The θ dependence of $SU(N)$ gauge theories at finite temperature

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Phys. Rev. Lett. **110**, 252003 (2013) [arXiv:1301.7640 [hep-lat]]

Cortona, XXXIV Convegno Nazionale di Fisica Teorica, 28-31 May 2014

Outline

The θ parameter

The Euclidean version of the QCD Lagrangian is

$$\mathcal{L}_\theta = \frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) - i\theta q(x); \quad q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$$

θ is a dimensionless RG-invariant parameter and a nonzero θ value would violate P and CP. Experimentally its value is bounded by $|\theta| \lesssim 10^{-9}$ (Reasons for $\theta = 0$? Strong CP problem)

Nevertheless θ related physics is interesting from various point of view:

- theoretical: the θ dependence is completely nonperturbative
- practical: some features of the hadron spectrum are related to θ (Witten, Veneziano 1979)
- phenomenologial: e.g. axions to resolve the strong CP problem

The free energy

At temperature T the free energy $F(\theta, T)$ is given by

$$F(\theta, T) = -\frac{1}{V_4} \log \int [\mathscr{D}A] \exp\left(- \int_0^{1/T} \mathrm{d}t \int \mathrm{d}^3x \, \mathcal{L}_\theta \right)$$

$$V_4 = T/V, \quad A_\mu(0, \mathbf{x}) = A_\mu(1/T, \mathbf{x})$$

and $F(\theta, 0)$ is just the ground state energy.

If we denote by Q the topological charge, we have

$$\left. \frac{\partial^n F(\theta, T)}{\partial \theta^n} \right|_{\theta=0} \propto \langle Q^n \rangle_0 \qquad (\langle \cdot \rangle_0 \equiv \langle \cdot \rangle_{\theta=0})$$

and by P invariance at $\theta = 0$ we have $\langle Q^{2k+1} \rangle_0 = 0$ (P cannot be spontaneously broken, Vafa Witten 1984)

The θ dependence

The θ dependence of the free energy can be parametrized in the following way

$$F(\theta, T) - F(0, T) = \frac{1}{2}\chi(T)\theta^2\left[1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \cdots\right]$$

where

$$\chi = \frac{1}{V_4}\langle Q^2\rangle_0 \qquad b_2 = -\frac{\langle Q^4\rangle_0 - 3\langle Q^2\rangle_0^2}{12\langle Q^2\rangle_0}$$

$$b_4 = \frac{\langle Q^6\rangle_0 - 15\langle Q^2\rangle_0\langle Q^4\rangle_0 + 30\langle Q^2\rangle_0^3}{360\langle Q^2\rangle_0}$$

Coefficients b_{2n} parametrize deviations of the distribution of topological charge from the Gaussian one in the theory at $\theta = 0$.

Large N versus instantons

Indication that large-N can be problematic at $T \neq 0$

By using factorization and translation invariance we get

$$\langle \mathscr{O}(0)\mathscr{O}(Rx)\rangle = \langle \mathscr{O}(0)\rangle\langle \mathscr{O}(Rx)\rangle = \langle \mathscr{O}(0)\rangle\langle \mathscr{O}(x)\rangle = \langle \mathscr{O}(0)\mathscr{O}(x)\rangle$$

thus correlators are $O(4)$ invariant also at finite temperature.

Indication that instanton calculus can be problematic at $T = 0$

Infrared divergences are present which has to be removed by introducing *ad hoc* a confinement length scale.
For $T > T_c$ no additional confinement length scale is present and T works as an infrared regulator.

What large-N tell us

$$\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \text{ and } \epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x) \text{ scale as } N^2$$

to have a nontrivial θ dependence in the large-N limit we have to keep $\bar{\theta} \equiv \theta/N$ fixed, in such a way that θg^2 does not scale with N.

The large-N scaling form of the free energy is thus (Witten 1980)

$$F(\theta, T) - F(0, T) = N^2 \bar{F}(\bar{\theta})$$

where $\bar{F}$ is nontrivial for $N \to \infty$:

$$\bar{F}(\bar{\theta}) = \frac{1}{2}\bar{\chi}\bar{\theta}^2\left[1 + \bar{b}_2\bar{\theta}^2 + \bar{b}_4\bar{\theta}^4 + \cdots\right]$$

By matching the power of θ we obtain

$$\chi = \bar{\chi} + c/N^2 + \cdots$$
$$b_{2n} = \bar{b}_{2n}/N^{2n} + \cdots$$

What instanton calculus tell us

For weakly interacting instantons we have (Gross, Pisarski, Yaffe 1981)

$$Z_\theta = \mathrm{Tr} e^{-H_\theta/T} \approx \sum \frac{1}{n_+! n_-!} (V_4 D)^{n_+ + n_-} e^{-\frac{8\pi}{g^2}(n_+ + n_-) + i\theta(n_+ - n_-)}$$
$$= \exp\left[2 V_4 D e^{-8\pi^2/g^2} \cos\theta\right]$$

where D is a typical volume. Thus

$$F(\theta, T) - F(0, T) \approx \chi(T)(1 - \cos\theta)$$

and (remember that $8\pi^2/g^2(T) \approx \frac{11}{3} N \log(T/\Lambda)$ in perturbation theory)

$$\chi(T) \approx T^4 \exp\left[-8\pi^2/g^2(T)\right] \sim T^{-\frac{11}{3}N+4}$$
$$b_2 = -\frac{1}{12} \quad b_4 = \frac{1}{360} \quad b_{2n} = (-1)^n \frac{2}{(2n+2)!}$$

Note: perturbation theory needed in $\chi(T)$ only!

Summary of theoretical expectations

Low temperature, large-N behaviour

$$\chi = \bar{\chi} + c/N^2 + \cdots$$
$$b_{2n} = \bar{b}_{2n}/N^{2n} + \cdots$$

High temperature, instanton behaviour

$$\chi(T) \sim T^{-\frac{11}{3}N+4} \quad \text{(semiclasical + PT)}$$
$$b_2 = -\frac{1}{12} \quad b_4 = \frac{1}{360} \quad \text{(semiclassical)}$$

Questions

- Which is the temperature of the regime change? (natural answer T_c)
- How fast (in T) is the convergence to the instanton gas results?

Topology on the lattice

The topological charge is well defined only for smooth enough gauge configuration, so its definition on the lattice require some care.

Several methods have been devised during the years to study topology on the lattice:

- Field theoretical methods (perturbative/nonperturbative computation of the renormalization constants)
- Fermionic methods (using the lattice index theorem for Ginsparg-Wilson fermions)
- Smoothing methods

All these methods have advantages and drawbacks, nevertheless they have been proven to give compatible results for the physical observables (see e.g. Panagopoulos, Vicari 0803.1593).

Smoothing methods

smoothing: any method that reduce the roughness of a configuration

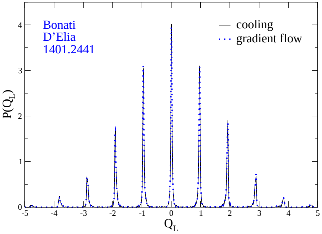
Different implementations of the same idea:

smearing : the gauge field is locally averaged

cooling : the gauge field is modified in order to locally minimize the action

gradient flow : the gauge field is modified in order to minimize the action proceeding along the steepest descendent

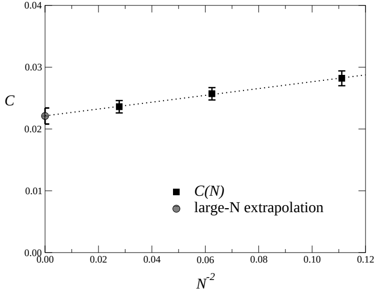
The final result is a smoothed configuration on which the topological charge is almost integer valued (i.e. up to lattice spacing corrections)



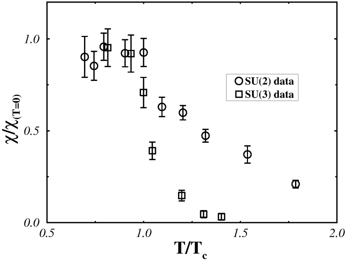
The topological susceptibility

$C = \chi/\sigma^2,\ T = 0$

Del Debbio, Panagopoulos, Vicari 0204125

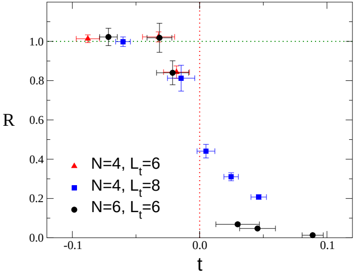


Alles, D'Elia, Di Giacomo 9605013



$R = \chi(T)/\chi(T = 0)$

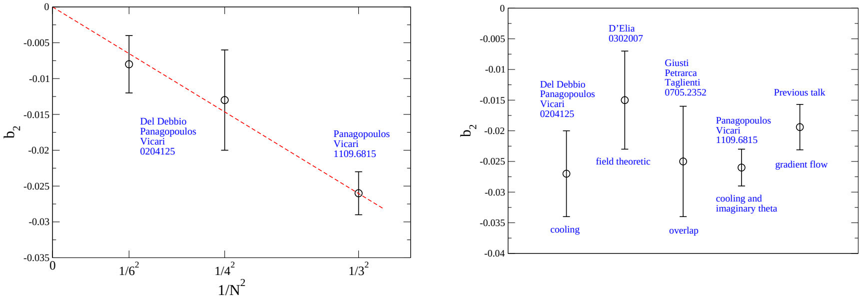
Del Debbio, Panagopoulos, Vicari 0407068



Low-T : good agreement with large-N

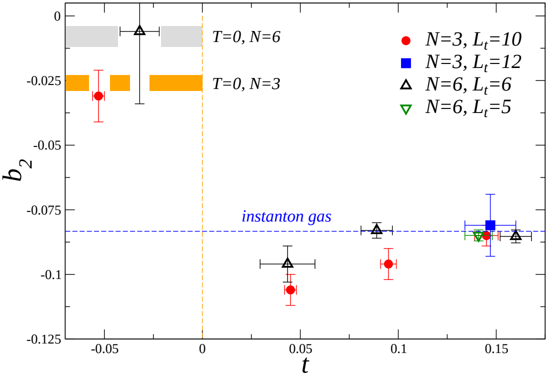
High-T : qualitative agreement with instanton result

b_2 at $T = 0$



- Good agreement with large-N scaling (i.e. $b_2 \sim 1/N^2$).
- Determinations obtained by using different approaches agree very well with each other.

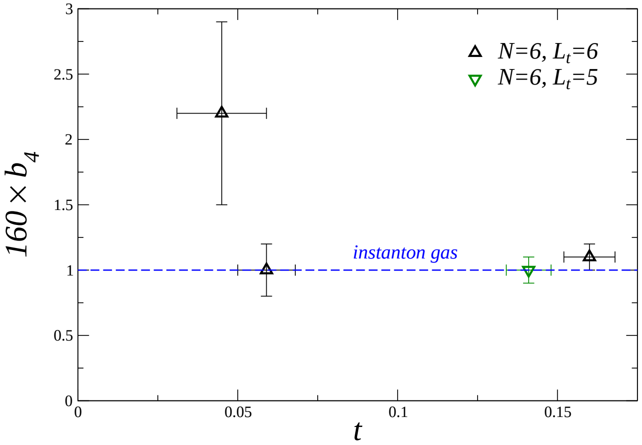
b_2 at $T \neq 0$



For $t < 0$ $b_2(t) \approx b_2(0)$, while for $t > 0$

- no large-N scaling
- data converge quickly to the istanton gas result $(-1/12)$

b_4 at $T \neq 0$



To our knowledge this is the first time someone gets an estimate of b_4 non compatible with zero and the b_4 value approaches quickly the instanton gas result (1/160).

Conclusions

We have shown that to study the θ dependence of the free energy it is convenient to look at b_{2n} instead of looking at χ, as was done in the past, in order to disentangle the semiclassical an perturbation theory contributions.

Our numerical results provide convincing evidence that

- for $T < T_c$ large-N scaling applies, the relevant variable is $\bar{\theta} = \theta/N$
- for $T > T_c$ the free energy is a function of θ and not of $\bar{\theta}$
- the approach to the instanton gas result is very fast, requiring $T \gtrsim 1.15 T_c$ for $SU(3)$ and $T \gtrsim 1.1 T_c$ for $SU(6)$, for observables not "contaminated" by perturbation theory