_j^\mu \right) \prod_{j=1}^{n} \frac{\mathrm{d}^d p_j}{(2\pi)^{d-1}} \delta_+ \left(p_j^2 - m_j^2\right)$$

Scattering Amplitude

- **Feynman Diagrams**

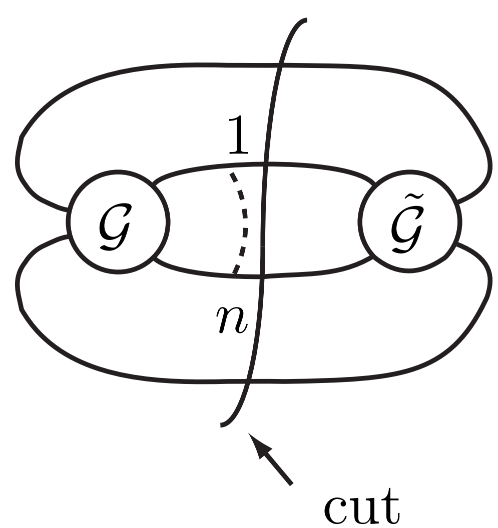
$$i\mathcal{M} = \sum_{\mathcal{G}}$$

Graphs

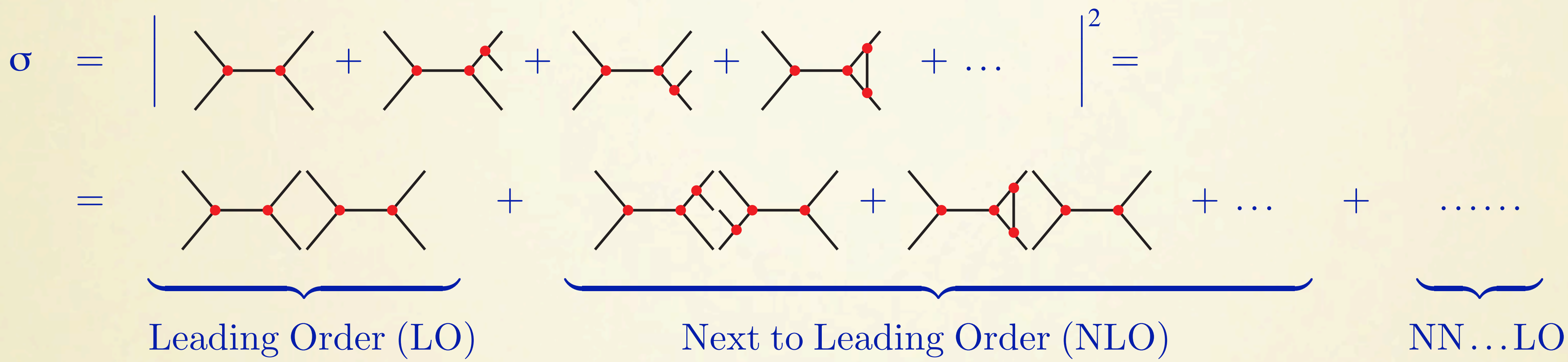


- **Squared Amplitude**

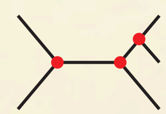
$$\sum |\mathcal{M}_n|^2 = \sum_{\text{cuts},\, \mathcal{G}}$$



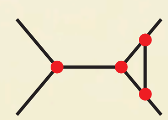
Perturbation Theory & Feynman Diagrams

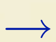


- ☑ tree-graphs with (n+1)-partons
  - soft/collinear divergences



- ☐ virtual-graphs with n-partons
  - divergences from loop-integration





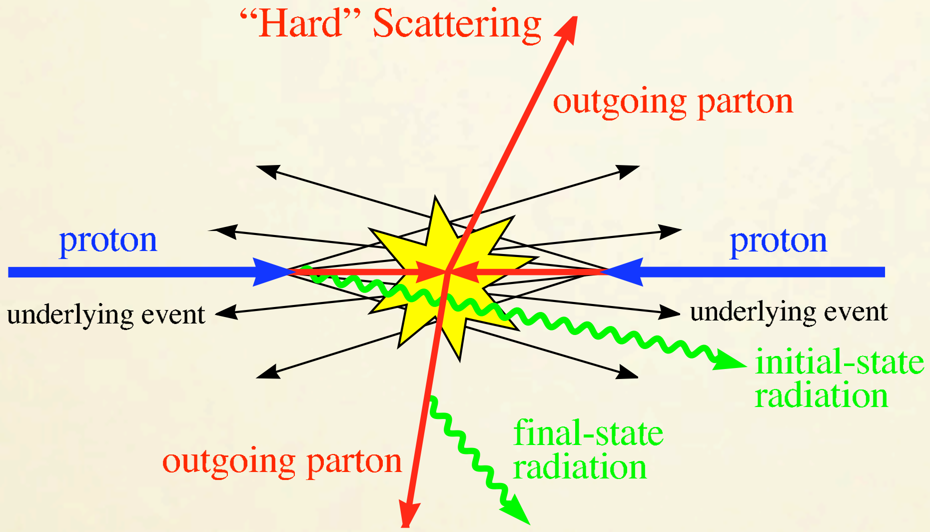
$$I^{\mu\nu\rho\dots} = \int d^D\ell \, \frac{\ell^\mu \ell^\nu \ell^\rho \dots}{D_1 D_2 \dots}$$

- ☑ extracting IR-singularities from both and combining them
  - phase-space slicing, subtractions, dipoles, antennas

Scattering Amplitudes

- Front-line in Theoretical Particle Physics

@ LHC Phenomenology



p-p collision @ 14 TeV c.m.e.

Signals:

- Decays: $H \to VV \qquad (V = \gamma, W, Z)$
- $PP \to H + 0,1,2$ jets (Gluon Fusion)
- $PP \to H + 2$ jets (Weak Boson Fusion)
- $PP \to H + t\bar{t}$
- $PP \to H + W, Z$

Backgrounds:

- $PP \to t\bar{t} + 0,1,2$ jets
- $PP \to VV + 0,1,2$ jets
- $PP \to V + 0,1,2,3$ jets
- $PP \to VVV + 0,1,2,3$ jets

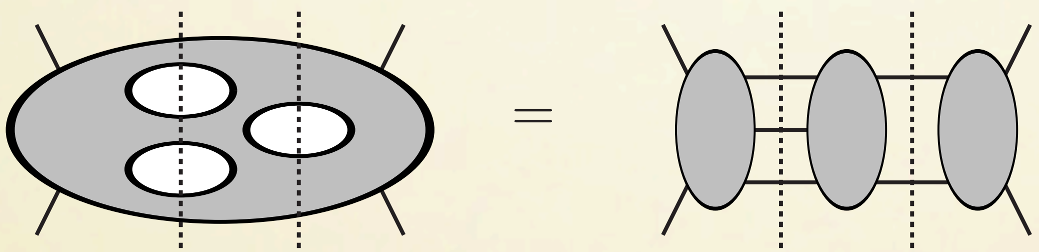
Scattering Amplitudes

- Front-line in Theoretical Particle Physics

@ LHC Phenomenology

@ QFT Stucture

- ElectroWeak Symmetry Breaking: Higgs mechanism
- Beyond the Standard Model (SuSy, Dark Matter, . . .)
- Unveiling the *Iterative Structure* of Scattering Amplitudes in gauge-Theory



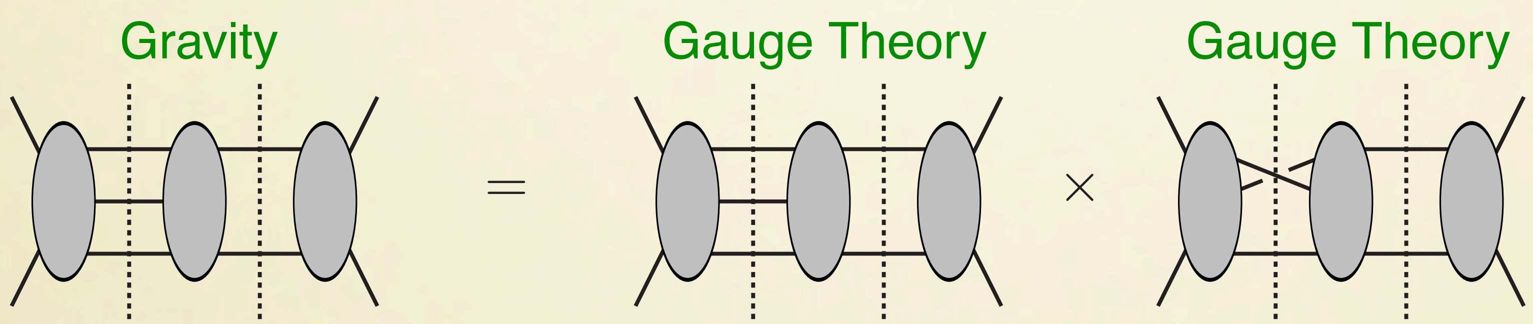
Scattering Amplitudes

Front-line in Theoretical Particle Physics

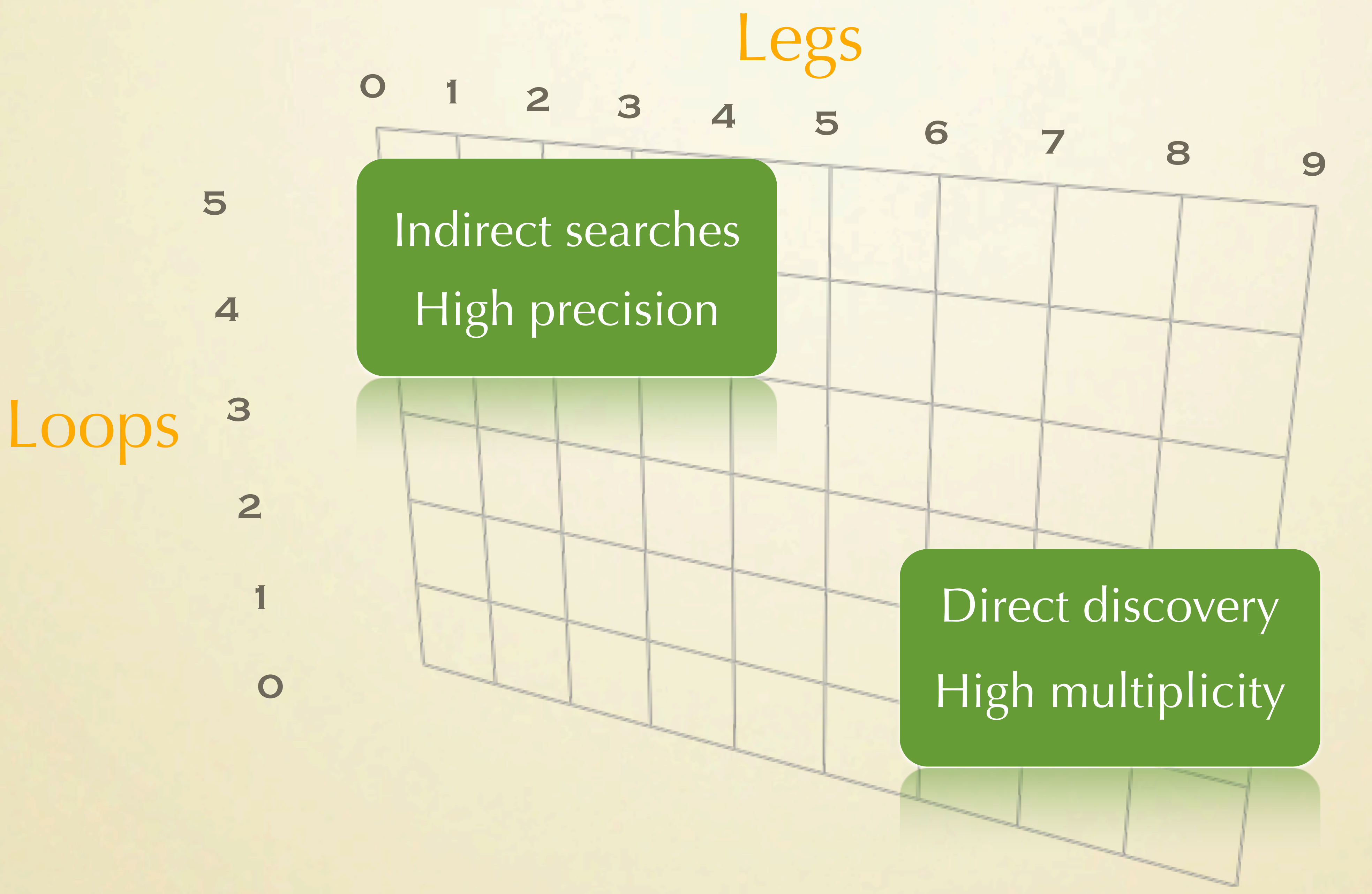
@ LHC Phenomenology

@ QFT Stucture

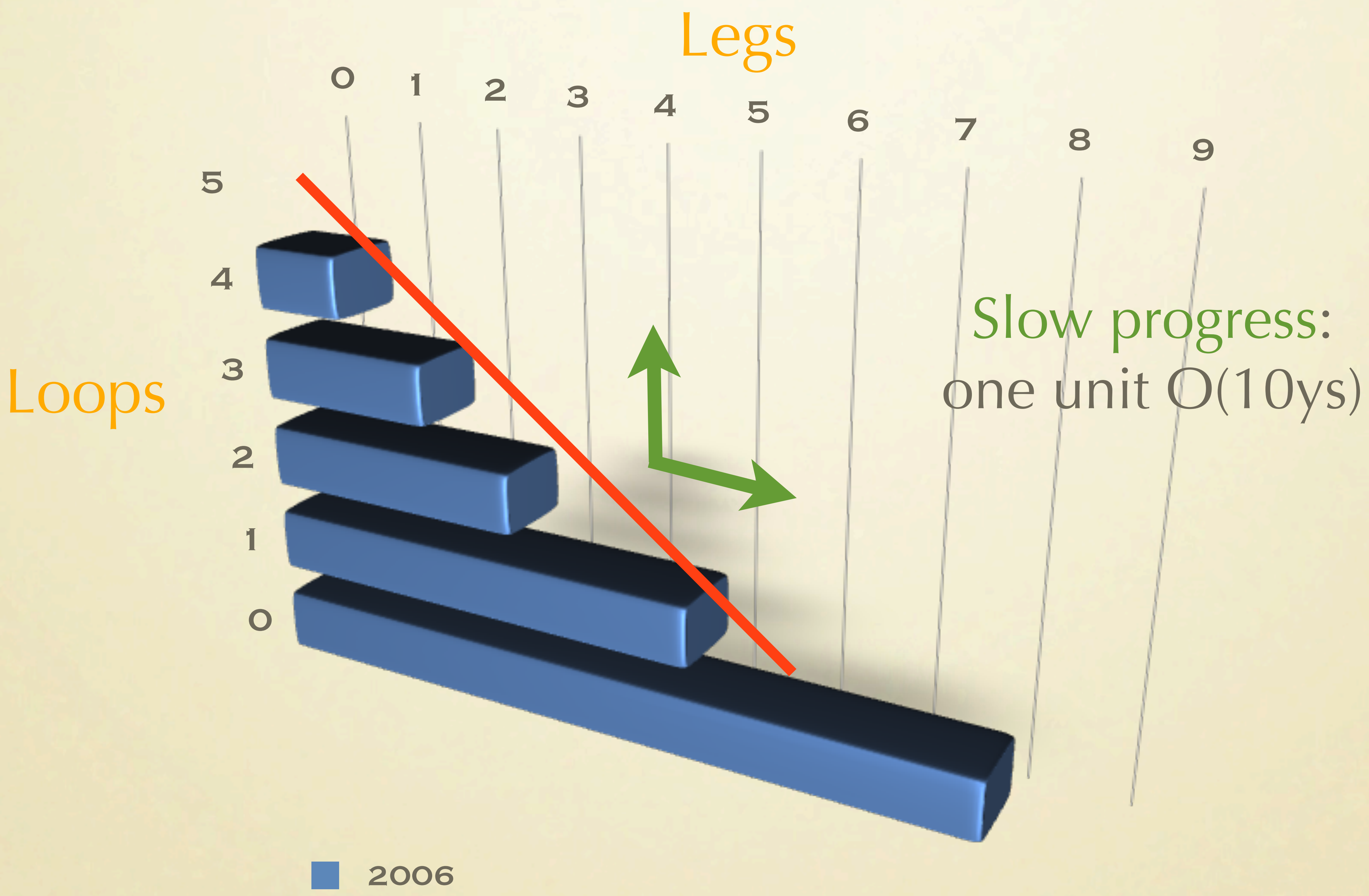
- ElectroWeak Symmetry Breaking: Higgs mechanism
- Beyond the Standard Model (SuSy, Dark Matter, . . .)
- Unveiling the Iterative Structure of Scattering Amplitudes in gauge-Theory
- Exploring the *Finiteness of Supergravity*



High Energy Physics Goals: Loops vs Legs



Complexity: Loops vs Legs



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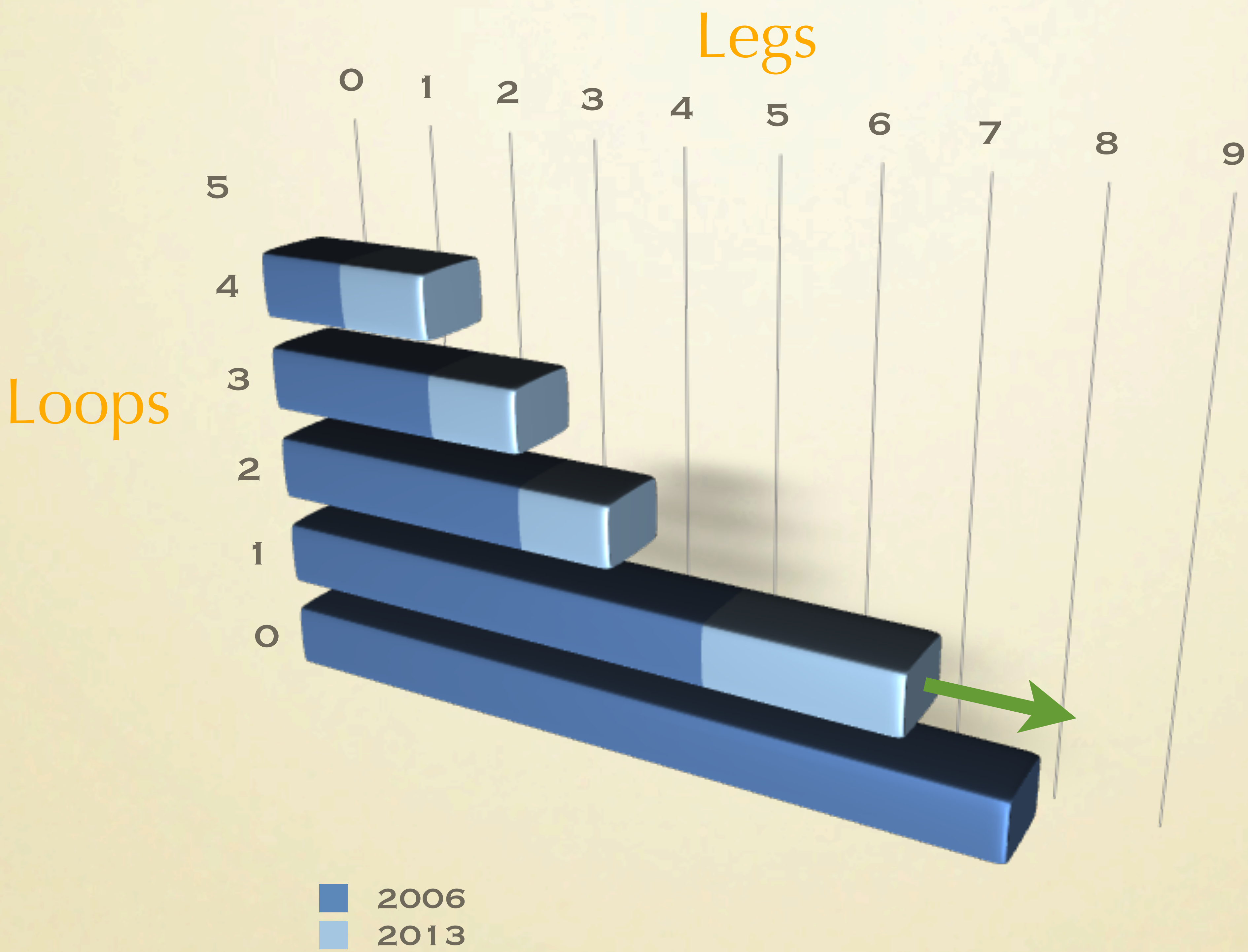
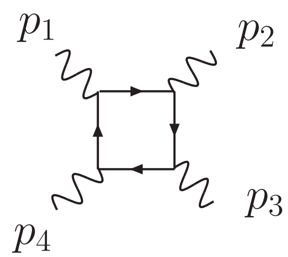
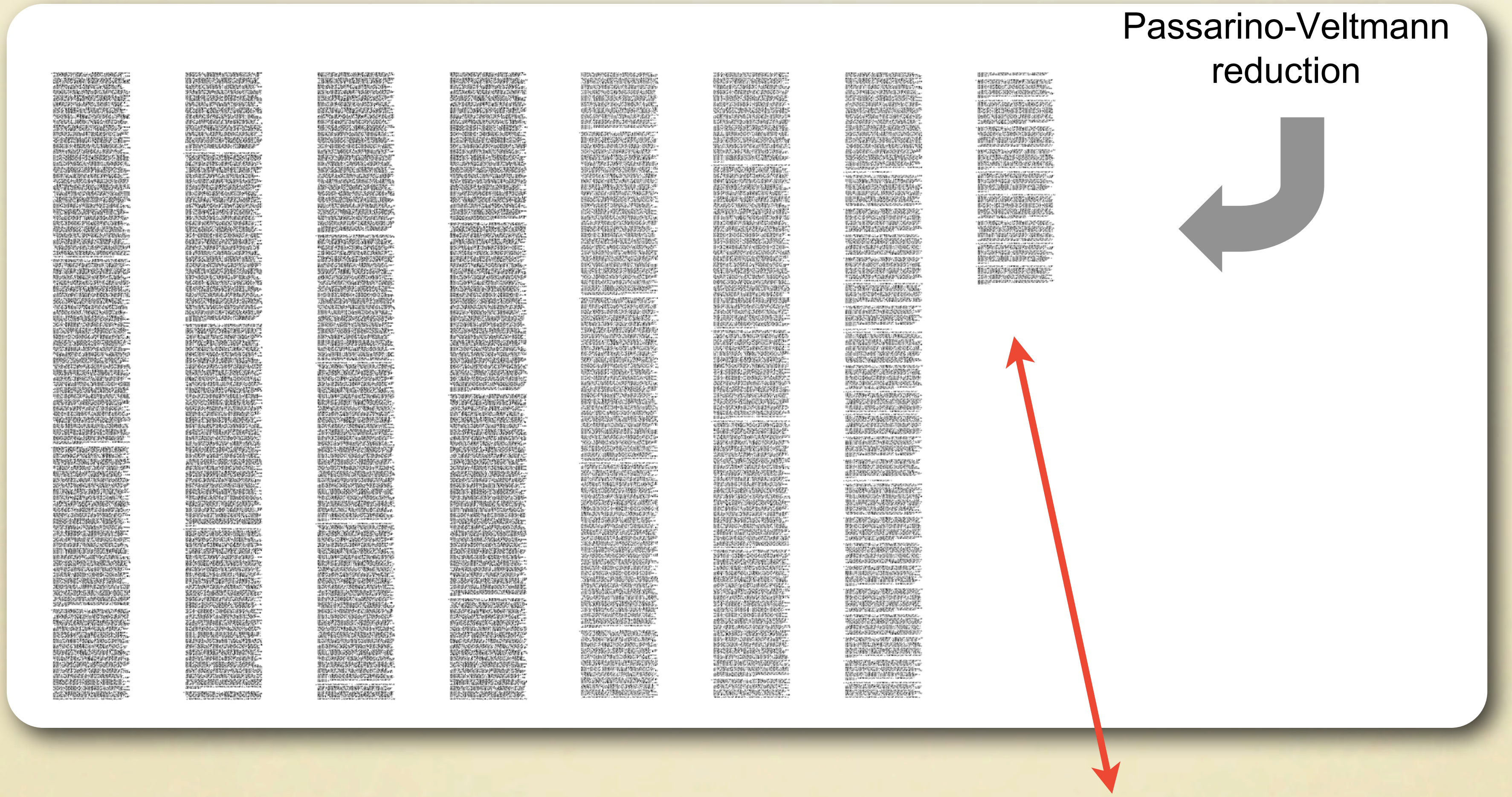


Feynman Diagrams Complexity

- four photon amplitude



$$\int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-\epsilon}} \frac{\ell^{\mu}\,\ell^{\nu}\,\ell^{\rho}\,\ell^{\lambda}}{\ell^2\,(\ell-k_1)^2\,(\ell-k_1-k_2)^2\,(\ell+k_4)^2}$$



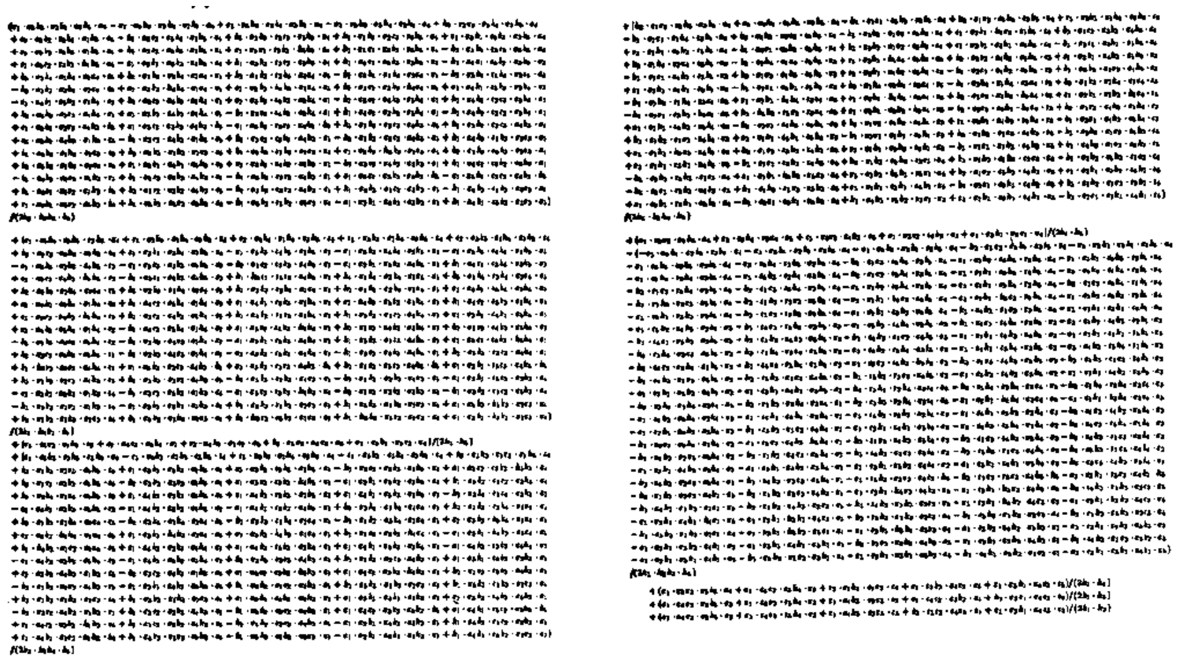
All-plus photon helicity-amplitude = -8 + O(ε)

Feynman Diagrams Complexity

- **n+2 gluon tree-amplitude** gg --> gg...g

n	2	3	4	5	6	7	8
# of diagrams	4	25	220	2485	34300	559405	10525900

5-gluon case (n=3)



All-plus helicity = 0

Single-minus helicity = 0

Two-minus => $\mathcal{A}_n(1^-,2^+,\ldots,m^-,\ldots,n^+) = i g^{n-2} \frac{\langle 1m\rangle^4}{\langle 12\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}$

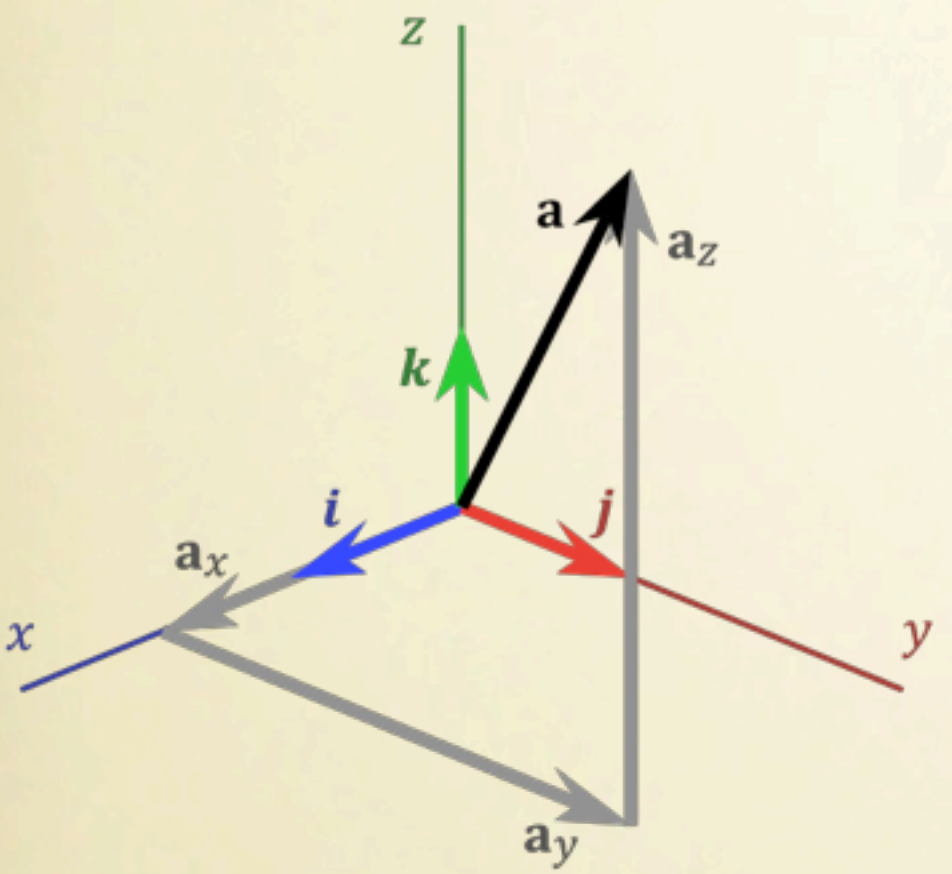
Looking for Simplicity behind Complexity?

Process-Independent Strategy

- Properties of the S-Matrix
  - a general mathematical property: **Analyticity** of Scattering-Amplitudes
    - ▷ *Scattering Amplitudes are determined by their poles and branch-cuts*
  - a general physical property: **Unitarity** of Scattering-Amplitudes
    - ▷ *The residues at poles and branch-points are products of simpler amplitudes, with lower number of particles and/or less loops*
- Multi-pole expansion of Scattering Amplitudes

Amplitudes Decomposition:

the algebraic way



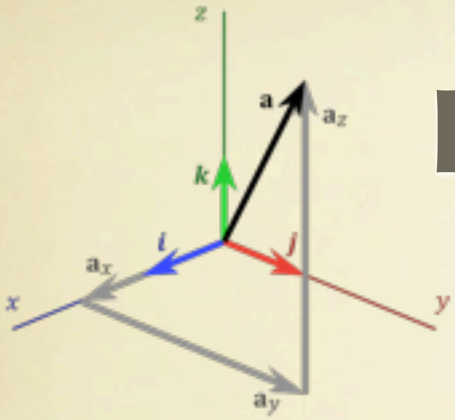
$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$

- Basis: {i j k}
- Scalar product/Projection: to extract the components

$a_x = \mathbf{a}.\mathbf{i}$

$a_y = \mathbf{a}.\mathbf{j}$

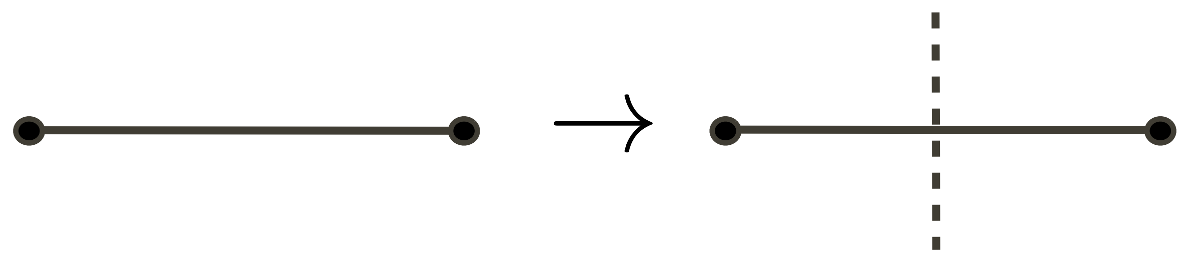
$a_z = \mathbf{a}.\mathbf{k}$



Projections :: On-Shell Cut-Conditions

vanishing denominators

$$\frac{1}{p^2 - m^2 - i0} \;\rightarrow\; \delta(p^2 - m^2)$$



Completeness Relations: cutting "1"

- the richness of factorization

$$i\,(-i) = 1$$

$$\sum_n |\,\psi_{\mathrm{n}}\,\rangle\,\langle\,\psi_{\mathrm{n}}| \;=\; \mathbb{1}$$

$$\sum_{n=0}^{N-1} \frac{e^{2\pi i \frac{k}{N} n}}{\sqrt{N}} \, \frac{e^{-2\pi i \frac{k'}{N} n}}{\sqrt{N}} \;=\; \delta_{kk'}$$

Completeness Relations: cutting propagators

- **massless Spin-1**

$$-i\frac{g^{\mu\nu}}{k^2 - i0}$$

$$\frac{1}{k^2 - i0} \to \delta(k^2) \qquad \Rightarrow \qquad -g^{\mu\nu} \to \sum_{\text{polarization}-\lambda} \epsilon_\lambda^\mu(k) \left(\epsilon_\lambda^\nu(k)\right)^*$$

on-shell

residue

- **massive fermions**

$$i\frac{(\not p + m)}{p^2 - m^2 - i0}$$

$$\frac{1}{p^2 - m^2 - i0} \to \delta(p^2 - m^2) \qquad \Rightarrow \qquad (\not p + m) \to \sum_{\text{spin}-s} u_s(p)\, \bar{u}_s(p)$$

on-shell

residue

On-shellness for Tree-Level Amplitudes

- **Cauchy's Residue Theorem**

$$\oint \frac{dz}{z(z-z_1)(z-z_2)\cdots(z-z_n)} = 0 \quad \Longleftrightarrow \quad \begin{aligned} \frac{(-1)}{z_1 z_2 \cdots z_n} &= \frac{1}{z_1(z_1-z_2)\cdots(z_1-z_n)} \\ &+ \frac{1}{(z_2-z_1)z_2\cdots(z_2-z_n)} \\ &+ \cdots\cdots \\ &+ \frac{1}{(z_n-z_1)(z_n-z_2)\cdots(z_n-z_{n-1})z_n} \end{aligned}$$

On-shellness for Tree-Level Amplitudes

- **Cauchy's Residue Theorem**

$$\oint \frac{dz}{z(z-z_1)(z-z_2)\cdots(z-z_n)} = 0$$

$$\Longleftrightarrow$$

- **Partial Fractioning**

$$\begin{aligned}\frac{(-1)}{z_1 z_2 \cdots z_n} &= \frac{1}{z_1(z_1-z_2)\cdots(z_1-z_n)} \\ &+ \frac{1}{(z_2-z_1)z_2\cdots(z_2-z_n)} \\ &+ \cdots\cdots \\ &+ \frac{1}{(z_n-z_1)(z_n-z_2)\cdots(z_n-z_{n-1})z_n}\end{aligned}$$

On-shellness for Tree-Level Amplitudes

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- **On-shell condition (cuts)**

$$(q_i - z_i\eta)^2 - m_i^2 = 0\ , \qquad z_i = \frac{q_i^2 - m_i^2}{2\eta . q_i}\ ,$$

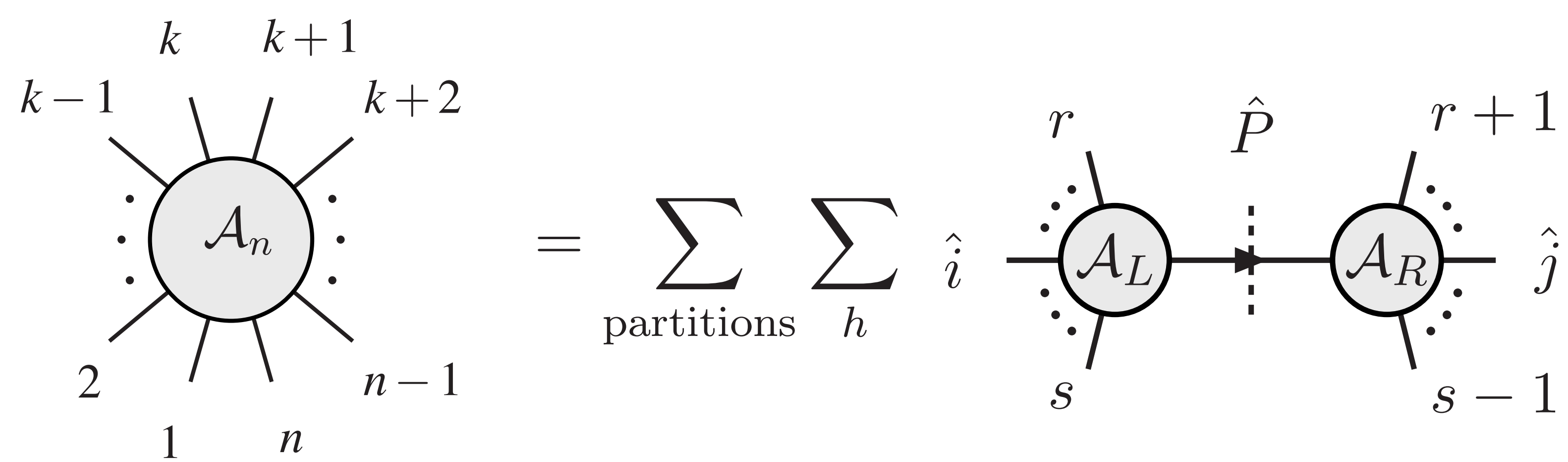
- **Denominator decomposition**

$$(-1)\frac{1}{q_1^2-m_1^2}\frac{1}{q_2^2-m_2^2}\cdots\frac{1}{q_n^2-m_n^2} = \frac{1}{q_1^2-m_1^2}\frac{1}{(q_2-z_1\eta)^2-m_2^2}\cdots\frac{1}{(q_n-z_1\eta)^2-m_n^2}$$
$$+ \frac{1}{(q_1-z_2\eta)^2-m_1^2}\frac{1}{q_2^2-m_2^2}\cdots\frac{1}{(q_n-z_2\eta)^2-m_n^2}$$
$$+ \cdots\cdots$$
$$+ \frac{1}{(q_1-z_n\eta)^2-m_1^2}\frac{1}{(q_2-z_n\eta)^2-m_2^2}\cdots\frac{1}{q_n^2-m_n^2}$$

Tree-Level Amplitudes

- Cauchy's Residue Theorem $$\frac{1}{2\pi i}\oint \frac{\mathcal{A}_n(z)}{z} = \mathcal{A}_n(\infty) = \mathcal{A}_n(0) + \sum_{\text{poles}} \text{Res}\mathcal{A}_n(z)$$

If $\mathcal{A}_n(\infty)=0$, then one obtaines the relation $$\mathcal{A}_n(0) = -\sum_{\text{poles}} \text{Res}\mathcal{A}_n(z)$$



$$\mathcal{A}_n(p_1^{h_1},\dots,p_n^{h_n}) = \sum_{\text{partition}}\sum_{h} \mathcal{A}_L(p_r,\dots,\hat{p}_i,\dots,p_s,-\hat{P}^h_{r:s})\,\frac{1}{P^2}\,\mathcal{A}_R(\hat{P}^h_{r:s},p_{s+1},\dots,\hat{p}_j,\dots,p_{r-1})$$

BCFW Recurrence Relation

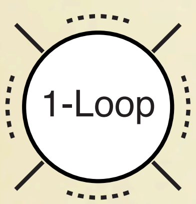
Britto, Cachazo, Feng, Witten

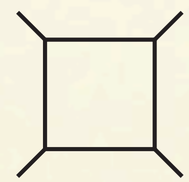
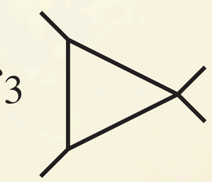
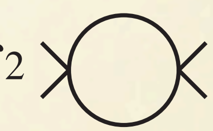
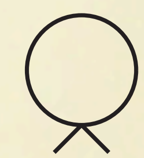
- Multi-pole expansion of Tree-level Amplitudes!

Tree-level decomposition
by
partial fractioning:
is this an accident?

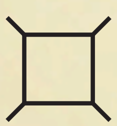
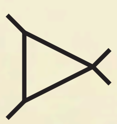
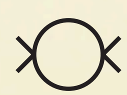
One-Loop Scattering Amplitudes

- *n*-particle Scattering: $1+2 \to 3+4+\ldots+n$
- Reduction to a Scalar-Integral Basis   Passarino-Veltman



$$= \sum_{10^2-10^3} \int d^D\ell \, \frac{\ell^\mu \ell^\nu \ell^\rho \ldots}{D_1 D_2 \ldots D_n} = c_4$$  $$+ c_3$$  $$+ c_2$$  $$+ c_1$$ 

- Known: Master Integrals

 $$= \int d^D\ell \, \frac{1}{D_1 D_2 D_3 D_4} \, ,$$  $$= \int d^D\ell \, \frac{1}{D_1 D_2 D_3} \, ,$$  $$= \int d^D\ell \, \frac{1}{D_1 D_2} \, ,$$  $$= \int d^D\ell \, \frac{1}{D_1}$$

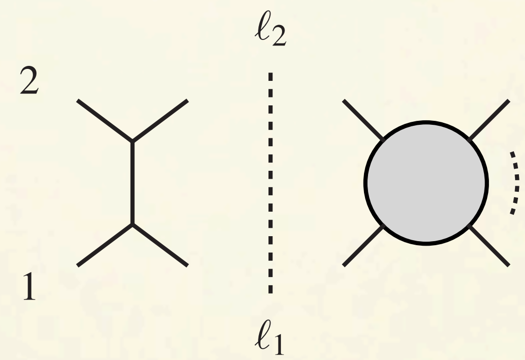
- Unknowns: c_i are rational functions of external kinematic invariants

Cutting Rules

- Discontinuity of Feynman Integrals Landau & Cutkosky

Cut Integral in the P_{12}^2-channel

$$\Delta(\Gamma_1) =$$



$$d^4\Phi \;\equiv\; d^4\ell_1 \; d^4\ell_2 \; \delta^{(4)}\Big(\ell_1 + \ell_2 - P_{12}\Big) \; \delta^{(+)}\Big(\ell_1^2 - m_1^2\Big) \; \delta^{(+)}\Big(\ell_2^2 - m_2^2\Big)$$

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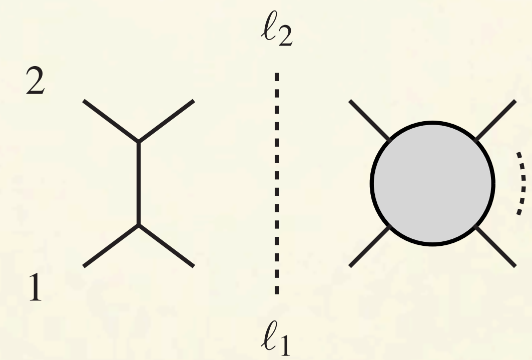
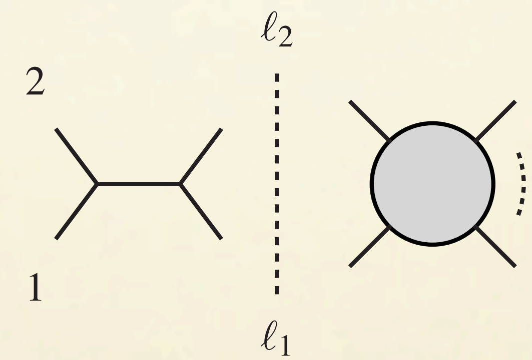


$\Delta(\Gamma_2) =$



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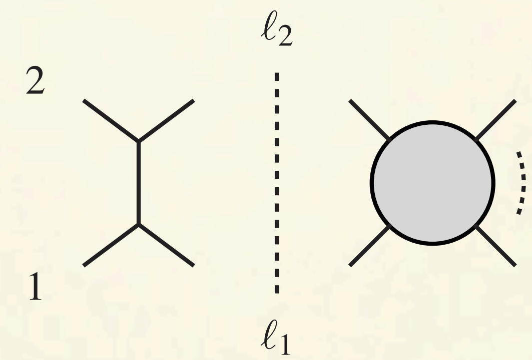
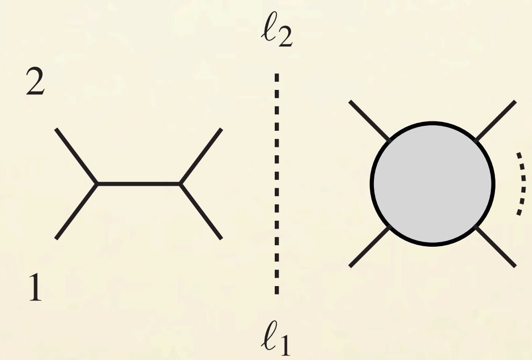
Cut Integral in the P_{12}^2-channel

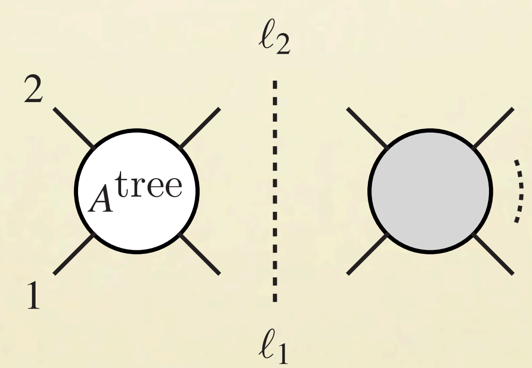
$$\Delta(\Gamma_1) =$$



$$\Delta(\Gamma_2) =$$



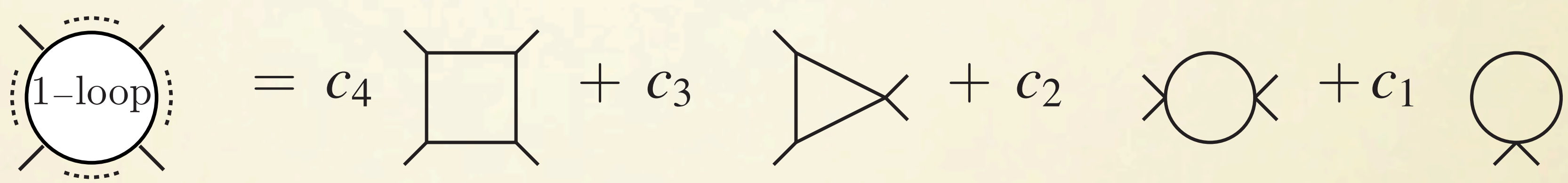
$$\Delta(\Gamma_1) + \Delta(\Gamma_2) =$$



Unitarity & Cutting Rules

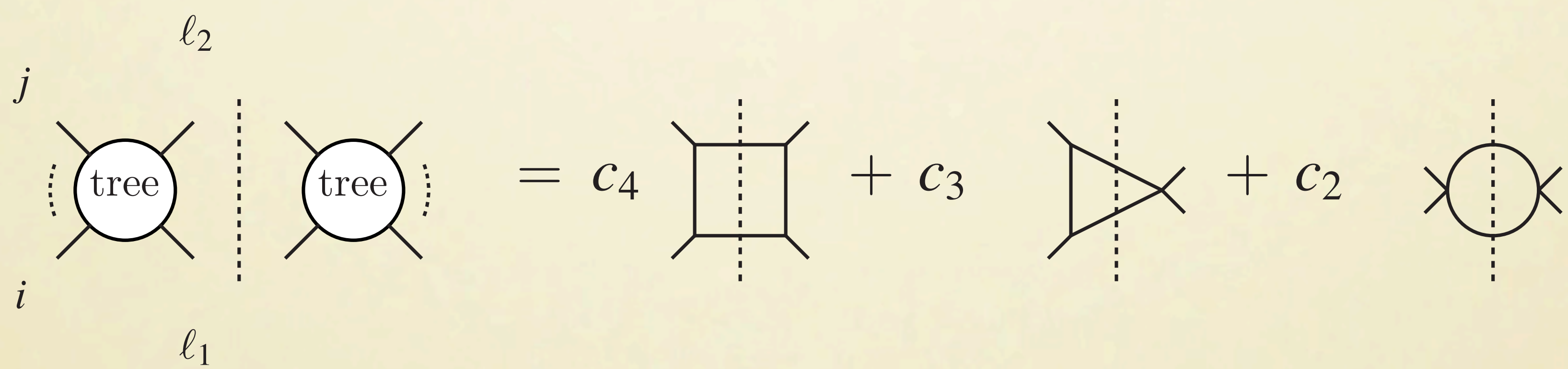
- Optical Theorem from Unitarity $\quad S \equiv 1 + iT \;:\; S^\dagger S = 1 \quad \Rightarrow \quad 2\mathrm{Im}T = -i(T - T^\dagger) = T^\dagger T$
- One-loop Amplitude:

$A_n^{\text{1-loop}} =$



- Discontinuity of Feynman Amplitudes Cutkosky-Veltman; Bern, Dixon, Dunbar & Kosower

$2\mathrm{Im}\left\{A_n^{\text{1-loop}}\right\} =$

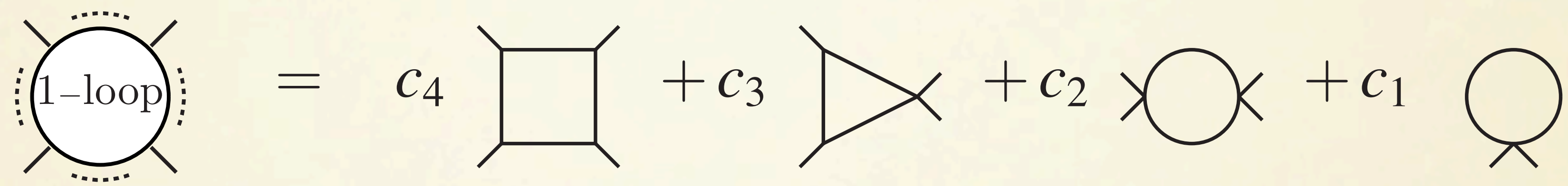


on − shell condition : $\dfrac{1}{(\ell_i^2 - m_i^2 + i0)} \rightarrow \delta\left(\ell_i^2 - m_i^2\right) \qquad (i = 1, 2)$

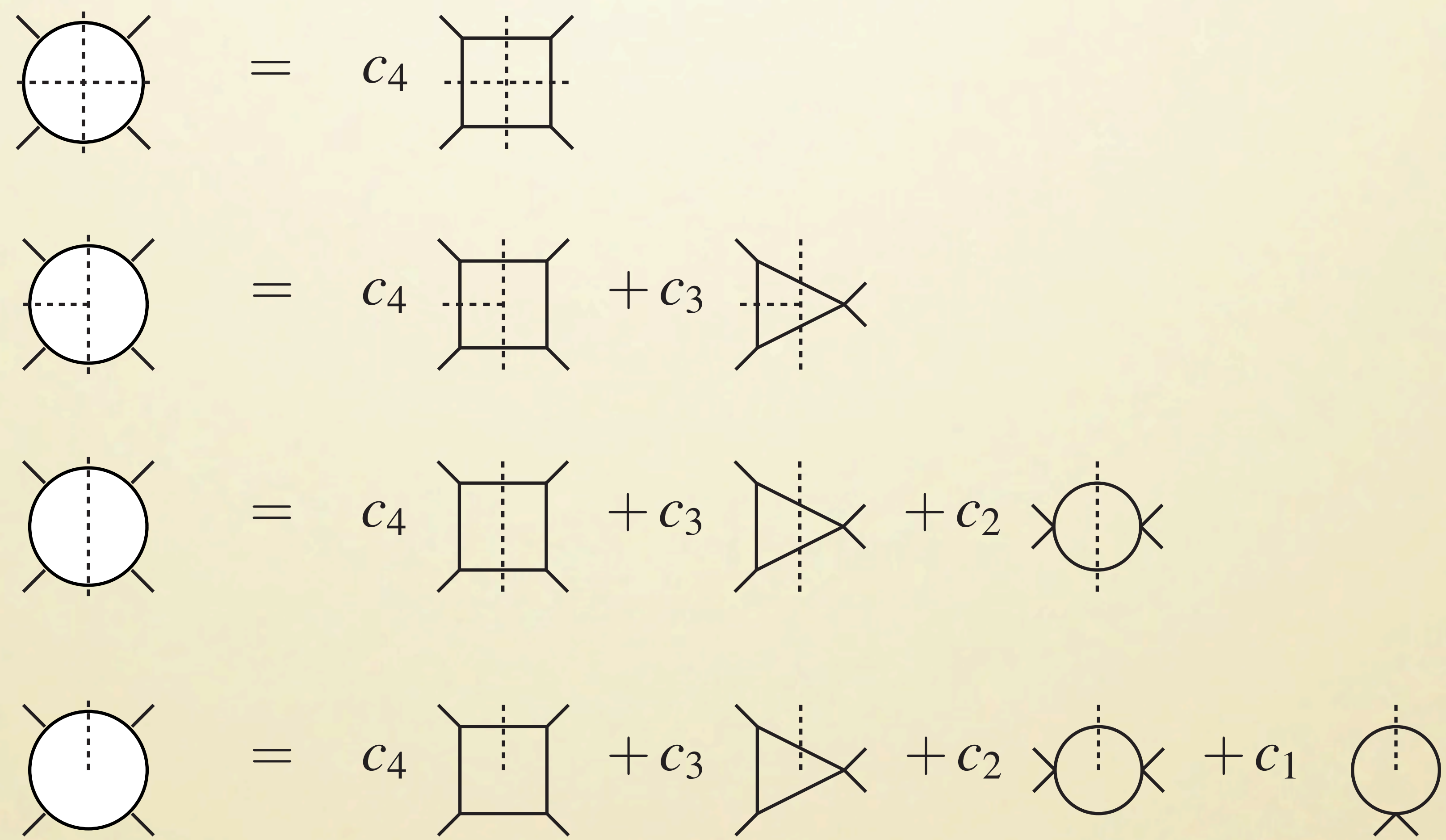
The Strategy: Generalised Unitarity

- One-loop Amplitude:

$A_n^{\text{1-loop}} =$



☑ Multiple-cut as projectors



The more you cut, the more you loose, the simpler it gets

Cut-Conditions

- Loop momentum decomposition $\ell_\mu = x_1 \, p_\mu + x_2 \, q_\mu + x_3 \, \varepsilon_\mu^+ + x_4 \, \varepsilon_\mu^-$
- On-shell condition $\delta\left(\ell_i^2 - m_i^2\right)$

- under Multiple On-shellness Conditions :

- the loop-momentum becomes complex ;
- some of its components (if not all) are frozen;
- the left over free components are *integration*-variable

Cut-Conditions

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- under Multiple On-shellness Conditions :

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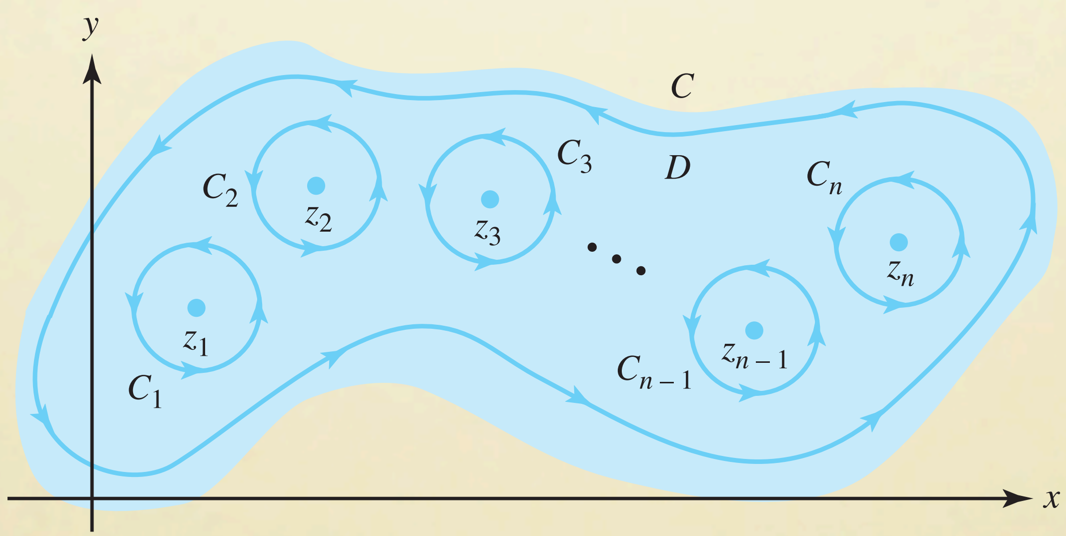
- the left over free components are *integration*-variable

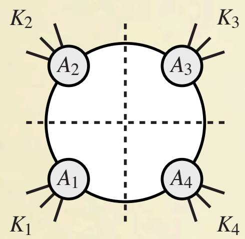
To *integrate* or not to *integrate*:
that is the question

To *integrate...*

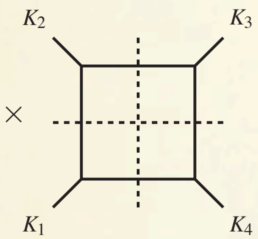
Cut-Integration
by
Cauchy's residue theorem
(and its generalization)



4ple-cut Britto, Cachazo, Feng



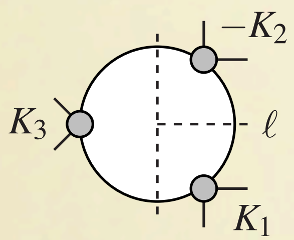
$= c_{[K_1|K_2|K_3|K_4]} \times$



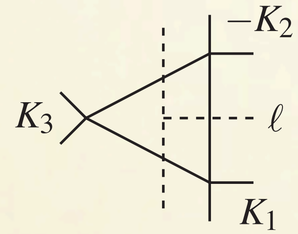
$$c_{[K_1|K_2|K_3|K_4]} = \frac{I_4(\ell_+) + I_4(\ell_-)}{2}$$

Cauchy's formula

3ple-cut Forde



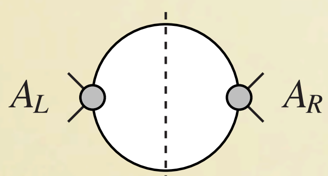
$= c_{[K_1|K_2|K_3]} \times$



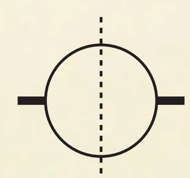
$$c_{[K_1,K_2,K_3]} = \frac{\mathrm{Res}_{t=0}\left\{ I_3(\ell^+) + I_3(\ell^-) \right\}}{2}$$

Laurent series

2ple-cut P.M.



$$= \int d^4\Phi \; A_L^{\rm tree}(\ell_1) \; A_R^{\rm tree}(\ell_1) = c_{[K]} \times K$$

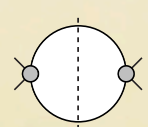


$$\int d^4\Phi = (1-2\rho) \iint \frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2}$$

$$2\pi i \mathcal{F}(z_0) = \int_{\partial D} \frac{\mathcal{F}(z)}{z - z_0} dz - \iint_D \frac{\mathcal{F}_{\bar{z}}}{z - z_0} d\bar{z} \wedge dz.$$

Stokes' Thm
Cauchy-Pompeiu formula

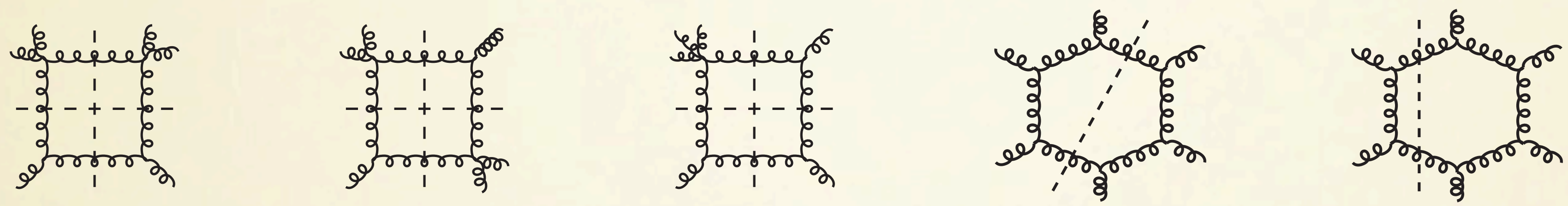
$c_{[K]} =$



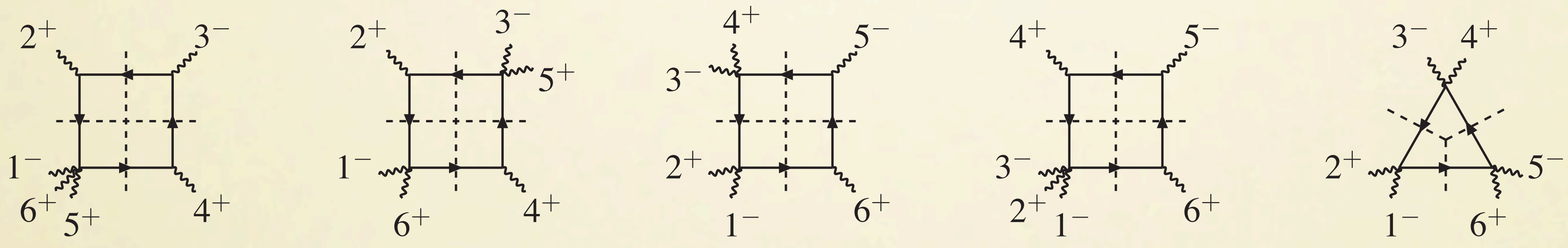
$$\Big|_{\rm rat} = \oint dz \, F^{\rm rat}(z, z^*) = \mathrm{Res}_{z=0} \, F^{\rm rat}(z, z^*) + \mathrm{Res}_{z\neq 0} \, F^{\rm rat}(z, z^*)$$

Analytic Calculations: state-of-the-art

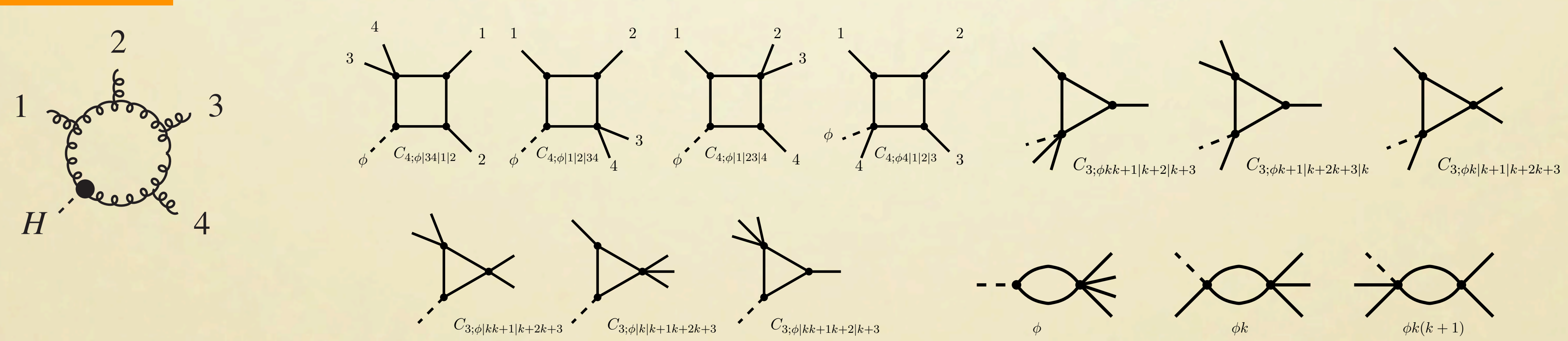
$gg \longrightarrow gggg$ Britto, Feng & P.M. (2006)



$\gamma\gamma \longrightarrow \gamma\gamma\gamma\gamma$ Binoth, Gehrmann, Heinrich & P.M. (2007)



$gg \longrightarrow Hgg$ Badger, Glover, Williams, P.M. (2008-2009)

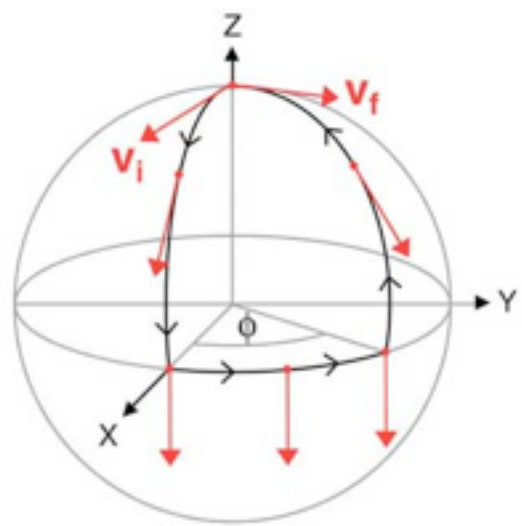


Optical-Thm and Berry Phase

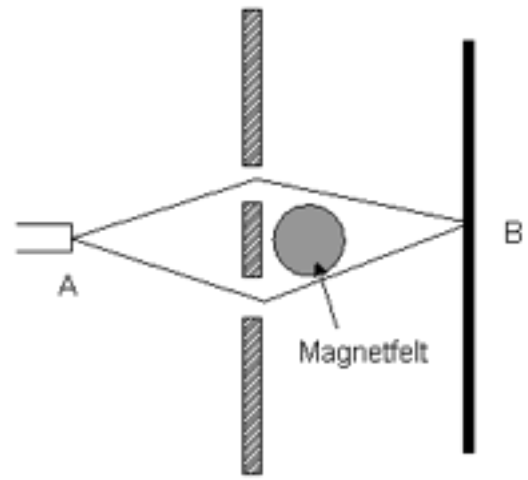
P.M. (2009)

- Geometric Phases

Simple Geometry



Aharonov-Bohm effect



$$\int_\Sigma \nabla \times \mathbf{F} \cdot d\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r},$$

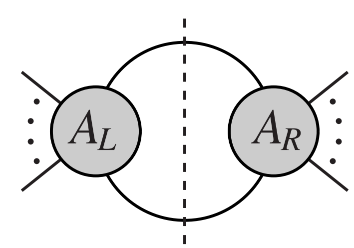
$$\nabla \times \mathbf{A} = \mathbf{B}$$

$$\varphi = \frac{q}{\hbar} \int_P \mathbf{A} \cdot d\mathbf{x},$$

Optical Theorem

$$\begin{aligned} \Delta &= \int d^4\Phi \; A^{*,\mathrm{tree}}_{m\to 2} \; A^{\mathrm{tree}}_{n\to 2} = \\ &= -i\Big[A^{\mathrm{one-loop}}_{n\to m} - A^{*,\mathrm{one-loop}}_{m\to n}\Big] = \\ &= 2\,\mathrm{Im}\Big\{A^{\mathrm{one-loop}}_{n\to m}\Big\}\,, \end{aligned}$$

$$\begin{aligned} \Delta &= (1-2\rho) \iint dz \wedge d\bar{z} \; \frac{A^{*,\mathrm{tree}}_{m\to 2} \; A^{\mathrm{tree}}_{n\to 2}}{(1+z\bar{z})^2} = \\ &= (1-2\rho) \oint dz \int d\bar{z} \; \frac{A^{*,\mathrm{tree}}_{m\to 2} \; A^{\mathrm{tree}}_{n\to 2}}{(1+z\bar{z})^2}\,, \end{aligned}$$



The double-cut is the flux of a 2-form.
The anholonomy phase shift is a consequence of Stokes' Theorem.

…or not to *integrate*

Cut-Integration replaced
by
partial fractioning
(and its generalization)

Multi-Loop Integrand-Reduction
by
Polynomial Division

Ossola & P.M. (2011)

Badger, Frellesvig, Zhang (2011)

Zhang (2012)

Mirabella, Ossola, Peraro, & P.M (2012)

Problem: what is the form of the residues?

"find the right variables encoding the cut-structure"

variables

- ISP's = Irreducible Scalar Products:
  - q-components which can variate under cut-conditions
  - spurious: vanishing upon integration
  - non-spurious: non-vanishing upon integration $\Rightarrow$ MI's

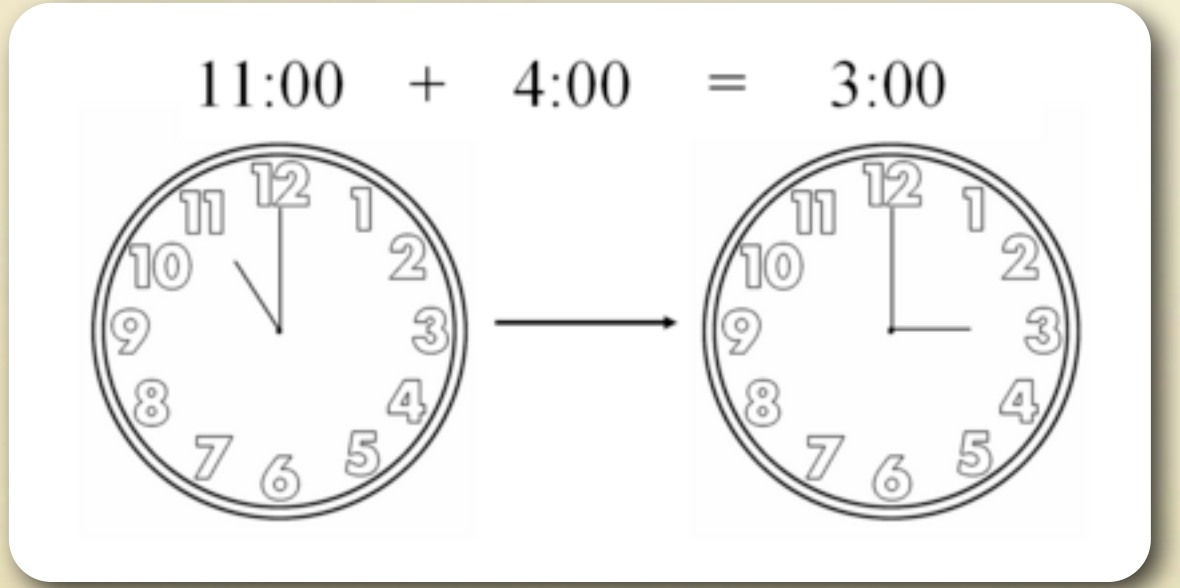
Ossola & P.M. (2011)

A simple idea from Modular Arithmetic

Division Modulo *n*

The following statements are all equivalent:

(i) $a \equiv b (mod\ n)$
(ii) $n | (a - b)$
(iii) $a - b = nt$ for some $t \in \mathbb{Z}$
(iv) $a = b + nt$ for some $t \in \mathbb{Z}$.



hold the *remainder* !

Multivariate Polynomial Division

Zhang (2012);
Mirabella, Ossola, Peraro, & P.M. (2012)

- **Ideal** $\mathcal{J}_{i_1\cdots i_n} = \langle D_{i_1}, \cdots, D_{i_n} \rangle \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$

- **Groebner Basis** $\mathcal{G}_{i_1\cdots i_n} = \{g_1(\mathbf{z}), \ldots, g_m(\mathbf{z})\}$

  n-ple cut-conditions $D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$

Multivariate Polynomial Division

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n-ple cut-conditions $$D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$$

Polynomial Division $$\mathcal{N}_{i_1 \cdots i_n}(\mathbf{z}) = \Gamma_{i_1 \cdots i_n} + \Delta_{i_1 \cdots i_n}(\mathbf{z}) \; ,$$

Remainder ~ Residue $$\Delta_{i_1 \cdots i_n}(\mathbf{z})$$

Quotients $$\Gamma_{i_1 \cdots i_n} = \sum_{i=1}^{m} \mathcal{Q}_i(\mathbf{z}) g_i(\mathbf{z})$$ belongs to the ideal $\mathcal{J}_{i_1 \cdots i_n}$,

$$= \sum_{\kappa=1}^{n} \mathcal{N}_{i_1 \cdots i_{\kappa-1} i_{\kappa+1} \cdots i_n}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \; .$$

Multi-Loop Integrand Recurrence

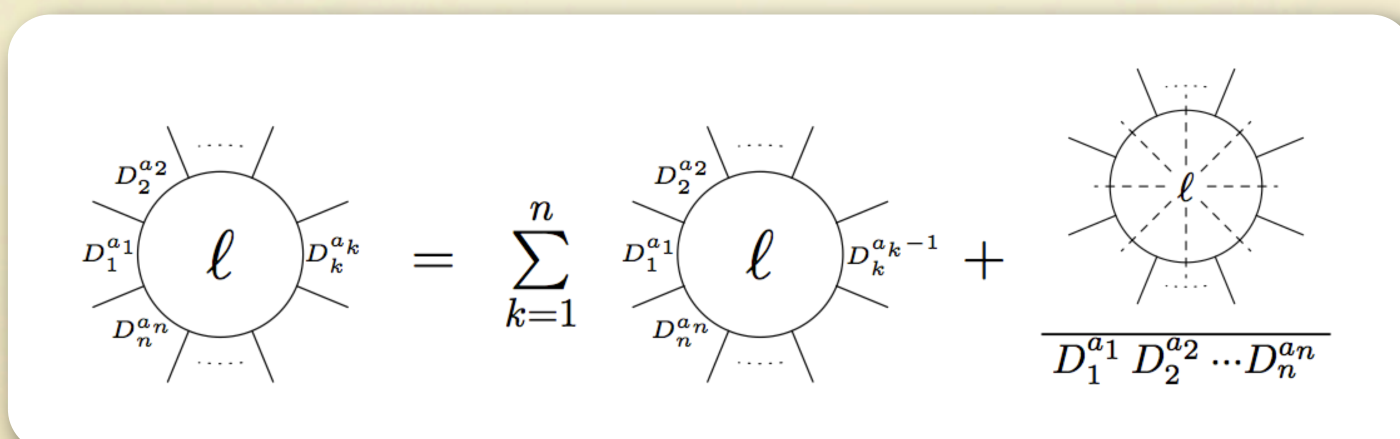
Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

remainder = residue

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{\kappa=1}^{k} \mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1} i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} \,.$$

n-denominator integrand

(n-1)-denominator integrand



Multi-Loop Integrand Decomposition

- Multi-(particle)-pole decomposition

$$\mathcal{I}_{i_1\cdots i_n} = \frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1} D_{i_2} \cdots D_{i_n}}$$

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-1}}$$
$$+ \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

Multi-Loop Integrand Decomposition

☑ Multi-(particle)-pole decomposition

$$\mathcal{I}_{i_1 \cdots i_n} = \frac{\mathcal{N}_{i_1 \cdots i_n}}{D_{i_1} D_{i_2} \cdots D_{i_n}}$$

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$$+ \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

Tree-level
decomposition
by
partial fractioning:
is this an accident?

Apparently no!

Parametric ***form of the residues*** is ***process independent***

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max-1}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max-1}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max-1}}} + \sum_{1=i_1<<i_{\max-2}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max-2}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max-2}}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

The actual ***values of the coefficients*** in the residues are are ***process dependent***

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-1}}$$

$$+ \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

☑ Parametric form of the residues is process independent.

Knowing the parametric form of residues is ***mandatory***!!!

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-1}}$$

$$+ \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

Use your favourite generator,
(Feynman diagrams, tree-amplitudes, currents,...),
and sample *I*(q's) as many time as the
number of unknown coefficients

☑ Parametric form of the residues is process independent.

☑ Actual values of the coefficients is process dependent.

The Maximum-Cut Theorem

Mirabella, Ossola, Peraro, & P.M. (2012)

At ℓ loops, in four dimensions, we define a *maximum-cut* as a (4ℓ)-ple cut

$$D_{i_1} = D_{i_2} = \cdots = D_{i_{4\ell}} = 0 \, ,$$

which constrains completely the components of the loop momenta. In four dimensions this implies the presence of four constraints for each loop momenta.

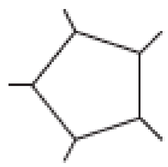
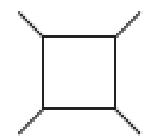
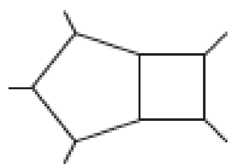
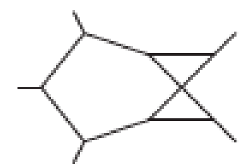
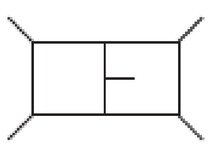
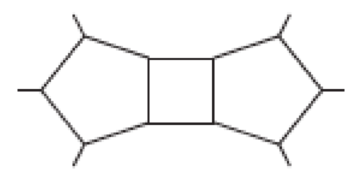
We assume that:

in non-exceptional phase-space points, a maximum-cut has a finite number n_s of solutions, each with multiplicity one.

Under this assumption we have the following

Theorem 4.1 (Maximum cut)**.** *The residue at the maximum-cut is a polynomial paramatrised by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.*

Examples of Maximum-Cuts

diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^{3} c_i z^i$	4		$\sum_{i=0}^{3} c_i z^{\,i}$	4
	$\sum_{i=0}^{7} c_i z^{\,i}$	8		$\sum_{i=0}^{7} c_i z^{\,i}$	8

One-Loop Integrand-Reduction

One-Loop Integrand Decomposition

- Choice of 4-dimensional basis for an m-point residue

$$e_1^2 = e_2^2 = 0\,, \qquad e_1 \cdot e_2 = 1\,, \qquad e_3^2 = e_4^2 = \delta_{m4}\,, \qquad e_3 \cdot e_4 = -(1-\delta_{m4})$$

- Coordinates: $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5) \equiv (x_1, x_2, x_3, x_4, \mu^2)$

$$q^\mu_{4\text{-dim}} = -p^\mu_{i_1} + x_1\, e^\mu_1 + x_2\, e^\mu_2 + x_3\, e^\mu_3 + x_4\, e^\mu_4\,, \qquad q^2 = q^2_{4\text{-dim}} - \mu^2$$

- Generic numerator

$$\mathcal{N}_{i_1\cdots i_m} = \sum_{j_1,\ldots,j_5} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5}\,, \qquad (j_1 \ldots j_5) \quad \text{such that} \quad \text{rank}(\mathcal{N}_{i_1\cdots i_m}) \leq m$$

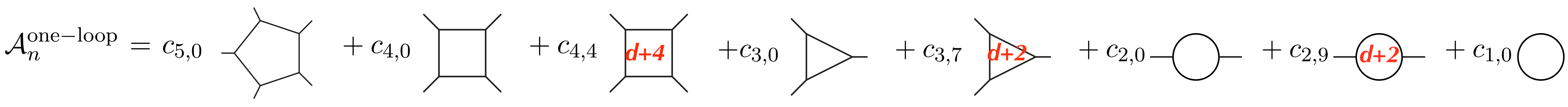
- Residues

$$\begin{aligned}
\Delta_{i_1 i_2 i_3 i_4 i_5} &= c_0 \\
\Delta_{i_1 i_2 i_3 i_4} &= c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4) \\
\Delta_{i_1 i_2 i_3} &= c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4) \\
\Delta_{i_1 i_2} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \\
\Delta_{i_1} &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}$$

One-Loop Integrand Decomposition

$$\mathcal{A}_n^{\rm one-loop} = \int d^{-2\epsilon}\mu \int d^4 q \; A_n(q,\mu^2)\;, \qquad A_n(q,\mu^2) \equiv \frac{\mathcal{N}_n(q,\mu^2)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{n-1}} \qquad \bar{D}_i \;=\; (\bar{q}+p_i)^2 - m_i^2 = (q+p_i)^2 - m_i^2 - \mu^2$$

We use a bar to denote objects living in $d = \; 4-2\epsilon$ dimensions $\qquad \bar{\not{q}} = \not{q} + \not{\mu}\,, \quad$ with $\quad \bar{q}^2 = q^2 - \mu^2\,.$



@ the integrand-level

Ossola, Papadopoulos, Pittau

$$A_n(q,\mu^2) \neq \frac{c_{5,0}}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3\bar{D}_4} + \frac{c_{4,0}+c_{4,4}\mu^4}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3} + \frac{c_{3,0}+c_{3,7}\mu^2}{\bar{D}_0\bar{D}_1\bar{D}_2} + \frac{c_{2,0}+c_{2,9}\mu^2}{\bar{D}_0\bar{D}_1} + \frac{c_{1,0}}{\bar{D}_0}$$

$$= \frac{c_{5,0}+f_{01234}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3\bar{D}_4} + \frac{c_{4,0}+c_{4,4}\mu^4+f_{0123}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3} + \frac{c_{3,0}+c_{3,7}\mu^2+f_{012}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2} + \frac{c_{2,0}+c_{2,9}\mu^2+f_{01}(q,\mu^2)}{\bar{D}_0\bar{D}_1} + \frac{c_{1,0}+f_0(q,\mu^2)}{\bar{D}_0}$$

method:

Reconstruct the complete polynomial residues to extract the coefficients of MI's

Ossola, Papadopoulos, Pittau

2.2.2 Quintuple cut

The residue of the quintuple-cut, $\bar{D}_i = \ldots = \bar{D}_m = 0$, defined as,

$$\Delta_{ijk\ell m}(\bar{q}) = \mathrm{Res}_{ijk\ell m}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}}\right\} = c_{5,0}^{(ijk\ell m)}\,\mu^2 \,.$$

2.2.3 Quadruple cut

The residue of the quadruple-cut, $\bar{D}_i = \ldots = \bar{D}_\ell = 0$, defined as,

$$\Delta_{ijk\ell}(\bar{q}) = \mathrm{Res}_{ijk\ell}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m}\right\} = c_{4,0}^{(ijk\ell)} + c_{4,2}^{(ijk\ell)}\mu^2 + c_{4,4}^{(ijk\ell)}\mu^4 - \left(c_{4,1}^{(ijk\ell)} + c_{4,3}^{(ijk\ell)}\,\mu^2\right)\Big[(K_3\cdot e_4)x_4 - (K_3\cdot e_3)x_3\Big](e_1\cdot e_2)\,,$$

2.2.4 Triple cut

The residue of the triple-cut, $\bar{D}_i = \bar{D}_j = \bar{D}_k = 0$, defined as,

$$\Delta_{ijk}(\bar{q}) = \mathrm{Res}_{ijk}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell}\right\}$$

SAMURAI Ossola, Reiter, Tramontano, & P.M. (2010)

Scattering **AM**plitudes from **U**nitarity-based **R**eduction **A**lgorithm at the **I**ntegrand-level

2.

T

$$\Delta \ldots \sum_{i<<m} \ldots \sum_{i<<\ell} \ldots \sum_{i<<k} \ldots$$

$$= c_{2,0}^{(ij)} + c_{2,9}^{(ij)}\mu^2 + \left(c_{2,1}^{(ij)}x_1 - c_{2,3}^{(ij)}x_4 - c_{2,5}^{(ij)}x_3\right)(e_1\cdot e_2) + \left(c_{2,2}^{(ij)}x_1^2 + c_{2,4}^{(ij)}x_4^2 + c_{2,6}^{(ij)}x_3^2 - c_{2,7}^{(ij)}x_1x_4 - c_{2,8}^{(ij)}x_1x_3\right)(e_1\cdot e_2)^2 \,.$$

2.2.6 Single cut

The residue of the single-cut, $\bar{D}_i = 0$, defined as,

$$\Delta_i(\bar{q}) = \mathrm{Res}_i\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell} - \sum_{i<<k}^{n-1} \frac{\Delta_{ijk}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k} - \sum_{i<j}^{n-1} \frac{\Delta_{ij}(\bar{q})}{\bar{D}_i\bar{D}_j}\right\}$$

$$= c_{1,0}^{(i)} + \left(c_{1,1}^{(i)}x_2 + c_{1,2}^{(i)}x_1 - c_{1,3}^{(i)}x_4 - c_{1,4}^{(i)}x_3\right)(e_1\cdot e_2)\,.$$

Improved Integrand Red'n

- **Integrand Reduction**

universal

$$\Delta_{i_1 \ldots i_m}(q,\mu^2) = \mathrm{Res}_{i_1 \ldots i_m}\left\{ \frac{\mathcal{N}(q,\mu^2)}{\bar{D}_{i_1}\bar{D}_{i_2}\ldots\bar{D}_{i_n}} - \sum_{k=(m+1)}^{5} \sum_{i_1<i_2<\ldots<i_k} \frac{\Delta_{i_1 i_2 \ldots i_k}(q,\mu^2)}{\bar{D}_{i_1}\bar{D}_{i_2}\ldots\bar{D}_{i_k}} \right\}$$

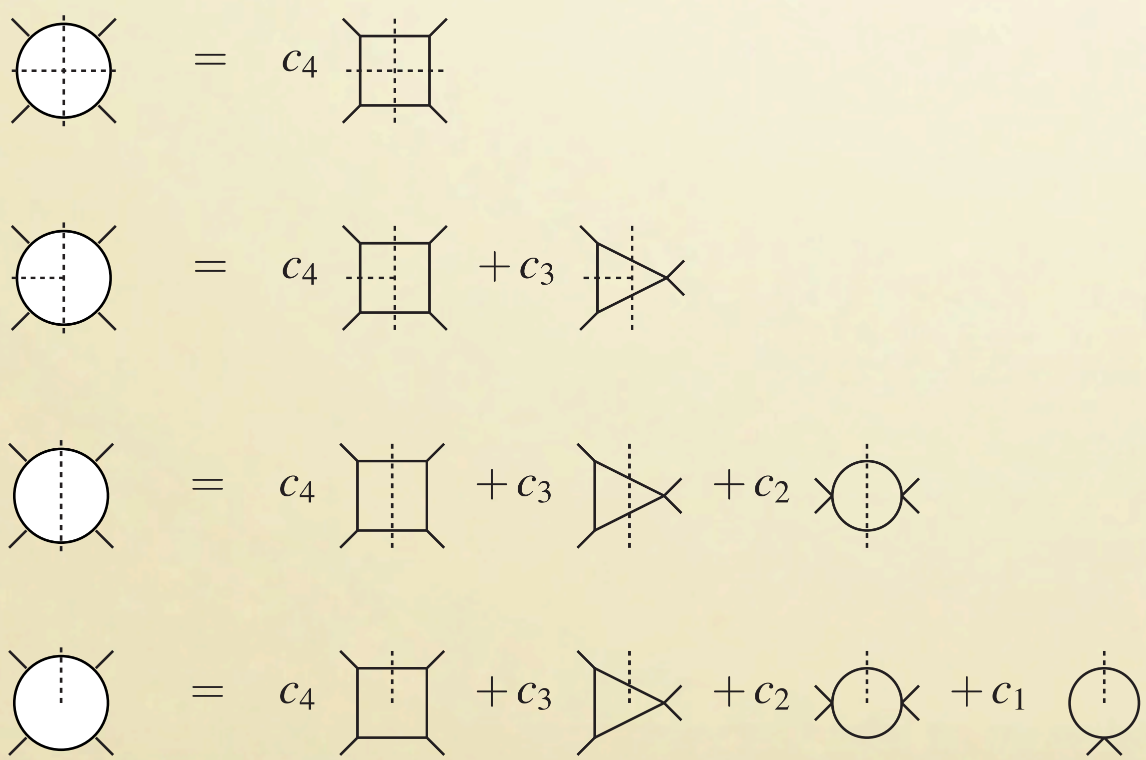
polynomial

non-polynomial

non-polynomial

a + b x + c x^2 + ...

Ossola Papadopoulos Pittau



Improved Integrand Red’n

- **Integrand Reduction** with **Laurent series expansion**

$$\Delta_{i_1\dots i_m}(q,\mu^2) = \mathrm{Res}_{i_1\dots i_m}\left\{ \frac{\mathcal{N}(q,\mu^2)}{\bar{D}_{i_1}\bar{D}_{i_2}\dots\bar{D}_{i_n}} - \sum_{k=(m+1)}^{5}\ \sum_{i_1<i_2<\dots<i_k} \frac{\Delta_{i_1 i_2\dots i_k}(q,\mu^2)}{\bar{D}_{i_1}\bar{D}_{i_2}\dots\bar{D}_{i_k}} \right\}$$

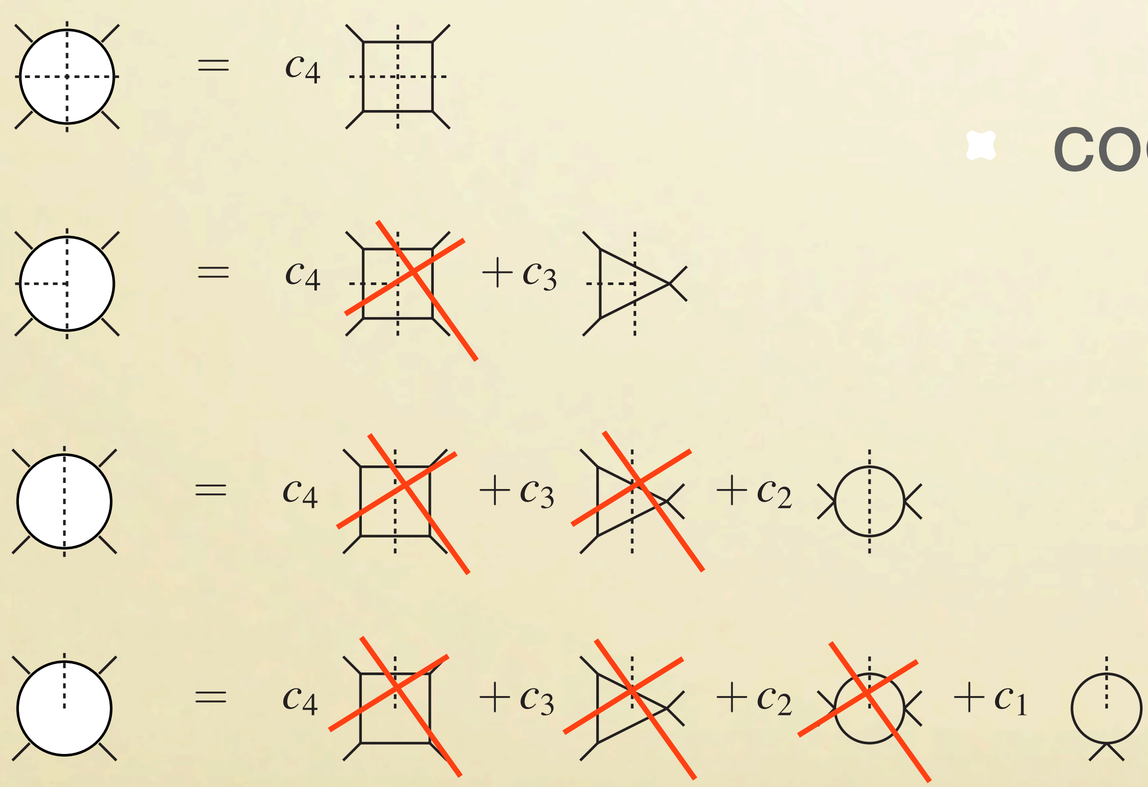
∞ universal ∞

polynomial: a + b x + c x^2 + ...

polynomial: a’+ b’ x + c’ x^2 + ...

universal polynomial: a” + b” x + c” x^2 + ...

Mirabella Peraro ***P.M.***



- coefficients of MI’s :: a = a’+ a”

Ninja C++ library$_{\text{Peraro}}$

Samurai... Ossola Reiter Tramontano ***P.M.***

- *Integrand Reduction* for One-Loop Integrals Ossola Papadopoulos Pittau

- *Generalised D-dim Unitarity*
  :: Complete reduction to D-reg Master Integrals
  :: cut-constructible & rational terms at once
  Anastasiou, Britto, Feng, Kunszt, ***P.M.***
  Ellis Giele Kunszt Melnikov

...meets **Golem** Binoth Guillet Heinrich Pilon Reiter

- *Integrand Generation*

- *Tensor Reduction*

Automatic one-loop calculations

The GoSam Project

Cullen van Deurzen Greiner Heinrich Luisoni
Mirabella Ossola Peraro Reichel Schlenk
von Soden-Fraunhofen Tramontano ***P.M.***

$$\sigma_{\text{NLO}} = \int_n \left(d\sigma_{\text{Born}} + d\sigma_{\text{Virtual}} + \int_1 d\sigma_{\text{Subtraction}} \right) + \int_{n+1} \left(d\sigma_{\text{Real}} - d\sigma_{\text{Subtraction}} \right)$$

Monte Carlo Generator

The GoSam Project

Cullen van Deurzen Greiner Heinrich Luisoni
Mirabella Ossola Peraro Reichel Schlenk
von Soden-Fraunhofen Tramontano ***P.M.***

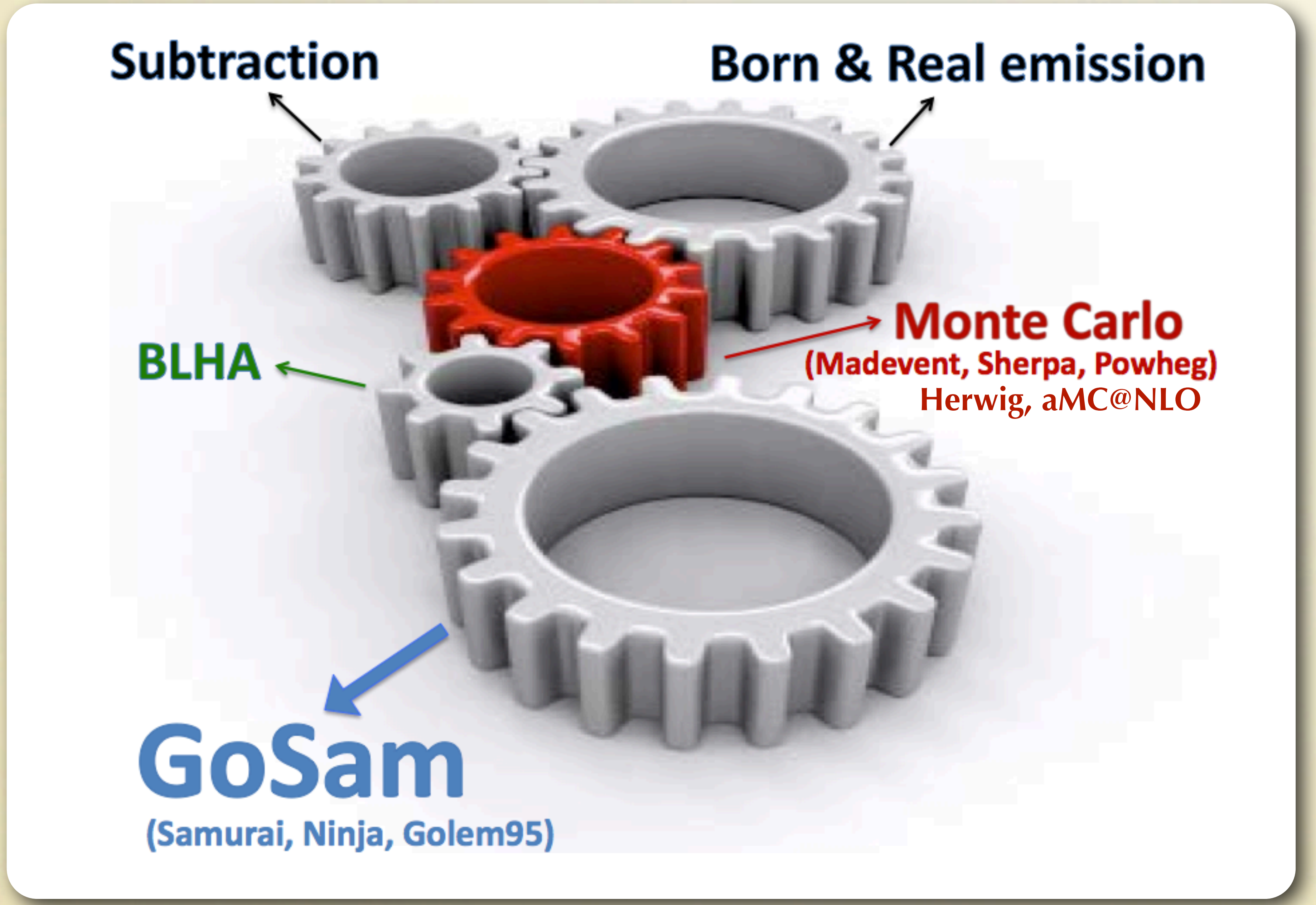
$$\sigma_{\text{NLO}} = \int_n \left(d\sigma_{\text{Born}} + d\sigma_{\text{Virtual}} + \int_1 d\sigma_{\text{Subtraction}} \right) + \int_{n+1} \left(d\sigma_{\text{Real}} - d\sigma_{\text{Subtraction}} \right)$$

Monte Carlo Generator

Multi Process One-Loop Provider

The GoSam Project

Cullen van Deurzen Greiner Heinrich Luisoni
Mirabella Ossola Peraro Reichel Schlenk
von Soden-Fraunhofen Tramontano ***P.M.***



MC Interfaces

Beyond SM

EW Physics

Top Physics

Diphoton and jets

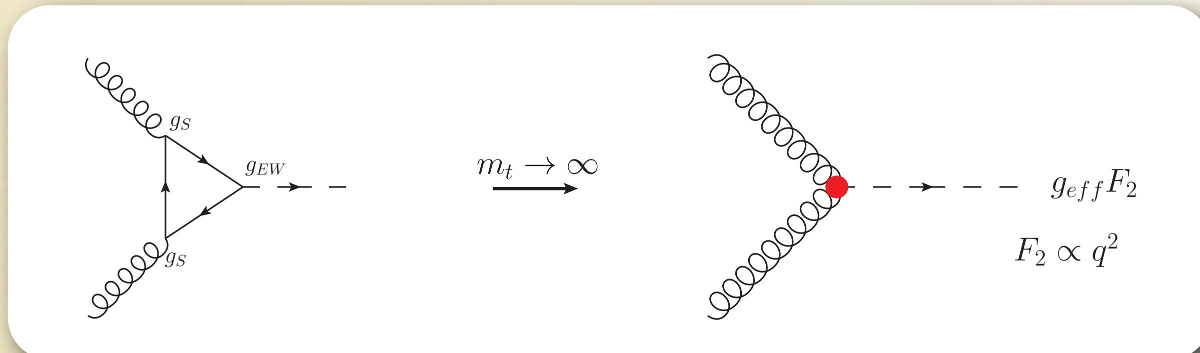
Higgs & Jets

Higgs & Jazz ?

The path to Hjjj @ NLO

Challenges

- *effective Hgg-coupling*:



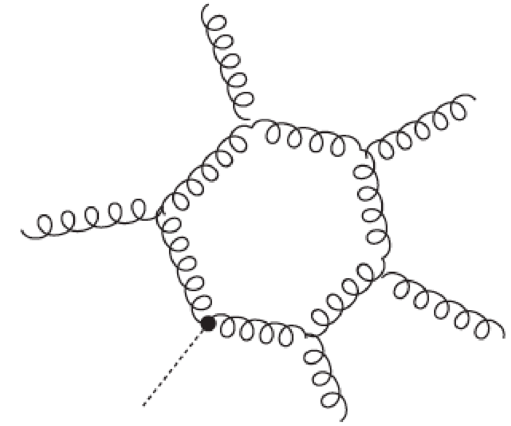
higher rank :: r < n+2

the rank *r* of the numerator can be larger than the number *n* of denominators

☑ Extending the Polynomial Residues

Mirabella Peraro ***P.M.***

Samurai > XSamurai van Deurzen

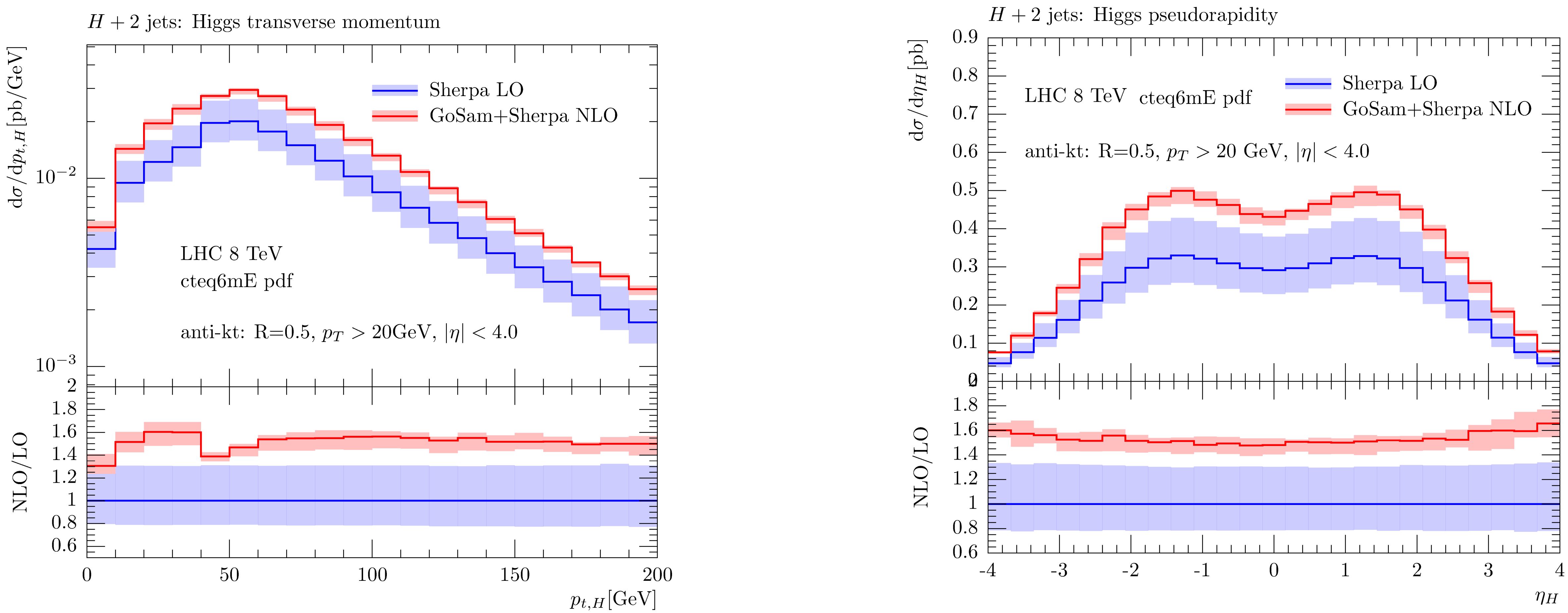


H+0j	**1 NLO**
$gg \to H$	1 NLO
H+1j	**62 NLO**
$qq \to Hqq$	14 NLO
$qg \to Hqg$	48 NLO
H+2j	**926 NLO**
$qq' \to Hqq'$	32 NLO
$qq \to Hqq$	64 NLO
$qg \to Hqg$	179 NLO
$gg \to Hgg$	651 NLO
H+3j	**13179 NLO**
$qq' \to Hqq'g$	467 NLO
$qq \to Hqqg$	868 NLO
$qg \to Hqgg$	2519 NLO
$gg \to Hggg$	9325 NLO

- Over 10,000 diagrams
- Higher-Rank terms
- 60 Rank-7 hexagons

pp-->Hjj @ NLO

v. Deurzen Greiner Luisoni Mirabella Ossola
Peraro v. Soden-Fraunhofen Tramontano ***P.M.***
Phys.Lett. B721 (2013) 74-81, 1301.0493 [hep-ph]



$$\mu = \mu_R = \mu_F = \hat{H}_t \qquad \hat{H}_t = \sqrt{M_H^2 + p_{t,H}^2} + \sum_j p_{t,j}\,,$$

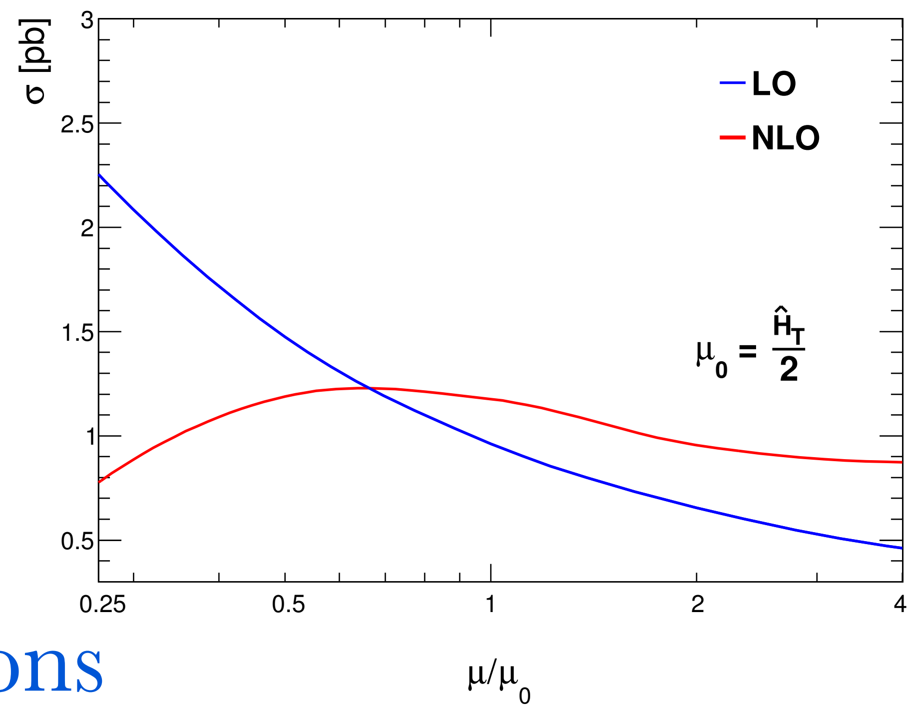
- our amplitudes confirmed by MCFM (v6.4)

Campbell, Ellis, Williams

pp-->Hjjj @ NLO

Cullen v. Deurzen Greiner Luisoni
Mirabella Ossola Peraro Tramontano ***P.M.***
1307.4737 to appear in PRL

xsection

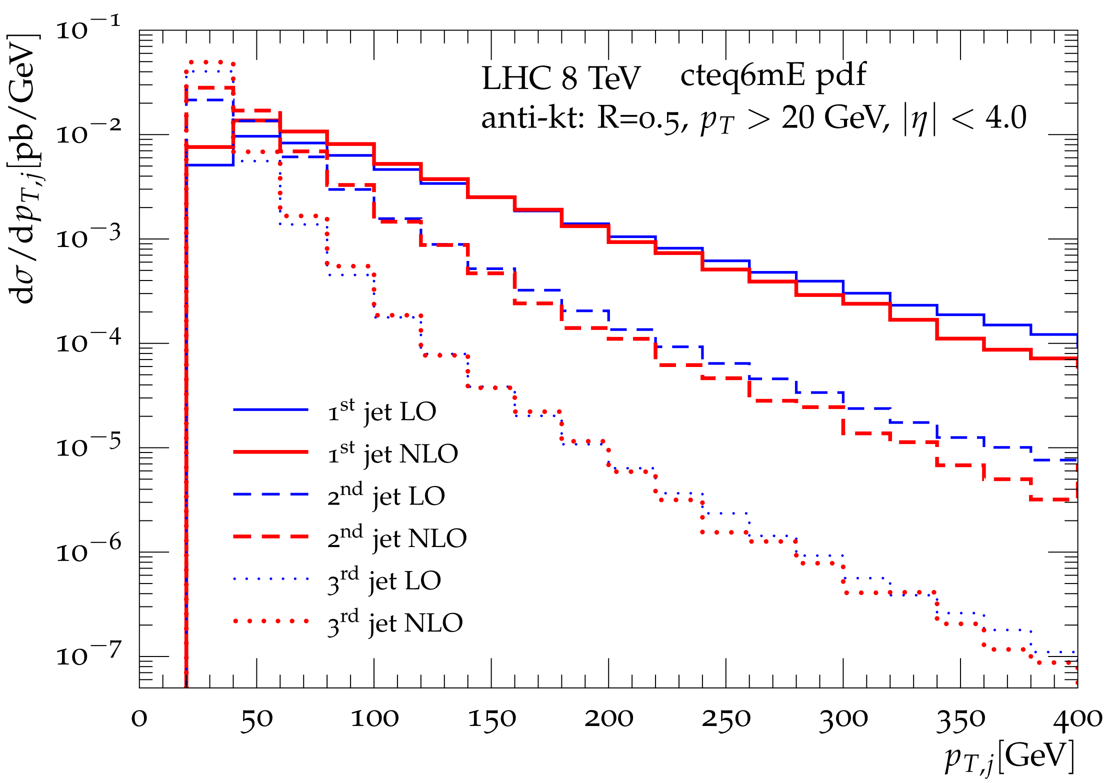


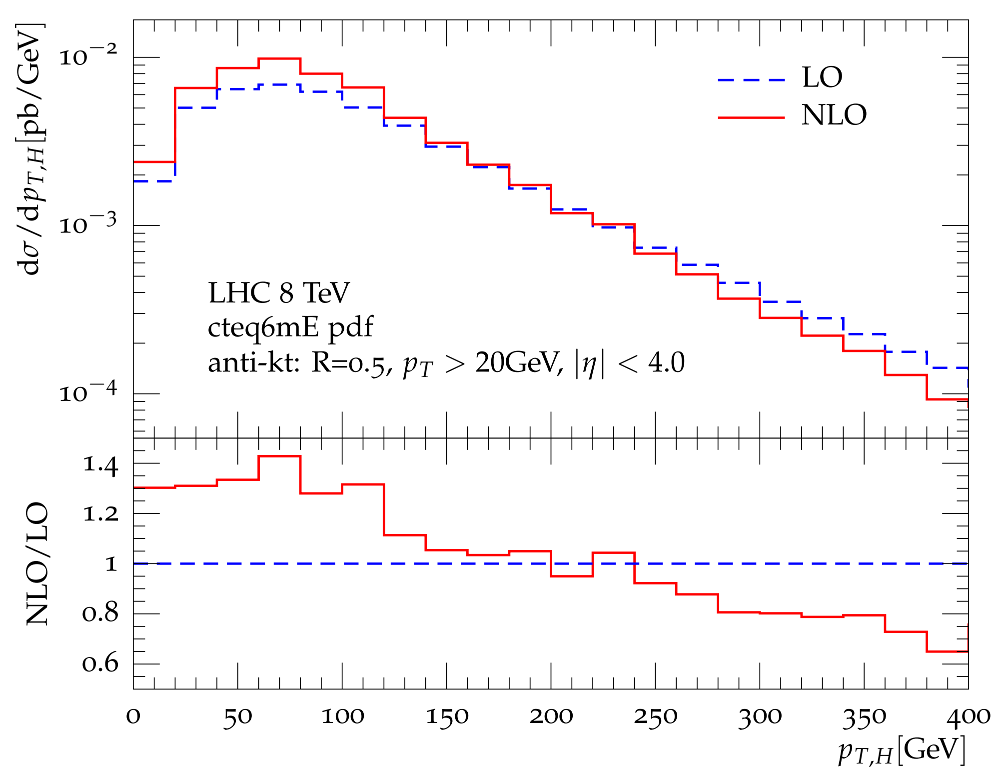
GoSam+Sherpa+MadDipole

$$\mu_F = \mu_R = \frac{\hat{H}_T}{2} = \mu_0$$

$$\hat{H}_T = \sqrt{m_H^2 + p_{T,H}^2} + \sum_i |p_{T,i}|$$

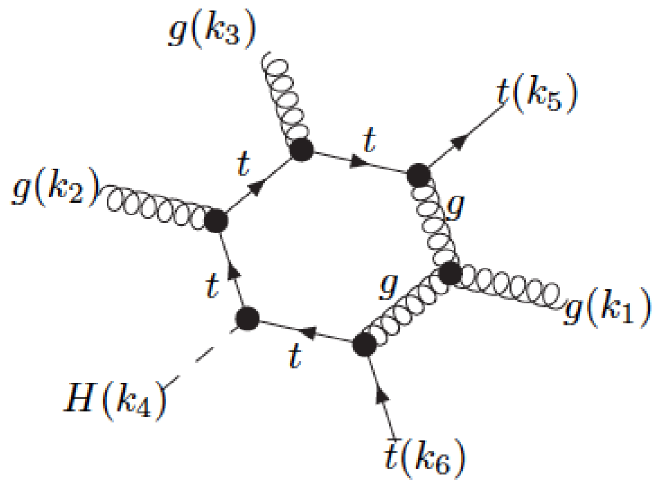
distributions





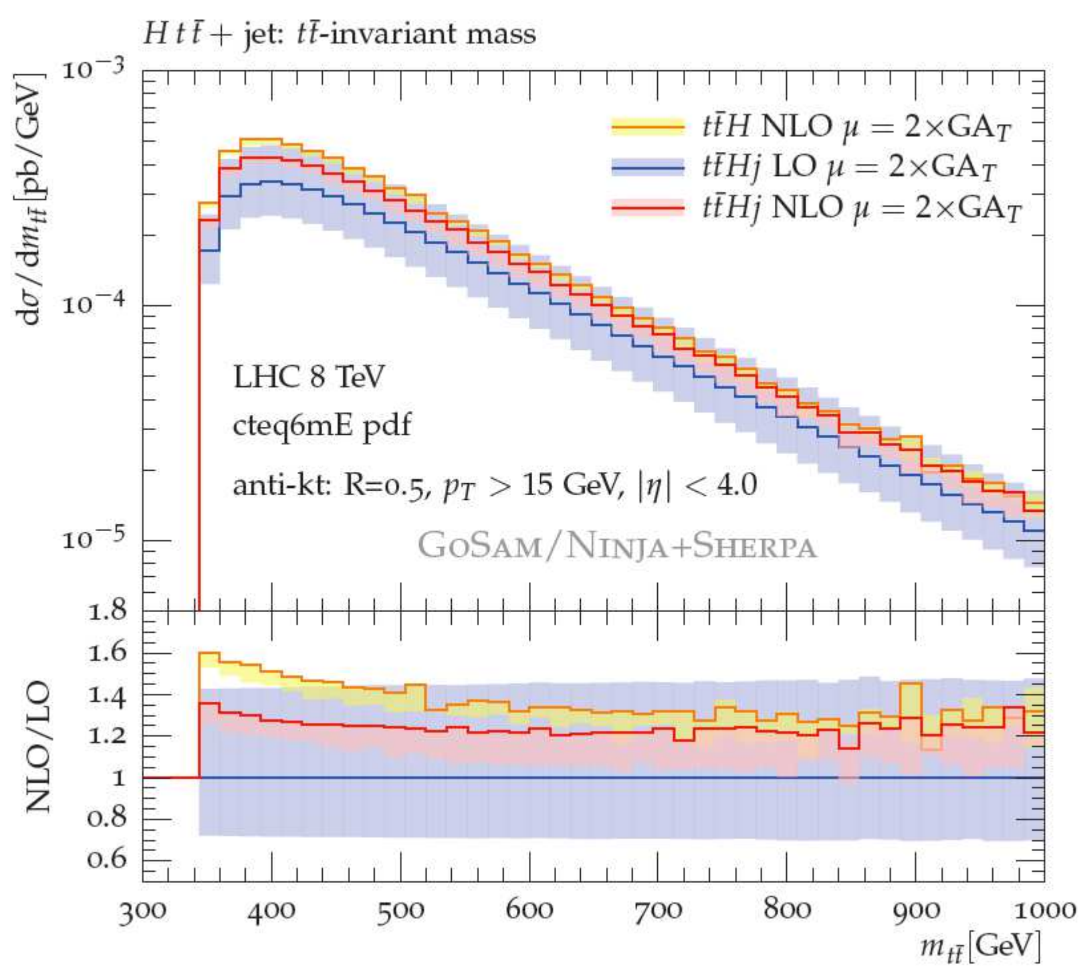
pp-->HtTj @ NLO

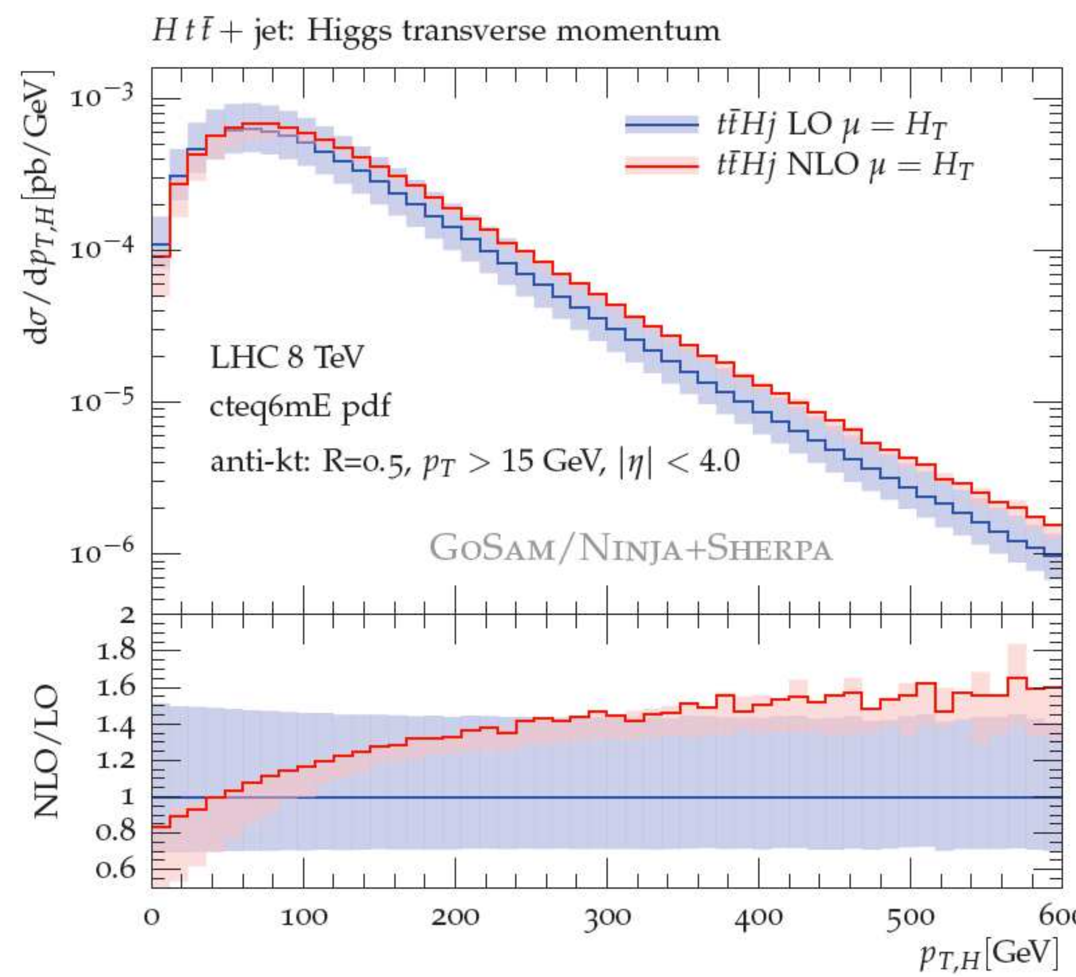
van Deurzen Luisoni Mirabella Ossola Peraro **P.M.**
1307.8437 to appear in PRL



GoSam+Ninja+Sherpa

$t\bar{t}H + 1j$	**1895 NLO**	**Time/psp**
$qq \to Ht\bar{t}g$	320 NLO	80 ms
$gg \to Ht\bar{t}g$	1575 NLO	1685 ms





$$\mathrm{GA}_T = \sqrt[3]{m_{T,H}\, m_{T,t}\, m_{T,\bar{t}}} + \sum_{\text{jets } j} |p_{T,j}|$$

$$H_T = \sum_{\substack{\text{final}\\ \text{states } f}} |p_{T,f}|$$

GoSam + Ninja: more app's

van Deurzen Luisoni Mirabella Ossola Peraro ***P.M***. (2013)

Subprocess	Time/PS-point [ms]
$\mathbf{pp \to Wjjj}$	
$d\bar{u} \to \bar{\nu}_e e^- ggg$	226
$\mathbf{pp \to Zjjj}$	
$d\bar{d} \to e^+ e^- ggg$	1911.4
$\mathbf{pp \to t\bar{t}b\bar{b}} \quad (\mathbf{m_b \neq 0})$	
$d\bar{d} \to t\bar{t}b\bar{b}$	178
$gg \to t\bar{t}b\bar{b}$	5685
$\mathbf{pp \to Wb\bar{b}j} \quad (\mathbf{m_b \neq 0})$	
$u\bar{d} \to e^+ \nu_e b\bar{b}g$	67
$\mathbf{pp \to Hjjj} \quad (\mathbf{GF, m_t \to \infty})$	
$gg \to Hggg$	11266
$gg \to Hgu\bar{u}$	999
$u\bar{u} \to Hgu\bar{u}$	157
$u\bar{u} \to Hgd\bar{d}$	68
$\mathbf{pp \to Hjjj} \quad (\mathbf{VBF})$	
$u\bar{u} \to Hgu\bar{u}$	101
$\mathbf{pp \to Hjjjj} \quad (\mathbf{VBF})$	
$u\bar{u} \to Hggu\bar{u}$	669
$u\bar{u} \to Hu\bar{u}u\bar{u}$	600

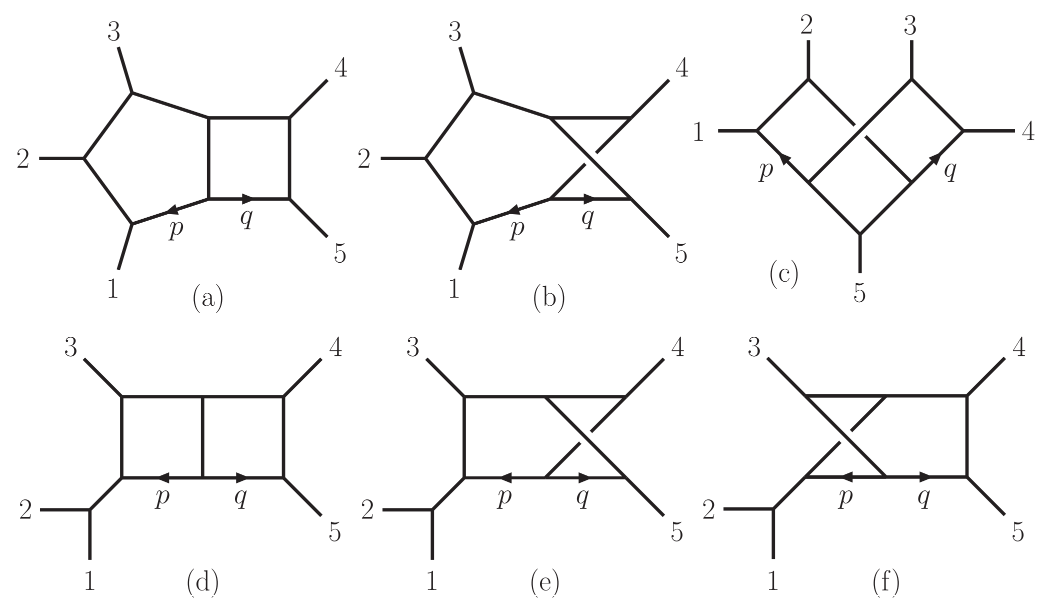
Intel i7 960 (3.20GHz) CPU + Intel fortran compiler ifort (with optimization O2).

faster,
higher accuracy,
more stable,
no-problem with
multiple masses

Two-Loop Integrand-Reduction

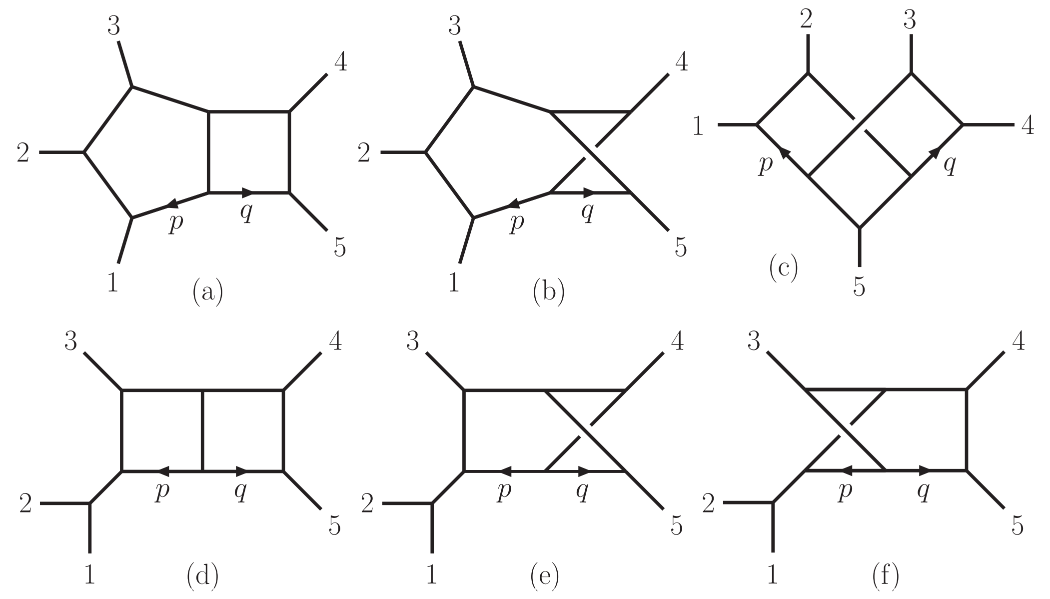
2-loop 5-point amplitudes in N=4 SYM & N=8 Sugra

Mirabella, Ossola, Peraro, & ***P.M***. (2012)

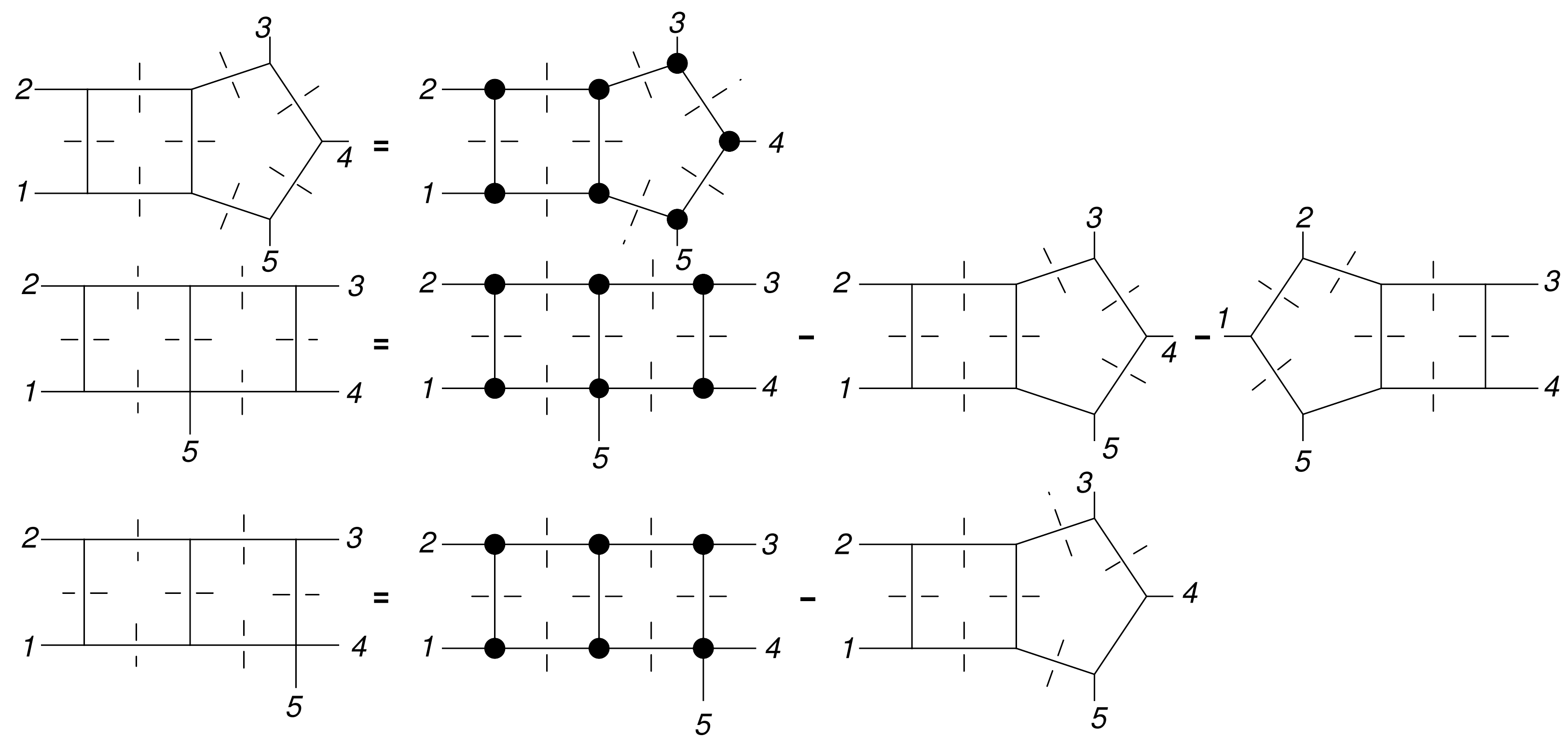


2-loop 5-point amplitudes in N=4 SYM & N=8 Sugra

Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

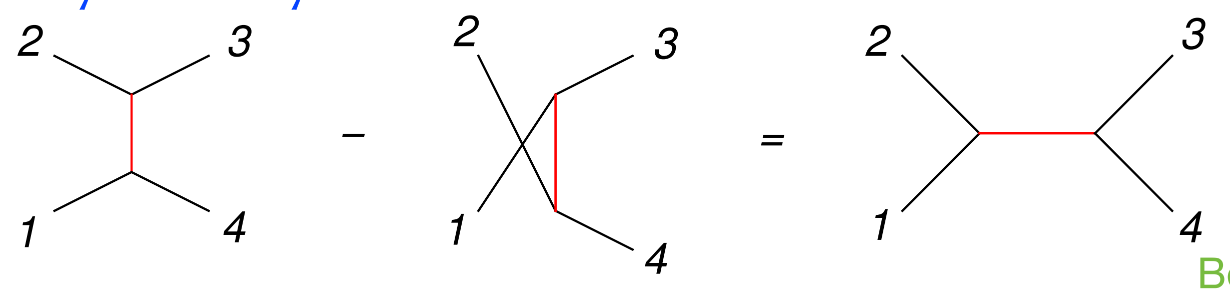


- [x] integrand-reduction



Integrand Red’n & Color-Kinematic Duality

Jacoby identity for trees



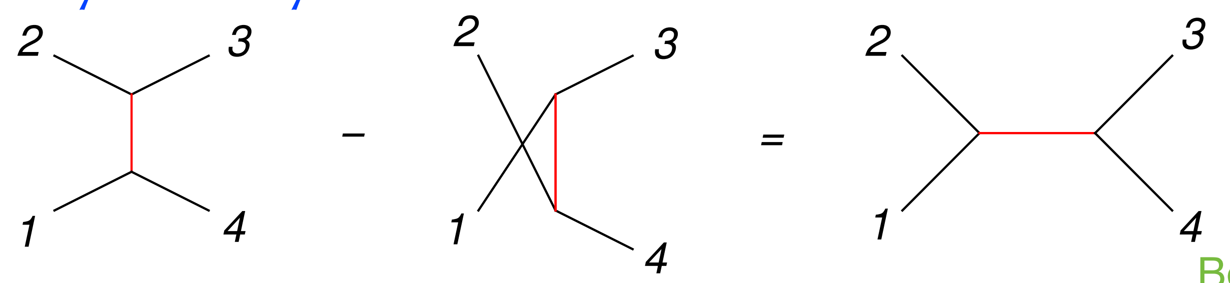
Bern Carrasco Johansson

kinematic term of scattering amplitudes fulfills the same algebra as the **color** term

Integrand Red'n & Color-Kinematic Duality

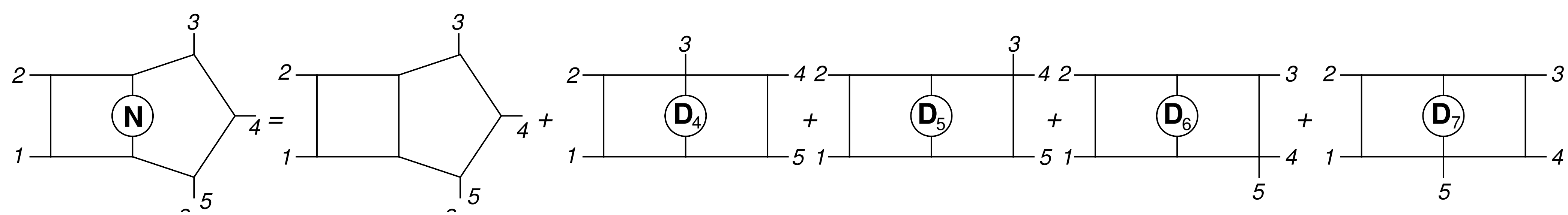
Schubert & ***P.M.*** (2013)

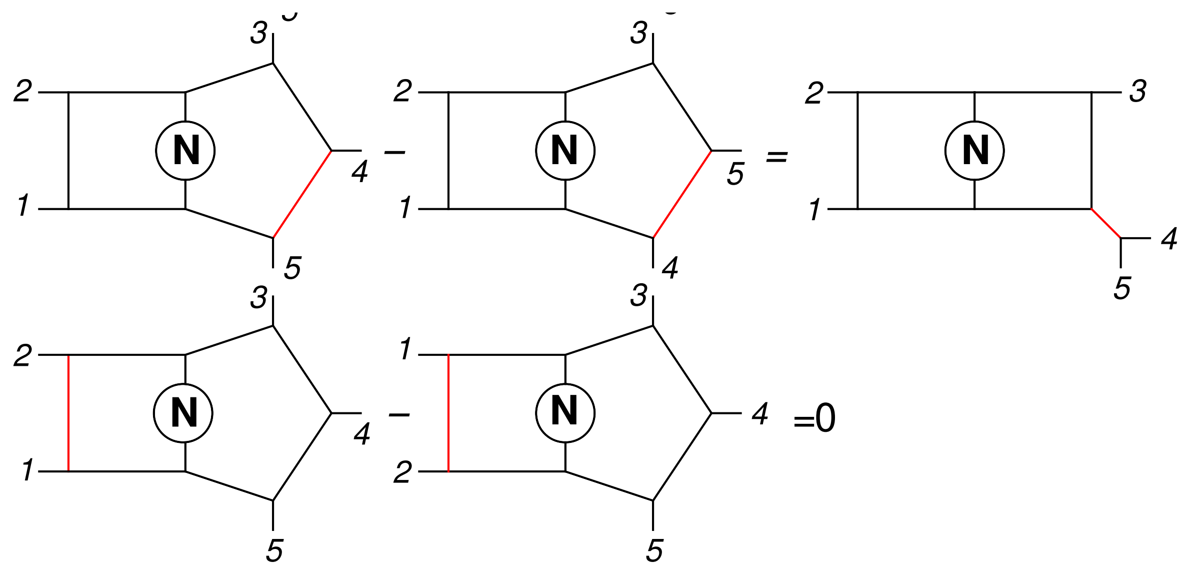
Jacoby identity for trees



Bern Carrasco Johansson

☑ integrand-reduction



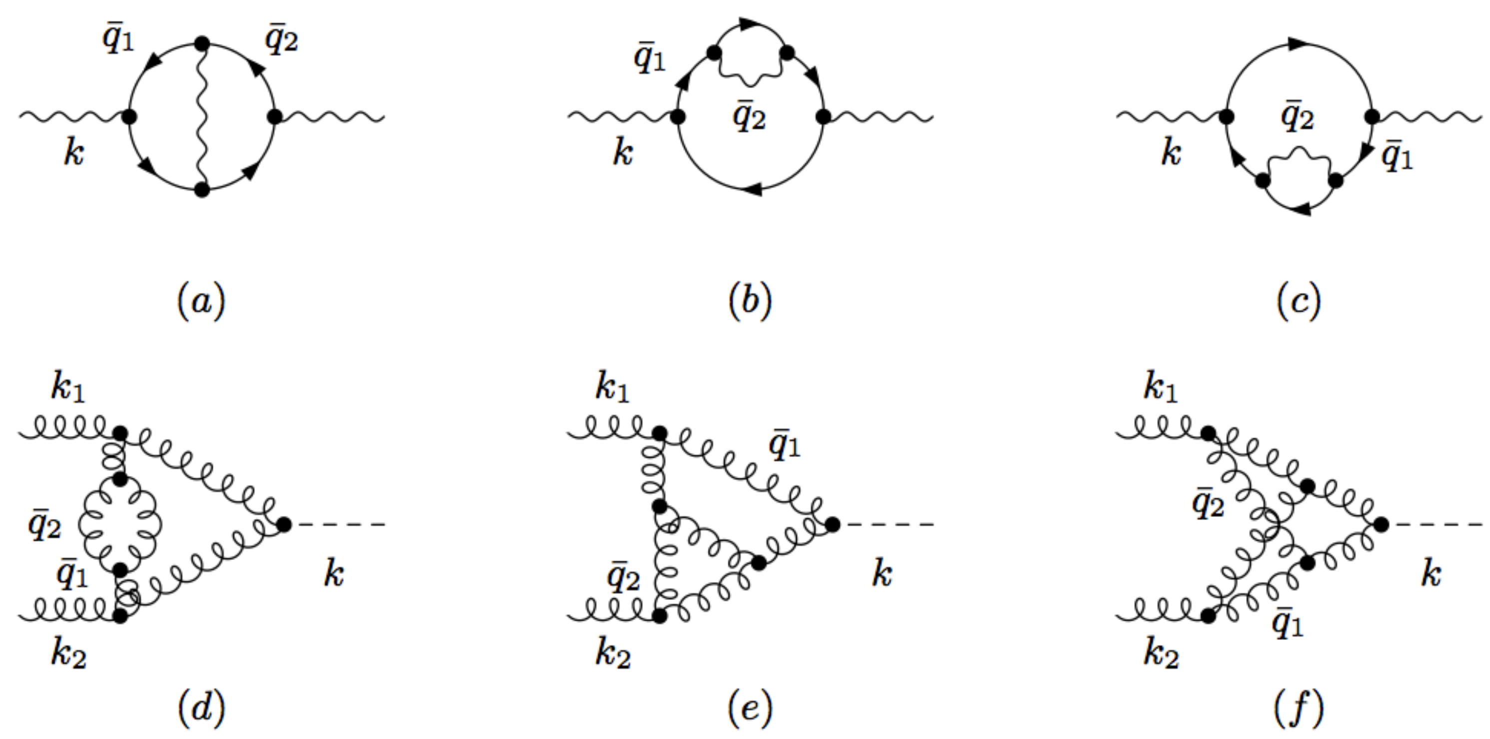


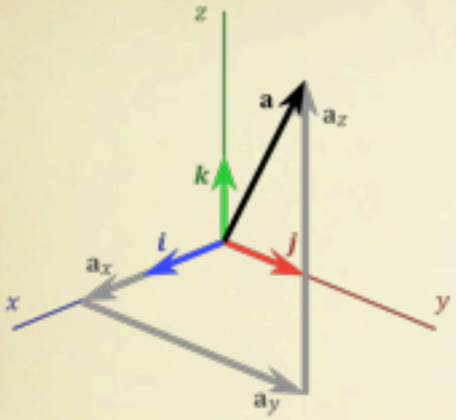
Schubert & ***P.M.***

confirming the result of Carrasco & Johansson

Integrand Red′n & Dim-reg Amplitudes

Mirabella, Ossola, Peraro, & ***P.M***. (2013)



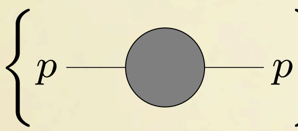


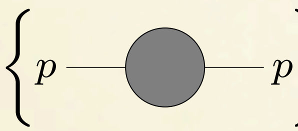
Basis :: Magnus Expansion for Feynman Integrals

$$(\exp X)(\exp Y) = \exp(X + Y + (1/2)[X,Y] + (1/12)[X,[X,Y]] - (1/12)[Y,[X,Y]] + \ldots).$$

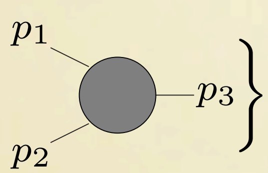
Differential Equations for Master Integrals

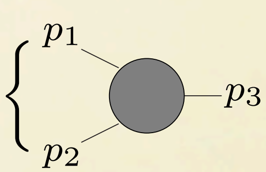
Kotikov; Remiddi;
Caffo, Czyc, Remiddi:
Gehrmann, Remiddi;
Bonciani, Remiddi, **P.M.**;
Argeri, Bonciani, Ferroglia, Remiddi, **P.M.**
Henn;
Henn, Smirnov & Smirnov

$p^2 \frac{\partial}{\partial p^2} \{$ 

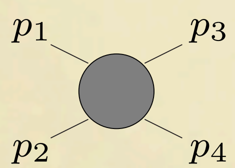
$\} = \frac{1}{2} p_\mu \frac{\partial}{\partial p_\mu} \{$ 

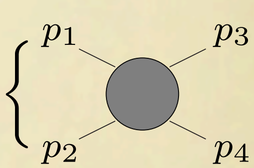
$\}$

$P^2 \frac{\partial}{\partial P^2} \{$ 

$\} = \left[A \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + B \left(p_{1,\mu} \frac{\partial}{\partial p_{2,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{1,\mu}} \right) \right] \{$ 

$\}$

$P^2 \frac{\partial}{\partial P^2} \{$ 

$\} = \left[C \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{3,\mu} \frac{\partial}{\partial p_{3,\mu}} \right) + D p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + E (p_{1,\mu} + p_{3,\mu}) \left(\frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} \right) \right] \{$ 

$\}$;

$$P = p_1 + p_2,$$

Magnus Expansion

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

System of 1st ODE

$$\partial_x Y(x) = A(x) Y(x) \ , \quad Y(x_0) = Y_0 \ .$$

A(x) non-commutative

solution: Matrix Exponential & Iterated Integrals

$$Y(x) = e^{\Omega(x,x_0)} \, Y(x_0) \equiv e^{\Omega(x)} \, Y_0 \, ,$$

$$\Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x) \ .$$

$$\Omega_1(x) = \int_{x_0}^{x} d\tau_1 A(\tau_1) \ ,$$

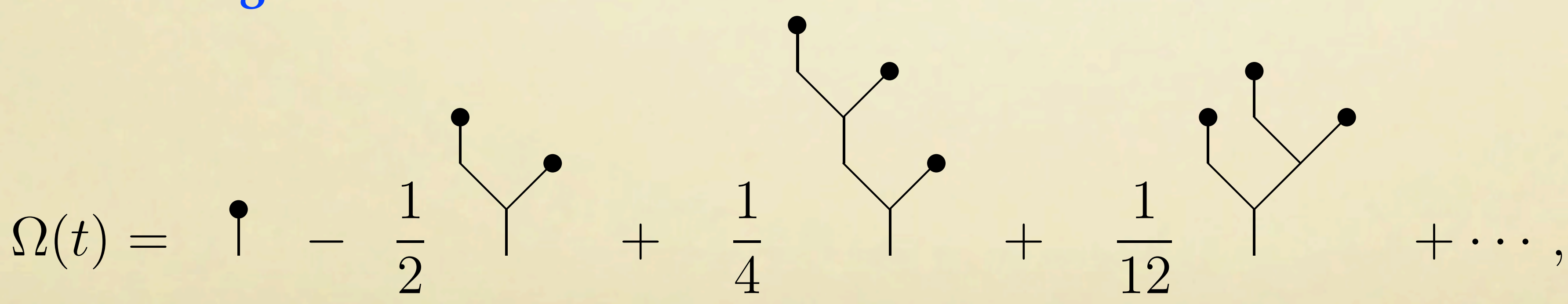
$$\Omega_2(x) = \frac{1}{2} \int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \, [A(\tau_1), A(\tau_2)] \ ,$$

$$\Omega_3(x) = \frac{1}{6} \int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 \, [A(\tau_1), [A(\tau_2), A(\tau_3)]] + [A(\tau_3), [A(\tau_2), A(\tau_1)]] \ .$$

....

BHC-formula

Iterated integrals and rooted trees



Magnus & Dyson Series

Magnus

$$Y(x) = e^{\Omega(x,x_0)}\, Y(x_0) \equiv e^{\Omega(x)}\, Y_0 \,,$$

Dyson

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x) \;, \qquad Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \ldots \int_{x_0}^{\tau_{n-1}} d\tau_n \; A(\tau_1)A(\tau_2)\cdots A(\tau_n)$$

$$\sum_{j=1}^{\infty} \Omega_j(x) = \log\left(Y_0 + \sum_{n=1}^{\infty} Y_n(x) \right)$$

$$\begin{aligned}
Y_1 &= \Omega_1 \;, \\
Y_2 &= \Omega_2 + \frac{1}{2!}\Omega_1^2 \;, \\
Y_3 &= \Omega_3 + \frac{1}{2!}(\Omega_1\Omega_2 + \Omega_2\Omega_1) + \frac{1}{3!}\Omega_1^3 \;, \\
&\;\vdots \qquad\quad \vdots \\
Y_n &= \Omega_n + \sum_{j=2}^{n} \frac{1}{j} Q_n^{(j)} \,.
\end{aligned}$$

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

- Linear-**eps** Matrix

$$\partial_x f(\epsilon, x) = A(\epsilon, x)\ f(\epsilon, x)\ , \qquad A(\epsilon, x) = A_0(x) + \epsilon A_1(x)\ ,$$

- change of basis :: Magnus #1

$$f(\epsilon, x) = B_0(x)\ g(\epsilon, x)\ , \qquad B_0(x) \equiv e^{\Omega[A_0](x, x_0)}\ . \qquad \partial_x B_0(x) = A_0(x) B_0(x)\ ,$$

- *Canonical form* Henn (2013)

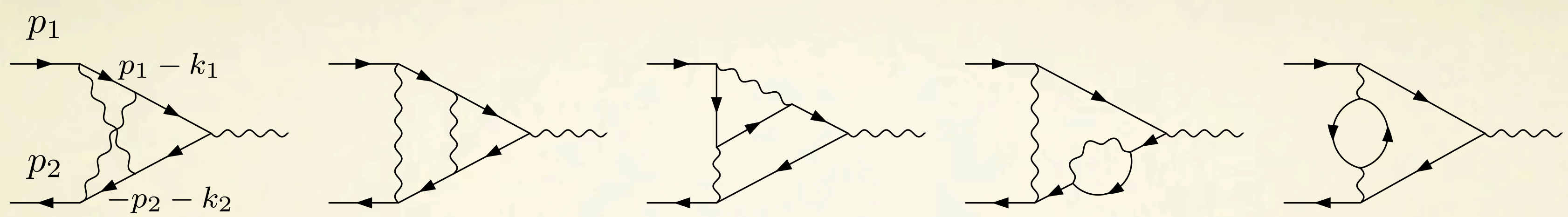
$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \qquad \hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x)\ .$$

- Solution :: Magnus #2 (or Dyson)

$$g(\epsilon, x) = B_1(\epsilon, x) g_0(\epsilon)\ , \qquad B_1(\epsilon, x) = e^{\Omega[\epsilon \hat{A}_1](x, x_0)}$$

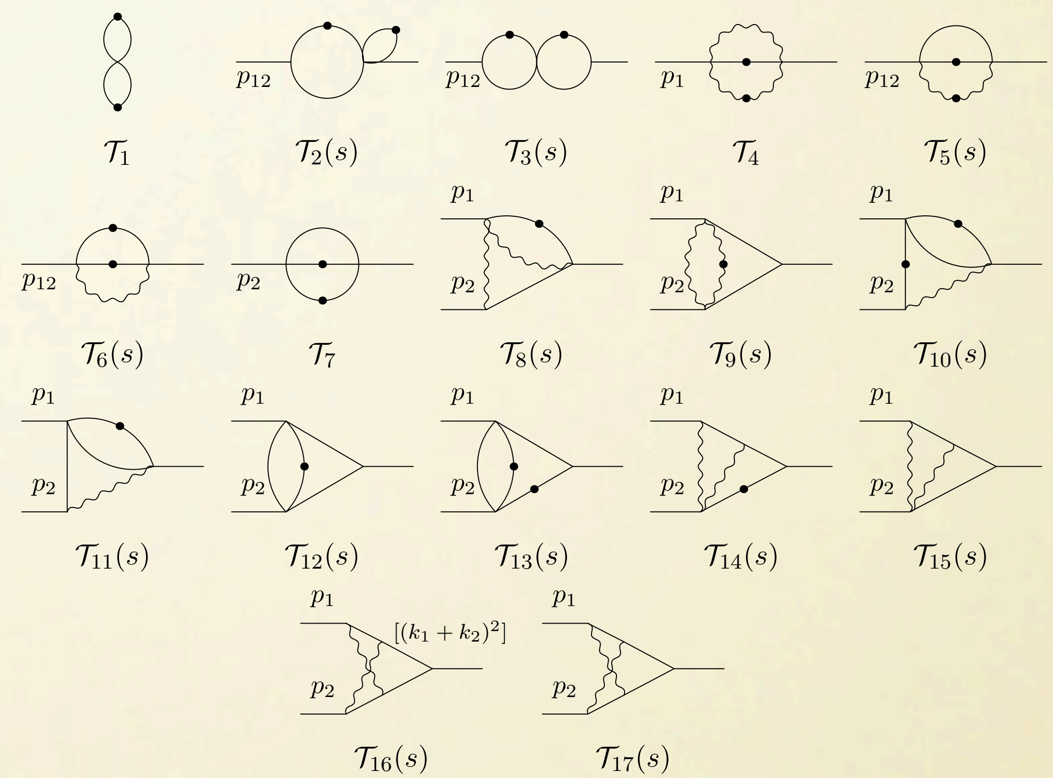
Uniform Transcendentality!

Feynman integrals can be determined from differential equations that looks like gauge transformations



- intial set of MI's

$$
\begin{aligned}
&f_1 = \epsilon^2 \mathcal{T}_1 , && f_2 = \epsilon^2 \mathcal{T}_2 , && f_3 = \epsilon^2 \mathcal{T}_3 , && f_4 = \epsilon^2 \mathcal{T}_4 , && f_5 = \epsilon^2 \mathcal{T}_5 , \\
&f_6 = \epsilon^2 \mathcal{T}_6 , && f_7 = \epsilon^2 \mathcal{T}_7 , && f_8 = \epsilon^3 \mathcal{T}_8 , && f_9 = \epsilon^3 \mathcal{T}_9 , && f_{10} = \epsilon^2 \mathcal{T}_{10} , \\
&f_{11} = \epsilon^3 \mathcal{T}_{11} , && f_{12} = \epsilon^3 \mathcal{T}_{12} , && f_{13} = \epsilon^2 \mathcal{T}_{13} , && f_{14} = \epsilon^3 \mathcal{T}_{14} , && f_{15} = \epsilon^4 \mathcal{T}_{15} , \\
&f_{16} = \epsilon^4 \mathcal{T}_{16} , && f_{17} = \epsilon^4 \mathcal{T}_{17} ,
\end{aligned}
$$



- after *rotation*

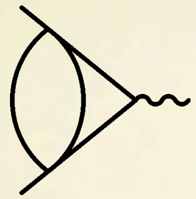
$$
\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \; , \qquad \hat{A}_1(x) = \frac{M_1}{x} + \frac{M_2}{1+x} + \frac{M_3}{1-x} ,
$$

$$
M_1 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
-1&0&0&0&5&-6&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&2&-2&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
-1&0&0&-4&0&-2&0&-2&0&0&0&0&0&0&0&0&0 \\
0&0&0&-2&0&0&0&0&2&0&0&0&0&0&0&0&0 \\
-\frac{1}{2}&0&0&0&1&-2&-3&0&0&3&3&0&0&0&0&0&0 \\
0&0&0&0&1&-1&2&0&0&-2&-2&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&-1&-1&0&0&0&0 \\
0&-1&0&0&0&0&-3&0&0&0&0&3&3&0&0&0&0 \\
0&-1&0&0&1&-\frac{1}{2}&0&2&2&0&0&0&0&2&2&0&0 \\
0&0&0&0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0&0&0&0&-1&-1&0&0 \\
-\frac{1}{2}&0&0&-2&-1&0&-2&1&0&2&0&-2&0&0&-2&-2&2 \\
0&0&0&0&-1&\frac{1}{2}&0&3&-2&0&-6&-2&0&0&-4&-4&4
\end{pmatrix}
$$

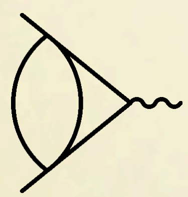
$$
M_2 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&-4&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&-6&3&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&-4&0&0&0&0&0&0&0&0 \\
0&0&0&0&-1&\frac{1}{2}&0&0&0&-4&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0 \\
0&0&0&0&0&0&-6&0&0&0&0&0&-2&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&-4&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-4
\end{pmatrix}
$$

$$
M_3 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&2&-2&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&-6&0&0&2&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&-12&0&0&0&0&0&4&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&2&0&0&0&-4&0&0&-4&-2&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4
\end{pmatrix}
$$

Bonciani, Remiddi, **P.M.** (2013)

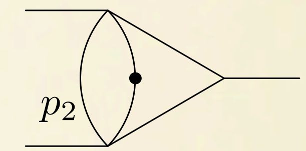


$$
\begin{aligned}
M_{-2} &= \frac{1}{2}\,, \\
M_{-1} &= \frac{5}{2} - \left[1 - \frac{2}{(1-x)}\right] H(0,x)\,, \\
M_0 &= \frac{19}{2} + \zeta(2) + \left[1 - \frac{2}{(1-x)}\right]\left[\zeta(2) - 5H(0,x) + 2H(-1,0,x)\right] \\
&\quad + \frac{2}{(1-x)} H(0,0,x) + \left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\left[\zeta(2) H(0,x)\right. \\
&\quad \left. + H(0,0,0,x)\right].
\end{aligned}
$$

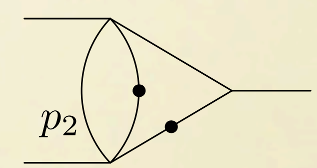
 $(p_2 \cdot k_1)$

$$
\begin{aligned}
\frac{N_{-2}}{a} &= \frac{1}{8} + \frac{1}{16}\left[x + \frac{1}{x}\right], \\
\frac{N_{-1}}{a} &= \frac{9}{32}\left[2 + x + \frac{1}{x}\right] - \frac{1}{8}\left[4 + x - \frac{1}{x}\right] H(0,x) + \frac{1}{(1-x)} H(0,x)\,, \\
\frac{N_0}{a} &= \frac{63}{32} + \frac{\zeta(2)}{2} + \frac{63}{64}\left[\left(1 + \frac{16}{63}\zeta(2)\right) x + \frac{1}{x}\right] - \frac{\zeta(2)}{(1-x)} - \frac{1}{16}\left[32 + 9x \right. \\
&\quad \left. - \frac{9}{x}\right] H(0,x) + \frac{(16 + \zeta(2))}{4(1-x)} H(0,x) - \frac{\zeta(2)}{4(1+x)} H(0,x) - \frac{1}{4}\left[2 - \frac{1}{x} \right. \\
&\quad \left. \frac{4}{(1-x)}\right] H(0,0,x) + \frac{1}{4}\left[4 + x - \frac{1}{x} - \frac{8}{(1-x)}\right] H(-1,0,x) \\
&\quad + \frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0,0,0,x)\,.
\end{aligned}
$$

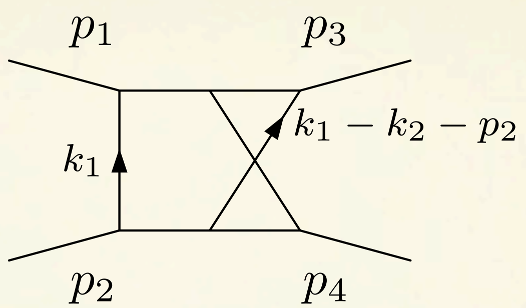
AdVMMSST (2014)



$$
\begin{aligned}
g_{12}^{(0)} &= 0\,, \\
g_{12}^{(1)} &= 0\,, \\
g_{12}^{(2)} &= 0\,, \\
g_{12}^{(3)} &= -\,\mathrm{H}(0,0,0;x) - \zeta_2 \mathrm{H}(0;x)\,, \\
g_{12}^{(4)} &= -\,2\,\mathrm{H}(-1,0,0,0;x) + 2\,\mathrm{H}(0,-1,0,0;x) + 2\,\mathrm{H}(0,0,-1,0;x) \\
&\quad - 3\,\mathrm{H}(0,0,0,0;x) - 4\,\mathrm{H}(0,1,0,0;x) + \zeta_2(-2\,\mathrm{H}(-1,0;x) \\
&\quad + 6\,\mathrm{H}(0,-1;x) - \mathrm{H}(0,0;x)) + 2\,\zeta_3\,\mathrm{H}(0;x) + \frac{\zeta_4}{4}\,,
\end{aligned}
$$

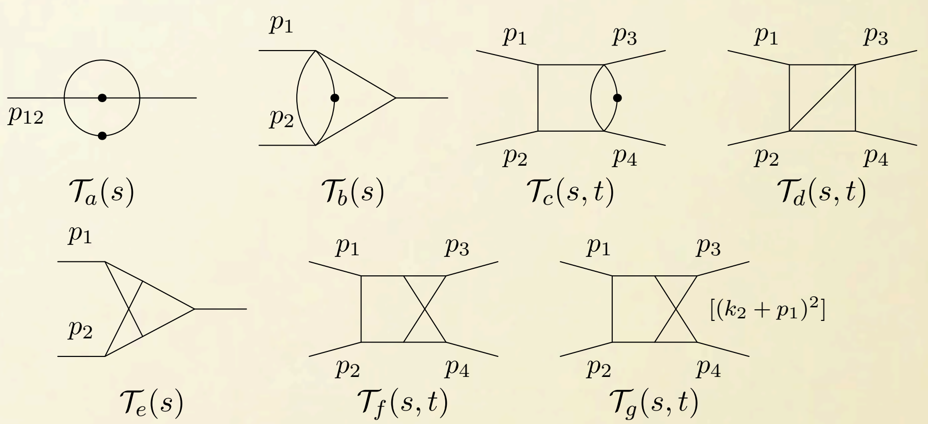


$$
\begin{aligned}
g_{13}^{(0)} &= 0\,, \\
g_{13}^{(1)} &= 0\,, \\
g_{13}^{(2)} &= \mathrm{H}(0,0;x) + \frac{3\,\zeta_2}{2}\,, \\
g_{13}^{(3)} &= -\,2\,\mathrm{H}(-1,0,0;x) - 2\,\mathrm{H}(0,-1,0;x) + 4\,\mathrm{H}(0,0,0;x) + 4\,\mathrm{H}(1,0,0;x) \\
&\quad + \zeta_2(-6\,\mathrm{H}(-1;x) + 2\,\mathrm{H}(0;x) - 3\log 2) - \frac{\zeta_3}{4}\,, \\
g_{13}^{(4)} &= 4\,\mathrm{H}(-1,-1,0,0;x) + 4\,\mathrm{H}(-1,0,-1,0;x) - 8\,\mathrm{H}(-1,0,0,0;x) \\
&\quad - 8\,\mathrm{H}(-1,1,0,0;x) + 4\,\mathrm{H}(0,-1,-1,0;x) - 8\,\mathrm{H}(0,-1,0,0;x) \\
&\quad - 8\,\mathrm{H}(0,0,-1,0;x) + 10\,\mathrm{H}(0,0,0,0;x) + 12\,\mathrm{H}(0,1,0,0;x) \\
&\quad - 8\,\mathrm{H}(1,-1,0,0;x) - 8\,\mathrm{H}(1,0,-1,0;x) + 16\,\mathrm{H}(1,0,0,0;x) \\
&\quad + 16\,\mathrm{H}(1,1,0,0;x) + 12\,\mathrm{Li}_4\frac{1}{2} + \frac{\log^4 2}{2} + 2\,\zeta_2\,(12\log 2\,\mathrm{H}(-1;x) \\
&\quad + 12\log 2\,\mathrm{H}(1;x) + 6\,\mathrm{H}(-1,-1;x) - 2\,\mathrm{H}(-1,0;x) - 8\,\mathrm{H}(0,-1;x) \\
&\quad + \mathrm{H}(0,0;x) - 12\,\mathrm{H}(1,-1;x) + 4\,\mathrm{H}(1,0;x) + 3\log^2 2) \\
&\quad - 2\,\zeta_3(5\,\mathrm{H}(-1;x) + 4\,\mathrm{H}(0;x) + 11\,\mathrm{H}(1;x)) - \frac{47\,\zeta_4}{4}\,,
\end{aligned}
$$



- intial set of MI's

$$
\begin{aligned}
f_1 &= \epsilon^2\, s\, \mathcal{T}_a(s)\,, \quad f_2 = \epsilon^2\, t\, \mathcal{T}_a(t)\,, \quad f_3 = \epsilon^2\, u\, \mathcal{T}_a(u)\,,\\
f_4 &= \epsilon^3\, s\, \mathcal{T}_b(s)\,, \quad f_5 = \epsilon^3\, s\, t\, \mathcal{T}_c(s,t)\,, \quad f_6 = \epsilon^3\, s\, u\, \mathcal{T}_c(s,u)\,,\\
f_7 &= \epsilon^4\, u\, \mathcal{T}_d(s,t)\,, \quad f_8 = \epsilon^4\, s\, \mathcal{T}_d(t,u)\,, \quad f_9 = \epsilon^4\, t\, \mathcal{T}_d(u,s)\,,\\
f_{10} &= \epsilon^4\, s^2\, \mathcal{T}_e(s)\,,\\
f_{11} &= \epsilon^4\, s\, t\, u\, \mathcal{T}_f(s,t) - \frac{3}{4\, s\, (4\epsilon+1)} \left[\epsilon^2 \left(s^2\, \mathcal{T}_a(s) + t^2\, \mathcal{T}_a(t) + u^2\, \mathcal{T}_a(u) \right) \right.\\
&\quad \left. - 4\epsilon^4 \left(u^2\, \mathcal{T}_d(s,t) + s^2\, \mathcal{T}_d(t,u) + t^2\, \mathcal{T}_d(u,s) \right) \right]\,,\\
f_{12} &= \epsilon^4\, s\, t\, \mathcal{T}_g(s,t) - \frac{3}{8\, u\, (4\epsilon+1)} \left[\epsilon^2 \left(s^2\, \mathcal{T}_a(s) + t^2\, \mathcal{T}_a(t) + u^2\, \mathcal{T}_a(u) \right) \right.\\
&\quad \left. - 4\epsilon^4 \left(u^2\, \mathcal{T}_d(s,t) + s^2\, \mathcal{T}_d(t,u) + t^2\, \mathcal{T}_d(u,s) \right) \right]\,,
\end{aligned}
$$



- after *rotation*

$$
\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x)\ , \qquad \hat{A}(x) = \frac{M_1}{x} + \frac{M_2}{1-x}\,,
$$

$$
M_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{3}{2} & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{3}{2} & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-6 & -6 & -\frac{9}{2} & 0 & -4 & -2 & -18 & -12 & -12 & 1 & 1 & -2\\
\frac{3}{4} & \frac{9}{4} & -\frac{21}{4} & 3 & 2 & -3 & 12 & -6 & -18 & 0 & 0 & -2
\end{pmatrix},
\qquad
M_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{3}{2} & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{3}{2} & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0\\
\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-6 & -6 & -\frac{9}{2} & 0 & -4 & -2 & -18 & -12 & -12 & 1 & 1 & -2\\
-\frac{21}{4} & \frac{9}{4} & -\frac{27}{4} & -6 & 2 & -4 & 12 & -6 & -24 & 1 & -1 & 0
\end{pmatrix}
$$

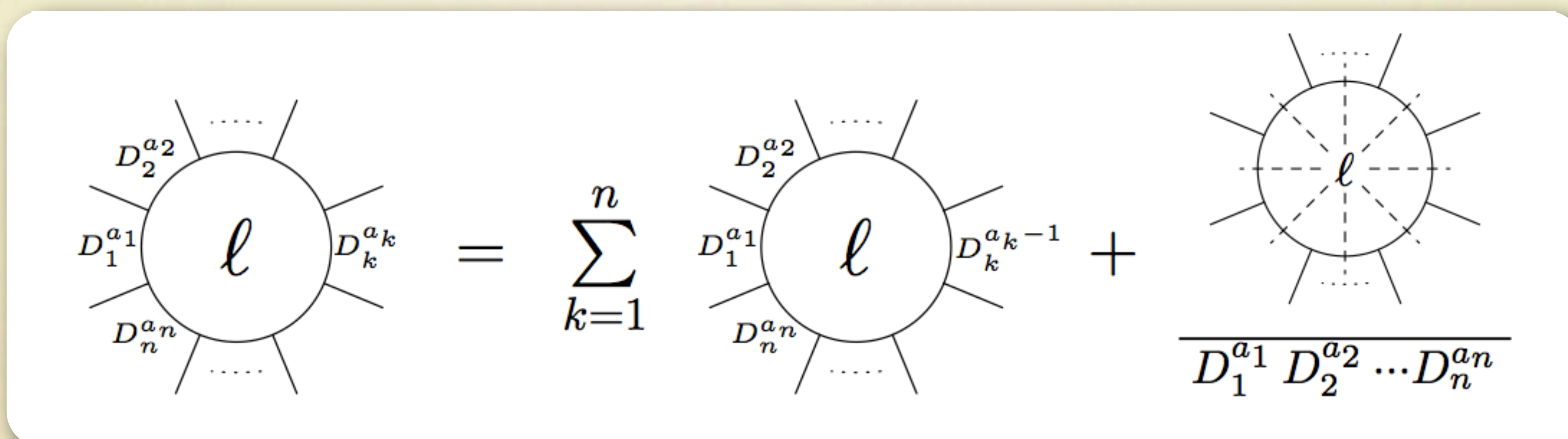
Conclusions

A new result in QFT

- A unique mathematical framework for Amplitudes at any order in Perturbation Theory
  - one ingredient: Feynman denominator
  - one operation: *partial fractioning*
- Multivariate Polynomial Division/Groebner-basis
  generates the ***residue*** at an arbitrary cut

A new computational method

- *Recursive* generation of the *Integrand-decomposition Formula* @ any loop
- Amplitude decomposition from the shape of ***residues***



Conclusions

- **A new result in QFT**
  - A unique mathematical framework for Amplitudes at any order in Perturbation Theory
    - one ingredient: Feynman denominator
    - one operation: *partial fractioning*
  - Multivariate Polynomial Division/Groebner-basis generates the ***residue*** at an arbitrary cut

- **A new computational method**
  - *Recursive* generation of the *Integrand-decomposition Formula* @ any loop
  - Amplitude decomposition from the shape of ***residues***

- Residues' ***classification*** complementary to Landau's singularity classification

ON ANALYTIC PROPERTIES OF VERTEX PARTS IN QUANTUM FIELD THEORY

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Received 27 April 1959

Abstract: A general method of finding the singularities of quantum field theory values on the basis of graph techniques is evolved.

Conclusions

- **A new result in QFT**
  - A unique mathematical framework for Amplitudes at any order in Perturbation Theory
    - one ingredient: Feynman denominator
    - one operation: *partial fractioning*
  - Multivariate Polynomial Division/Groebner-basis generates the ***residue*** at an arbitrary cut

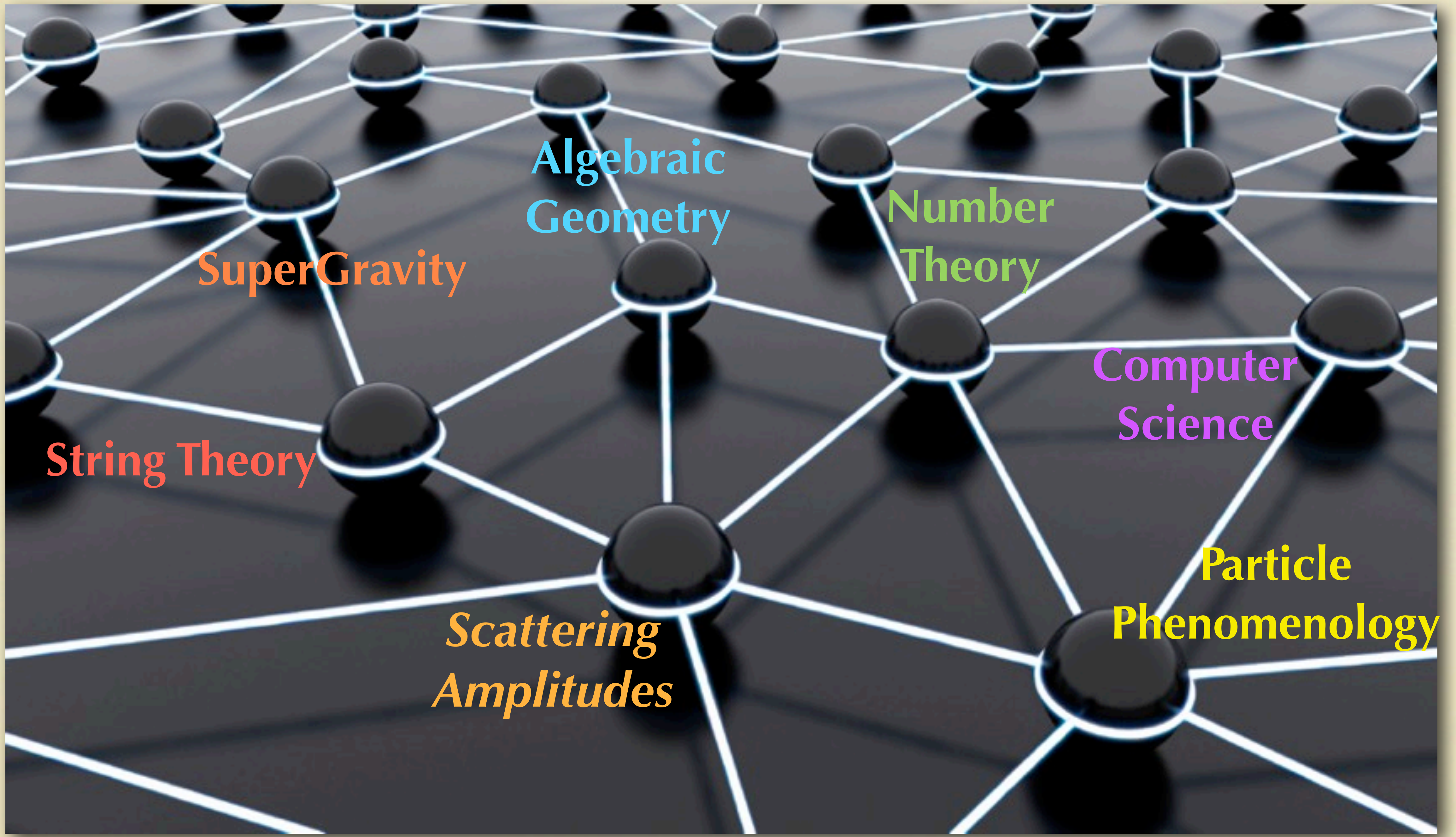
- **A new computational method**
  - *Recursive* generation of the *Integrand-decomposition Formula* @ any loop
  - Amplitude decomposition from the shape of ***residues***

- Residues' ***classification*** complementary to Landau's singularity classification

- **Pheno applications**
  - GoSam, Samurai, Ninja: multi-process automatic NLO calculations
  - main achievements:
    Higgs production in association with jets and heavy-quarks at NLO

OutLoo(k/p)

- One-Loop
  - GoSam2.0 @ LHC
    - New integrand generator (5D-unitarity)
    - EW and massive particles
  - a new horizon: Automating the integrand reduction analytically
- Beyond One-Loop
  - Combining Integrand Reduction & Integration-by-parts
  - Master Integrals from Magnus exponential
  - pp-->H+2 at NNLO
- a driving question:
  - QFT finiteness: KLN-theorem @ the integrand level



xing-path with many subjects

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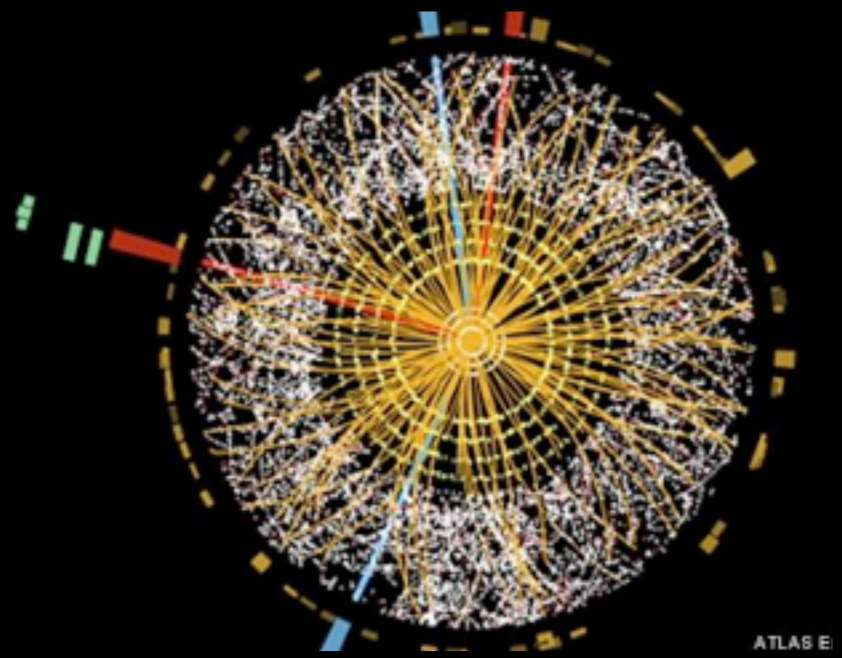
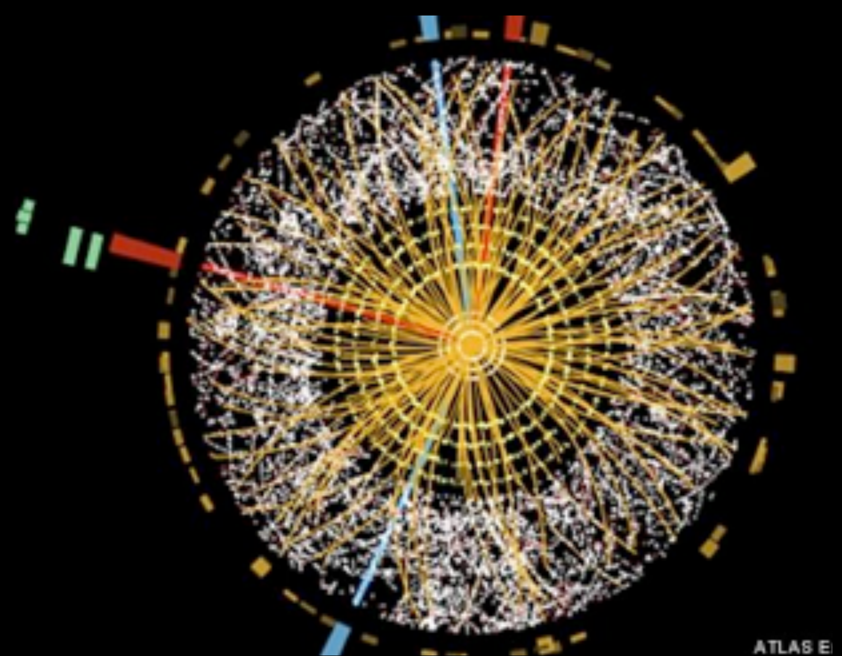
actractive for a wide spectrum of people

Scattering



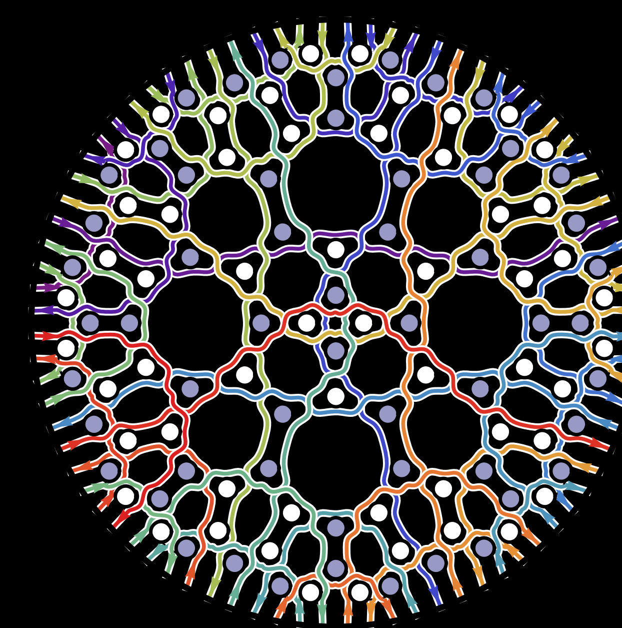
ATLAS

Scattering



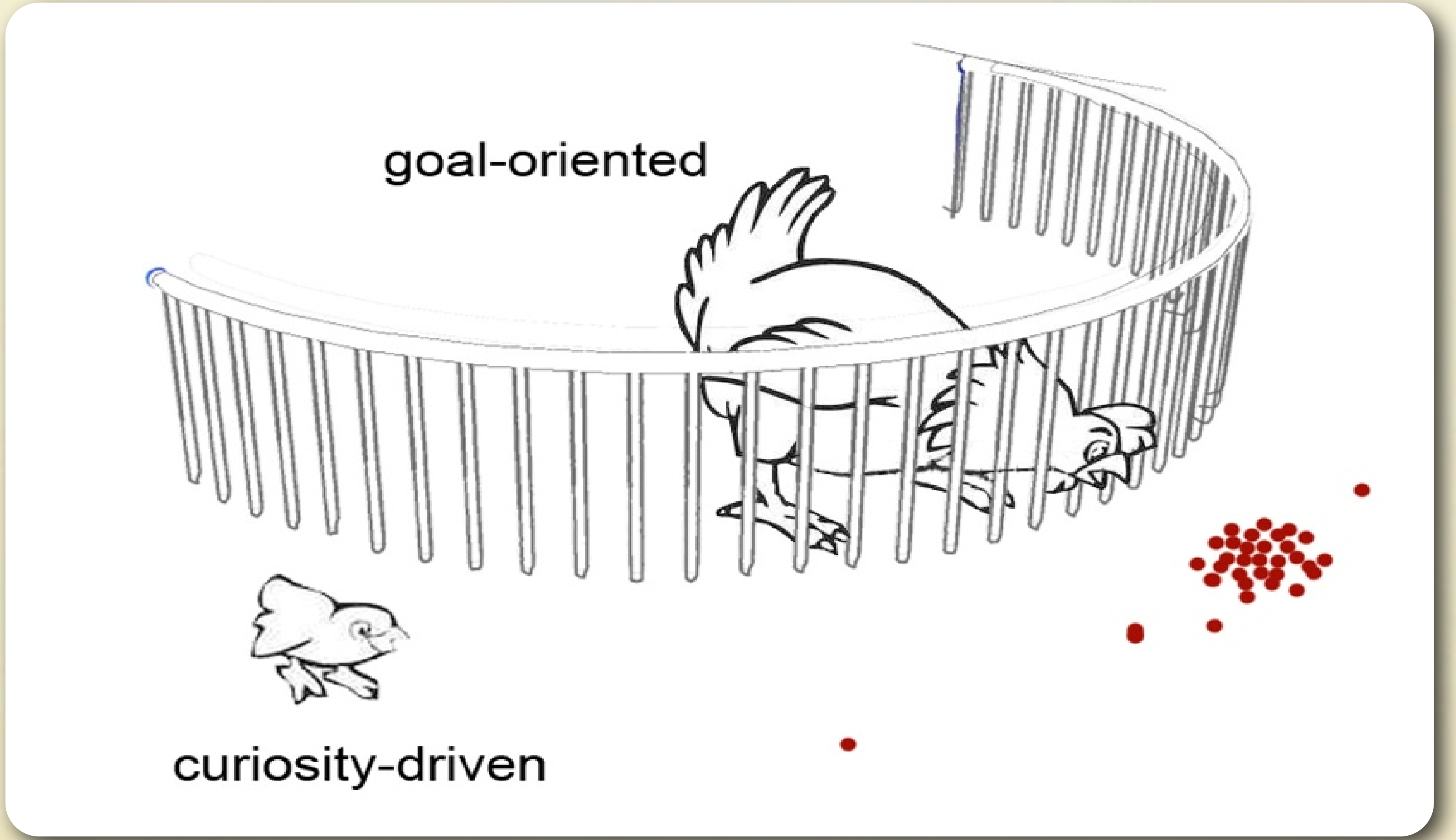
ATLAS

Scattering in $\mathcal{N}=4$ sYM



Arkani-Hamed, Bourjaily, Cachazo, Trnka

Precision Calculations to fill the gap!



Electric lamps were not invented by improving candles

T. Hänsch