Supersymmetric gauge theories and quantum hydrodynamics

Antonio Sciarappa

SISSA

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with Giulio Bonelli, Alessandro Tanzini, Petr Vasko

Outline of the talk

Connections among different fields:

- $\mathcal{N} = (2, 2)$ gauge theories on S^2
- Equivariant quantum cohomology
- Quantum integrable systems

Main example:

- ADHM gauged linear sigma model
- Quantum cohomology of instanton moduli space
- Periodic Intermediate Long Wave system

Main results:

- Spectrum ILW from gauge theory
- A gauge theoretical proof of AGT

$\mathcal{N} = (2,2)$ gauge theories on S^2

Supersymmetry algebra $SU(2|1)_{A(B)}$:

$$\{\overline{Q}_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2}\epsilon_{\alpha\beta} R_{V(A)}$$

J_m: $SU(2)$ isometry generators of S^2

- realized by conformal Killing spinors

$$\nabla_\mu \epsilon = \frac{i}{2r}\gamma_\mu \epsilon$$

- subalgebra of superconformal algebra preserving $SU(2)$
- Mirror symmetry: map $SU(2|1)_A \leftrightarrow SU(2|1)_B$

$$Q_A \leftrightarrow Q_B \ , \ \ R_V \leftrightarrow R_A$$

Representation of SUSY algebra on supermultiplets:

- vector multiplet: $(A_\mu, \sigma, \eta, \lambda, \overline{\lambda}, D)$
- chiral multiplet: $(\phi, \overline{\phi}, \psi, \overline{\psi}, F, \overline{F})$
- twisted multiplet: $(Y, \overline{Y}, \chi, \overline{\chi}, G, \overline{G})$

Superfield strength Σ is twisted chiral

$$\Sigma \ = \ (\sigma + i\eta, \lambda, \overline{\lambda}, D - iF_{01})$$

Mirror symmetry:

$$\text{chiral multiplet} \quad \Longleftrightarrow \quad \text{twisted chiral multiplet}$$

Most general Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_W + \mathcal{L}_{\widetilde{W}} + c.c.$$

Field content specified by:

- G: gauge group for vector multiplets (focus on $U(k)$)
- Fayet-Iliopoulos term $\mathcal{L}_{\text{FI}}(\xi, \theta)$, allowed since $U(1) \subset U(k)$
- $\mathbf{R}$: representation of G for matter multiplets
- Twisted masses $\vec{a}$, if a flavour group G_F is present

$\mathcal{L}$ depends on two holomorphic functions:

$W(\phi)$: superpotential

$\widetilde{W}(Y, \Sigma)$: twisted superpotential

Twisted Landau-Ginzburg model (LG):

- twisted chiral multiplets (also Σ)
- twisted superpotential $\widetilde{W}(Y, \Sigma)$

Gauged linear sigma model (GLSM):

- chiral + vector multiplets
- superpotential $W(\phi)$

Higgs branch GLSM: classical SUSY vacua = target space M

$$M = \{\text{constant } \phi \ / \ F = 0\,,\ D = 0\}/G$$

Mirror symmetry:

$$\text{GLSM} \quad \Longleftrightarrow \quad \text{LG}$$

Partition function Z_{S^2}

Exact computation of Z_{S^2} via SUSY localization

- GLSM case:

$$Z_{\text{GLSM}} = \sum_{m\in\mathbb{Z}} \int_{\mathbf{t}_G} d\sigma e^{-S_0}\, Z_{11}^{V}(\sigma, m) \prod_{\Phi} Z_{11}^{\Phi}(\sigma, m, a)$$

- LG case:

$$Z_{\text{LG}} = \int dY\, d\overline{Y}\, d\Sigma\, d\overline{\Sigma}\, e^{-4\pi i r\,\widetilde{W}(Y,\Sigma)-4\pi i r\,\overline{\widetilde{W}}(\overline{Y},\overline{\Sigma})}$$

Mirror symmetry on S^2 gives the LG mirror of a GLSM:

$$Z_{S^2}\,(\text{GLSM}) \;=\; Z_{S^2}\,(\text{mirror LG})$$

Gauge theories and integrable systems

Coulomb branch: take LG model, integrate out twisted matter

$$\frac{\partial \widetilde{W}}{\partial Y_i} = 0 \quad \Longrightarrow \quad Y_i = Y_i(\Sigma)$$

Obtain pure abelian gauge theory with effective $\widetilde{W}$

$$\widetilde{W}_{\text{eff}} = \widetilde{W}(\Sigma, Y_i(\Sigma))$$

Study vacua of the effective theory (symmetry $\theta \to \theta + 2\pi$)

$$\boxed{\frac{\partial \widetilde{W}_{\text{eff}}}{\partial \Sigma} = 2\pi i n \quad , \quad \exp\left(\frac{\partial \widetilde{W}_{\text{eff}}}{\partial \Sigma}\right) = 1}$$

Same as Bethe ansatz equations for quantum integrable systems

The model: ADHM GLSM

ADHM model: gauged linear sigma model with target $\mathcal{M}_{k,N}$

$$\mathcal{M}_{k,N} = \{\, \text{moduli space of } k \text{ instantons, gauge theory } SU(N)\,\}$$

ADHM construction of $\mathcal{M}_{k,N}$ via D-branes:

- System of N D5-branes + k D1-branes wrapped on S^2
- D1 theory: GLSM with $G = U(k)$, chiral χ, B_1, B_2, I, J, superpotential $W = \mathrm{Tr}_k \{\chi\,([B_1, B_2] + IJ)\}$
- Vacua (Higgs branch) $\Longrightarrow$ target $\mathcal{M}_{k,N}$

$$[B_1, B_2] + IJ = 0 \qquad (F\text{-equations})$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \xi \qquad (D\text{-equations})$$

ADHM model and equivariant quantum cohomology

Nekrasov partition function:

$$Z_N^{\text{Nek}} = \sum_k q^k Z_{k,N} = \exp\{-\frac{1}{\epsilon_1\epsilon_2}(\mathcal{F}_0 + \dots)\}$$

- $Z_{k,N}$: equivariant volume of $\mathcal{M}_{k,N}$

Spherical partition function $Z_{k,N}^{S^2}$ generalizes $Z_{k,N}$:

- Corrections in r to Nekrasov: $Z_{k,N}^{S^2} \to Z_{k,N}$ for $r \to 0$
- Small quantum cohomology $\mathcal{M}_{k,N}$ (maps $S^2 \to \mathcal{M}_{k,N}$):

$$\boxed{Z_{k,N}^{S^2} = e^{-\mathcal{K}_K(\mathcal{M}_{k,N})}}$$

$\mathcal{K}_K$: quantum Kähler potential of Kähler moduli space; contains information on genus 0 equivariant GW invariants

ADHM model - dual LG theory

LG dual theory, Coulomb branch ($t = \xi - i\frac{\theta}{2\pi}$) $\Longrightarrow$ find $\widetilde{W}_{\text{eff}}$

$\Longrightarrow$ Bethe ansatz equations for a quantum integrable system

$$\prod_{j=1}^{N}(\Sigma_s - a_j - \frac{\epsilon}{2})\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\Sigma_{st}-\epsilon_1)(\Sigma_{st}-\epsilon_2)}{(\Sigma_{st})(\Sigma_{st}-\epsilon)}$$

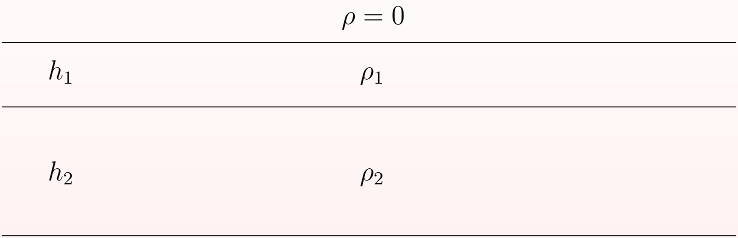
$$= e^{-2\pi t}\prod_{j=1}^{N}(-\Sigma_s + a_j - \frac{\epsilon}{2})\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(-\Sigma_{st}-\epsilon_1)(-\Sigma_{st}-\epsilon_2)}{(-\Sigma_{st})(-\Sigma_{st}-\epsilon)}$$

Associated integrable system:

periodic $gl(N)$ Intermediate Long Wave system

Cases of interest: $N = 1, 2$

Hydrodynamics: the two-layer configuration



Two fluids of densities $\rho_2 > \rho_1$, total depth $h = h_1 + h_2$

- Thin upper layer ($h_1 \ll h_2$), small wave amplitude ($\ll h_1$)
- Wavelength $\lambda \gg h_1$: long wave, compared to thin layer
  - $\boxed{\lambda \gg h}$, long wave: KdV regime
  - $\boxed{\lambda \ll h}$, short wave: Benjamin-Ono regime
  - $\boxed{\lambda \simeq h}$, intermediate wave: ILW regime

Quantum hydrodynamics equations ($2\alpha_0 = (\sqrt{\beta} - 1/\sqrt{\beta})$):

$$\boxed{u_t + uu_x - i\alpha_0 \partial_x^2 u^H = 0}$$

- Case $N = 1\,(N = 2)$: x , $u(x,t)$ real (complex)
- $u = u^+ + u^-$, Hilbert transform $u^H = i(u^+ - u^-)$
- Periodic ILW:

$$u^H = \frac{1}{2\pi} P.V. \int_0^{2\pi} dx' u(x')\zeta(x' - x) \quad \left(\text{or } \oint_C \text{ for } N = 2 \right)$$

$\zeta(x)$ Weierstrass function, $\zeta'(x) = \wp(x)$

- Periodic Benjamin-Ono ($\xi \to \infty$): $\zeta(x) \to \cot(x)$
- Periodic KdV ($\xi \to 0$): $u^H \to \partial_x u$

Moving pole ansatz

Idea: study motion of poles of $u(x,t)$ (soliton waves)

Example: BO $L \to \infty$ (non-periodic), $\cot(2\pi x/L) \to \frac{L}{2\pi}\frac{1}{x}$

- Ansatz for the solution:

$$u(x,t) = \sum_{j=1}^{N_1} \frac{i\sqrt{\beta}}{x - a_j(t)} - \sum_{k=1}^{N_2} \frac{i\sqrt{\beta}}{x - b_k(t)}$$

- Poles a_j, b_k below / above real axis $\Rightarrow$ $u^H = i(u^+ - u^-)$

$$\left(\frac{1}{x-a}\right)^H = \mp \frac{i}{x-a}\,, \quad \operatorname{Im} a < 0 \,/\, \operatorname{Im} a > 0$$

Insert in BO $\Longrightarrow$ EOM quantum rational Calogero

$$\ddot{a}_j = \sum_{l \neq j}^{N_1} \frac{2\beta(\beta-1)}{(a_j - a_l)^3} \quad , \quad \ddot{b}_k = \sum_{m \neq k}^{N_2} \frac{2\beta(\beta-1)}{(b_j - b_l)^3}$$

$$\mathcal{H}_{\mathrm{Cal}} = -\frac{1}{2} \sum_{i=1}^{N} \left(\frac{d}{dx_i} \right)^2 + \sum_{i<j} \frac{\beta(\beta-1)}{(x_i - x_j)^2}$$

Impose a_j real $\Longrightarrow$ one real Calogero system;

- $u(x,t)$ real $\Longrightarrow N_1 = N_2$, $b_j = a_j^* = a_j$ (case $N = 1$)
- $u(x,t)$ complex $\Longrightarrow b_k$ complex, $\int \to \oint_C$ (case $N = 2$)

Periodic case:

- BO $\longrightarrow$ trigonometric quantum Calogero
- ILW $\longrightarrow$ elliptic quantum Calogero

A gauge theory proof of AGT

AGT conjecture relates

- conformal block $\langle \Phi_1(z_1)\dots\Phi_n(z_n)\rangle$ Liouville CFT on S^2
- Z^{Nek} for $U(2)_1\otimes\dots\otimes U(2)_{n-3}$ + matter (bifund.)

CFT proof: consider Vir $\oplus$ H on a cylinder

- Find the unique basis $|\vec{\lambda}\rangle$ (bipartitions) such that

$$\langle\vec{\mu}|V_\alpha|\vec{\lambda}\rangle = Z_{\text{bif}}(\alpha|\vec{\mu};\vec{\lambda}) \;\;,\;\; \langle\vec{\lambda}|\vec{\lambda}\rangle = 1/Z_{\text{vec}}(\vec{\lambda})$$

- Diagonal basis for Integrals of Motion $\mathbf{I}_m$ of BO $(m\geqslant 1)$
- Spectrum $h_m^{(\vec{\lambda})}(a)$ of two copies trigonometric Calogero

$$\mathbf{I}_m|\vec{\lambda}\rangle = h_m^{(\vec{\lambda})}(a)|\vec{\lambda}\rangle \quad,\quad h_m^{(\vec{\lambda})}(a) \;=\; h_m^{(\lambda_1)}(a) + h_m^{(\lambda_2)}(-a)$$

Gauge theory proof: consider the BO limit $\xi \to \infty$

- Correspondence state $|\vec{\lambda}\rangle \longleftrightarrow$ vacuum $\vec{\lambda}$ (BA equations)
- Semiclassical approximation $Z_{\rm LG}$ around a vacuum $\vec{\lambda}$:

$$Z_{\rm LG} = 1/Z_{\rm vec}(\vec{\lambda})$$

- Relation $h_m^{(\vec{\lambda})}(a) \longleftrightarrow$ gauge theory observables $\mathrm{Tr}\,\Sigma^i$

$$h_m^{(\vec{\lambda})}(a) \;\; \text{linear combination of} \;\; \mathrm{Tr}\,\Sigma^i \;\; , \quad i < m$$

Valid for generic t - ILW case:

$$\mathfrak{h}_m^{(k)}(t) \longleftrightarrow \mathrm{Tr}\,\Sigma^i(t)$$

Gauge theory provides the spectrum of quantum ILW system

Summary

Z_{S^2} relates different fields of physics and mathematics:

- $\mathcal{N} = (2,2)$ theories (Coulomb) $\Longleftrightarrow$ integrable systems
- $\mathcal{N} = (2,2)$ theories (Higgs) $\Longleftrightarrow$ quantum cohomology
- Our example: ILW system $\Longleftrightarrow$ cohomology $\mathcal{M}_{k,N}$
- Gauge theory proof of AGT conjecture

Open questions:

- AGT equivalent for generic t
- Spectrum quantum dispersive KdV ($t \sim 0$)
- Relativistic elliptic Calogero (Ruijsenaars model): $S^2 \times S^1$?
- Other models: moduli space instantons $\mathbb{C}^2/\mathbb{Z}_p$, ...
- Physical proof generalized AGT (superLiouville, ...)