Recent results on the effective theories of confining strings. [1]

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[1] M. Billó, M. Caselle, F. Gliozzi, M. Meineri, R. Pellegrini JHEP05(2012)130
M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo JHEP07(2013)071
M. Caselle, R. Pellegrini Phys. Rev. Lett. 111 (2013) 132001

Summary:

1. Introduction and motivation
2. Lorentz invariance and "universality".
3. Application: the boundary term of the effective action

Lattice determination of the interquark potential.

In pure lattice gauge theories the interquark potential is usually extracted from two (almost) equivalent observables

- Wilson loop expectation values $< W(R,T) >$ ("zero temperature potential")

$$V(R) = \lim_{T\to\infty} -\frac{1}{T} \log < W(R,T) >$$

- Polyakov loop correlators $< P(0)P(R)^\dagger >$ ("finite temperature potential")

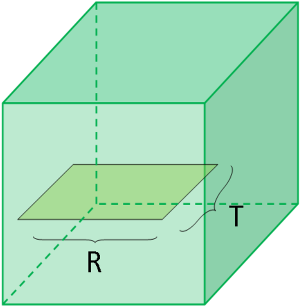
$$< P(0)P(R)^\dagger > \sim \sum_{n=0}^{\infty} c_n \; e^{-LE_n}$$

where L is the inverse temperature, i.e. the length of the lattice in the compactified imaginary time direction

$$E_0 = V(R) = - \lim_{L\to\infty} \frac{1}{L} \log < P(0)P(R)^\dagger >$$

Wilson Loop.

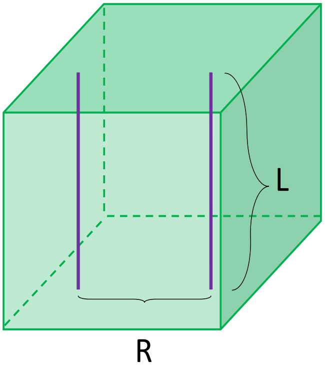
A Wilson loop of size $R \times T$



$$V(R) = \lim_{T \to \infty} -\frac{1}{T} \log < W(R, T) >$$

Polyakov loop correlator.

Expectation value of two Polyakov loops at distance R and Temperature $T = 1/L$



$$V(R) = -\lim_{L \to \infty} \frac{1}{L} \log < P(0)P(R)^{\dagger} >$$

Wilson Loops.

In the Wilson loop framework confinement is equivalent to the well known area-perimeter-constant law:

$$< W(R,T) >= e^{-(\sigma RL + c(R+T)+k)}$$

which implies $V(R) = \sigma R + c$.
Confinement is usually associated to the creation (via a mechanism which still has to be understood) of a thin flux tube joining the quark antiquark pair. (Nielsen-Olesen, 't Hooft, Wilson, Polyakov, Nambu ....) However if we accept this picture we cannot neglect the quantum fluctuations of this flux tube. The area law is thus only the classical contribution to the interquark potential and we should expect quantum corrections to its form. The theory which describes these quantum fluctuations is known as "effective string theory".

Effective string action

The simplest choice for the effective string action is to describe the quantum fluctuations of the flux tube as free massless bosonic degrees of freedom

$$S = S_{cl} + \frac{\sigma}{2} \int d^2\xi \, [\partial_\alpha X \cdot \partial^\alpha X] \, ,$$

where:

- S_{cl} describes the usual ("classical") area-perimeter term.
- $X_i(\xi_0, \xi_1)$ $(i = 1, \ldots, d-2)$ parametrize the displacements orthogonal to the surface of minimal area representing the configuration around which we expand
- ξ_0, ξ_1 are the world-sheet coordinates.

The Lüscher term.

- The first quantum correction to the interquark potential is obtained summing over all the possible string configuration compatible with the Wilson loop (i.e. with Dirichlet boundary conditions along the Wilson loop).
- This is equivalent to the sum over all the possible surfaces borderd by the Wilson loop i.e. to the partition function

$$< W(R,T) >= \int e^{-\sigma RT - \frac{\sigma}{2}\int d^2\xi X^i(-\partial^2)X^i}$$

- The functional integration is a trivial gaussian integral, the result is

$$V(R) = \sigma R - \frac{(d-2)\pi}{24R} + c$$

- This quantum correction is known as "Lüscher term" and is universal i.e. it does not depend on the ultraviolet details of the gauge theory but only on the geometric properties of the flux tube.

The Lüscher term.

This correction is in remarkable agreement with numerical simulations. First high precision test in d=4 SU(3) LGT more than ten years ago. [1]

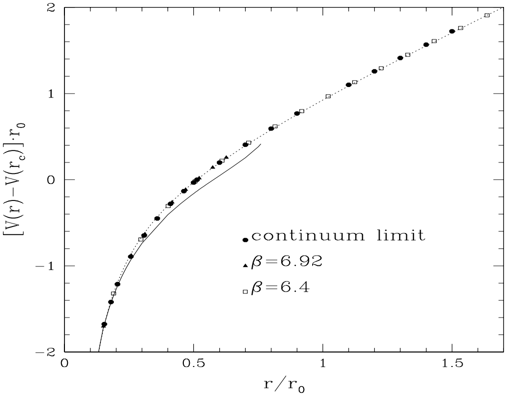


Figure : The static potential. The dashed line represents the bosonic string model and the solid line the prediction of perturbation theory.

[1] S. Necco and R. Sommer, Nucl.Phys. B622 (2002) 328

The Lüscher term.

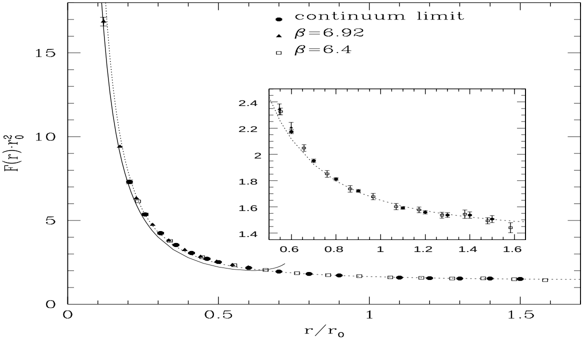


Figure : The force in the continuum limit and for finite resolution, where the discretization errors are estimated to be smaller than the statistical errors. The full line is the perturbative prediction. The dashed curve corresponds to the bosonic string model normalized by $r_0^2 F(r_0) = 1.65$.

The Nambu-Goto action.

- Evaluation of higher order quantum corrections requires further hypothesis on the nature of the flux tube. The simplest choice is the Nambu-Goto string in which quantum corrections are evaluated summing over all the possible surfaces bordered by the Wilson loop with a weight proportional to their area.

$$S = \sigma \int d^2\xi \sqrt{det(\eta_{\alpha\beta} + \partial_\alpha X \cdot \partial_\beta X)}$$

$$\sim \sigma RT + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8}(\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4}(\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right],$$

Interquark potential for the Nambu-Goto action.

- In the framework of the Nambu-Goto action one can evaluate exactly the energy of all the excited states of the flux tube:

$$E_n(R) = \sqrt{\sigma^2 R^2 + 2\pi\sigma\left(n - \frac{D-2}{24}\right)}$$

- In particular $E_0(R)$ corresponds to the interquark potential

$$V(R) = E_0(R) = \sqrt{\sigma^2 R^2 - 2\pi\sigma\frac{D-2}{24}} \,,$$

$$V(R) \;\sim\; \sigma R - \frac{\pi(D-2)}{24R} - \frac{1}{2\sigma R^3}\left(\frac{\pi(D-2)}{24}\right)^2 + O(1/R^5) \,,$$

The Nambu-Goto action.

High precision fit in the SU(2) case in 2+1 dimensions (A. Athenodorou, B. Bringoltz, M. Teper JHEP 1105:042 (2011))

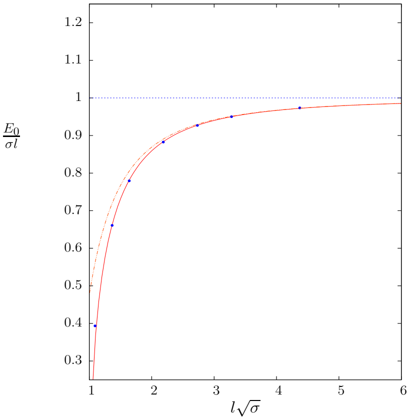
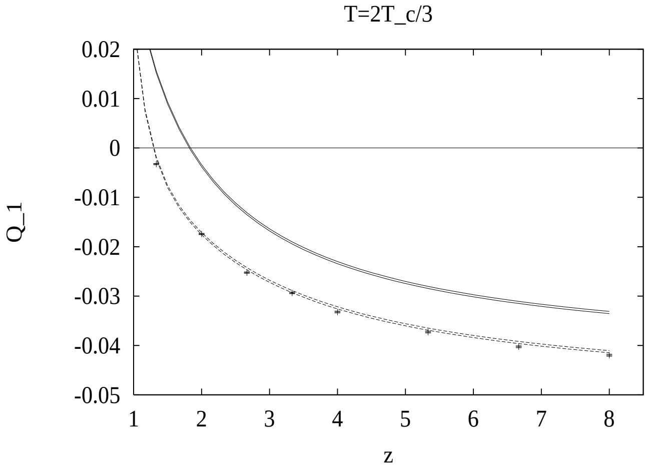


Figure 6: Energy of absolute ground state for SU(2) at $\beta = 5.6$. Compared to full Nambu-Goto (solid curve) and just the Lüscher correction (dashed curve).

The Nambu-Goto action.

High precision fit in the 2+1 dimensional Ising gauge model (M. Caselle, M. Hasenbusch, M. Panero JHEP 0301 (2003) 057)



Interquark potential via Polyakov Loop correlators.

- In this case we have different boundary conditions in the two directions (space R and inverse temperature L).
- The novel feature of this observable is that by exchanging R and L (the so called "open-closed string transformation") we can study the finite temperature behaviour of the string tension.

$$V(R) = \sigma(T)R, \quad \sigma(T) = \sigma_0\sqrt{1 - \frac{(d-2)\pi T^2}{3\sigma_0}}$$

where T is now the temperature and σ_0 the zero temperature string tension

- From this expression we may deduce a "Nambu-Goto" prediction for the critical temperature:

$$\frac{T_c}{\sqrt{\sigma_0}} = \sqrt{\frac{3}{(d-2)\pi}}$$

which turns out to be in remarkable agreement with LGT results both in d=3 and d=4.

Can we really trust these results?

- These results look nice, but they depend on a set of ad hoc assumptions on the behaviour of the flux tube. Why should we prefer the Nambu-Goto action to other possible choices for the flux tube action?
- They are "too universal" and show no dependence on the gauge group.
- It is somehow surprising that the Nambu-Goto model which looks so complex can be solved exactly at the quantum level (to all orders!!). How is it possible?
- Is there a "boundary" contribution due to the quarks at the flux tube boundaries?

In the past few years two important results changed our understanding of effective string theories and allowed us to answer to the above questions

Universality of effective string corrections.

- The Effective String action is strongly constrained by Lorentz invariance. The first few orders of the action are universal and coincide with those of the Nambu-Goto action. This explains why N.-G. describes so well the infrared regime of Wilson loops or Polyakov Loop correlators.[1] [2] [3]
- The Nambu-Goto effective theory can be described as a free 2d bosonic theory perturbed by the irrelevant operator $T\bar{T}$ (where T and $\bar{T}$ are the two chiral components of the energy momentum tensor). This perturbation turns out to be quantum integrable and yields, using the Thermodynamic Bethe Ansatz (TBA), a spectrum which, in a suitable limit, coincides with the Nambu-Goto one. [4]

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[1] M. Luscher and P. Weisz JHEP07(2004)014

[2] H. B. Meyer JHEP05(2006)066

[3] O. Aharony and M. Field JHEP01(2011)065

[4] M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo JHEP07(2013)071

Effective string action

The most general action for the effective string can be written as a low energy expansion in the number of derivatives of the transverse fields ("physical gauge").

$$S = S_{cl} + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + c_2(\partial_\alpha X \cdot \partial^\alpha X)^2 + c_3(\partial_\alpha X \cdot \partial_\beta X)^2 + \dots\right] + S_b \,,$$

where:

- S_{cl} describes the usual ("classical") perimeter-area term.
- S_b is the boundary contribution characterizing the open string
- In the Nambu-Goto case $c_2 = \frac{1}{8}$ and $c_3 = -\frac{1}{4}$

Effective string and spacetime symmetries.

- Symmetries of the action must hold in the low energy regime.
- String vacuum is not Poincaré invariant.

$\Longrightarrow$ Poincaré symmetry is broken spontaneously.

$$ISO(D-1,1) \rightarrow SO(D-2) \otimes ISO(1,1). \implies 3(D-2) \text{ Goldstone bosons?}$$

Only $D-2$ tranverse fluctuations of the string, where are the remaining Goldstone bosons?

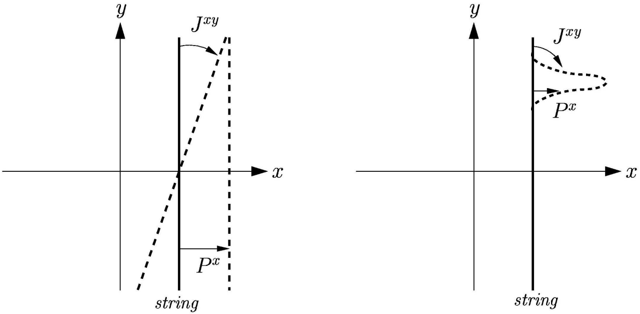
Goldstone's theorem states that there is a massless mode for each broken symmetry generator, but this counting cannot be naively extended to the case of spontaneously broken spacetime symmetries[1] .

[1] I. Low and A.V. Manohar, "Spontaneously broken spacetime symmetries and Goldstone's theorem" Phys.Rev.Lett. 88 (2002) 101602

Effective string and spacetime symmetries.

The remaining $2(D-2)$ **Lorentz transformations are realized non-linearly !** [1]

$$\delta_\epsilon^{bj} X_i = \epsilon\left(-\delta_{ij}\xi_b - X_j\partial_b X_i\right)$$



[1] I. Low and A.V. Manohar, "Spontaneously broken spacetime symmetries and Goldstone's theorem" Phys.Rev.Lett. 88 (2002) 101602

Non-linear realization and long-string expansion.

An internal transformation of the fields realizes the Poincaré group:

- Broken translations:
  $X^i \to X^i + a^i$. $\Longrightarrow$ Only field derivatives in the effective action.
- Broken rotation in the plane $(1,2)$:

$$\delta_\epsilon^{bj} X_i = \epsilon \left(-\delta_{ij} \xi_b - X_j \partial_b X_i \right)$$

Number of derivatives minus number of fields (scaling) preserved.

Fields and coordinates rescaling $\Longrightarrow$ Derivative expansion:

$$\partial_a X^i \longrightarrow \frac{1}{\sqrt{\sigma} R} \partial_a X^i .$$

Variations by broken rotation mix orders $\Longrightarrow$ Recurrence relations.

$ISO(1,1)$ and $SO(D-2)$ invariance $\Longrightarrow$ Contraction of indices.

Effective string action is strongly constrained! [1] [2] [3]

- the terms with only first derivatives coincide with the Nambu-Goto action to all orders in the derivative expansion.
- The first allowed correction to the Nambu-Goto action turns out to be the the six derivative term

$$c_4 \left(\partial_\alpha \partial_\beta X \cdot \partial^\alpha \partial^\beta X\right) \left(\partial_\gamma X \cdot \partial^\gamma X\right)$$

  with arbitrary coefficient c_4
- however this term is non-trivial only when $d > 3$. For $d = 3$ the first non-trivial deviation of the Nambu-Goto action is an eight-derivative term
- The fact that the first deviations from the Nambu-Goto string are of high order, especially in $d = 3$, explains why in early Monte Carlo calculations a good agreement with the Nambu-Goto string was observed.

[1] M. Luscher and P. Weisz JHEP07(2004)014
[2] H. B. Meyer JHEP05(2006)066
[3] O. Aharony and M. Field JHEP01(2011)065

Application: the boundary term of the effective action: Constraints imposed by the Lorentz invariance

If the boundary is a Polyakov line in the ξ_0 direction placed at $\xi_1 = 0$, on which we assume Dirichlet boundary conditions $X_i(\xi_0, 0) = 0$, the most general boundary action should be of this type

$$S_b = \int d\xi_0 \left[b_1 \partial_1 X \cdot \partial_1 X + b_2 \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X + b_3 (\partial_1 X \cdot \partial_1 X)^2 + \ldots \right] .$$

Imposing Lorentz invariance one finds that $b_1 = 0$ and that the b_2 term is only the first term of a Lorentz invariant expression[1] :

$$b_2 \int d\xi_0 \left[\frac{\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} - \frac{(\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} \right] .$$

which is the analogous in the case of the boundary action of the Nambu-Goto action for the "bulk" effective action.

[1] M. Billo, M. Caselle, F. Gliozzi, M. Meineri, R. Pellegrini JHEP05(2012)130

The boundary contribution to the interquark potential

Following the above discussion, the leading correction coming from the boundary turns out to be:

$$S_{b,2}^{(1)} = \int d\xi_0 \left[b_2 \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X \right] .$$

Its contribution to the interquark potential can be evaluated performing a simple gaussian functional integration[1]

$$\langle S_{b,2}^{(1)} \rangle \; = \; -b_2 \frac{\pi^3 L}{60 R^4} E_4 (\mathrm{i} \frac{L}{2R}) \; .$$

where the Eisenstein function E_4, is defined as

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \; ,$$

where $q = e \; 2\pi i \tau$ and $\sigma_p(n)$ is the sum of the p-th powers of the divisors of n:

$$\sigma_p(n) = \sum_{m|n} m^p \; .$$

[1] O. Aharony and M. Field JHEP01(2011)065

The boundary contribution to the interquark potential

- We end up with the following expression for the interquark potential

$$V(R) = \sigma R - \frac{\pi(D-2)}{24R} - \frac{1}{2\sigma R^3}\left(\frac{\pi(D-2)}{24}\right)^2 - b_2 \frac{\pi^3(D-2)}{60R^4} + O(1/R^5) ,$$

  where b_2 is a new physical parameter, similar to the string tension σ, which depends on the theory that we study and should be determined by simulations and comparison with experiments.

- To test this picture we performed a set of high precision simulations in the case of the 3d gauge Ising model, which is the simplest possible confining gauge theory.

Simulation I: Polyakov loops

- In order to eliminate the non-universal perimeter and constant terms from the expectation value of Polyakov loop correlators $P(R, L)$ (where L is the length of the two loops and R their distance) we measured the following ratio:

$$R_P(R, L) = \frac{P(R+1, L)}{P(R, L)} .$$

- Due to the peculiar nature of our algorithm, based on the dual transformation to the 3d spin Ising model, this ratio can be evaluated for large values of R and L with very high precision.

Simulation settings

- We performed our simulations in the 3d gauge Ising model, using a dual algorithm

data set	β	L	σ	$1/T_c$
1	0.743543	68	0.0228068(15)	5
2	0.751805	100	0.0105255(11)	8
3	0.754700	125	0.0067269(17)	10

Table : Some information on the data sample

Results

- The values of b_2 extracted from the data show the expected scaling behaviour $b_2 \sim \frac{1}{\sqrt{\sigma}^3}$

data set	b_2	$b_2\sqrt{\sigma}^3$	χ^2
1	7.25(15)	0.0250(5)	1.2
2	26.8(8)	0.0289(9)	1.8
3	57.9(12)	0.0319(7)	1.3

Table : Values of b_2 as a function of β

Simulation II: Wilson loops

As a check of our analysis we performed the same simulation for the Wilson loops fixing the value of b_2 obtained above. In this case there is no more parameter to fit and we can directly compare our predictions with the results of the simulations. To eliminate all the non-universal parameters we constructed the following combination:

$$R_W^{'}(L, Lu) = \frac{W(L,R)}{W(L+1, R-1)} - \exp\{-\sigma(1+L(1-u))\} \ , \qquad u = R/L$$

Simulation II: Wilson loops

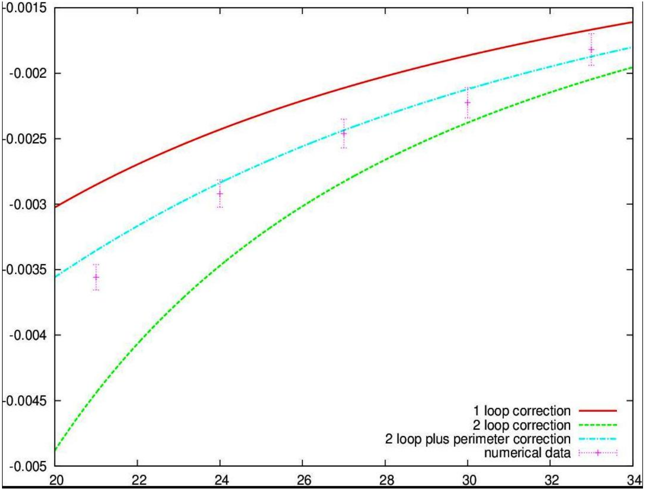


Figure : $R'_W(L, L^{\frac{4}{3}})$ at $\beta = 0.754700$.

Conclusions

- The Effective String action is strongly constrained by Lorentz invariance. The first few orders of the bulk and of the boundary action are universal This explains why the Nambu-Goto effective theory describes so well the infrared regime of the interquark potential
- In the 3d gauge Ising model also the first universal boundary correction can be reliably estimated and agrees with predictions
- The Nambu-Goto action can be described as a free 2d bosonic theory perturbed by the irrelevant operator $T\bar{T}$. This perturbation is quantum integrable and yields, via TBA, a spectrum which, in a suitable limit, coincides with the Nambu-Goto one
- A whole class of "Nambu-Goto like theories" can be constructed, as $T\bar{T}$ perturbations of generic 2d CFTs

Acknowledgements

Collaborators:

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Evaluation of the Lüscher term.

- The gaussain integration gives:

$$\int e^{-\frac{\sigma}{2}\int d^2\xi X^i(-\partial^2)X^i} \propto \left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} \quad .$$

- The determinant must be evaluated with Dirichlet boundary conditions. The spectrum of $-\partial^2$ with Dirichlet boundary conditions is:

$$\lambda_{mn} = \pi^2 \left(\frac{m^2}{T^2} + \frac{n^2}{R^2}\right)$$

corresponding to the normalized eigenfunctions

$$\psi_{mn}(\xi) = \frac{2}{\sqrt{RT}} \sin\frac{m\pi\tau}{T} \sin\frac{n\pi\varsigma}{R} \quad .$$

Evaluation of the Lüscher term.

- The determinant can be regularized with the ζ-function technique: defining

$$\zeta_{-\partial^2}(s) \equiv \sum_{mn=1}^{\infty} \lambda_{mn}^{-s}$$

the regularized determinant is defined through the analytic continuation of $\zeta'_{-\partial^2}(s)$ to $s = 0$:

$$\det(-\partial^2) = \exp\left[-\zeta'_{-\partial^2}(0)\right] \quad .$$

- The result is

$$\left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} = \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}} \quad .$$

where $\eta(\tau)$ is the Dedekind function

$$\eta(\tau) = q^{1/24}\Pi_{n=1}^{\infty}(1 - q^n)$$

with $q \equiv e^{2\pi i\tau}$ and $\tau = iT/R$.

Derivation of the Nambu-Goto action.

- The Nambu-Goto action is given by the area of the world–sheet:

$$S = \sigma \int_0^T d\tau \int_0^R d\varsigma \sqrt{g} \quad ,$$

where g is the determinant of the two–dimensional metric induced on the world–sheet by the embedding in R^d:

$$g = \det(g_{\alpha\beta}) = \det\ \partial_\alpha X^\mu \partial_\beta X^\mu \quad (\alpha, \beta = \tau, \varsigma,\ \mu = 1, \ldots, d)$$

- Choosing the "physical gauge"

$$X^1 = \tau \quad X^2 = \varsigma$$

g may be expressed as a function of the transverse degrees of freedom only:

$$\begin{aligned} g \;&=\; 1 + \partial_\tau X^i \partial_\tau X^i + \partial_\varsigma X^i \partial_\varsigma X^i \\ &\quad + \partial_\tau X^i \partial_\tau X^i \partial_\varsigma X^j \partial_\varsigma X^j - (\partial_\tau X^i \partial_\varsigma X^i)^2 \ (i = 3, \ldots, d)\ . \end{aligned}$$

- Expanding we find:

$$S \sim \sigma RT + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8} (\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4} (\partial_\alpha X \cdot \partial_\beta X)^2 + \ldots \right] ,$$