Higher poles and crossing phenomena from twisted elliptic genera

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based on arXiv:1404.7396 [hep-th]

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Cortona, Italy 2014

Plan of the talk

- The elliptic genus: definition and properties
- Elliptic genus of the cigar
  - Mock vs. modular: connection to Appell-Lerch sums
  - Effects of twisting the elliptic genus
- Generalization to higher pole Appell-Lerch sums
- Some interesting questions

Elliptic genus: definition

- The elliptic genus of $\mathcal{N} = 2$ CFT (in two dimensions):

$$\chi(\tau, \alpha) = \mathrm{Tr}_{\mathcal{H}} (-1)^F q^{L_0 - \frac{c}{24}} z^{J_0} \bar{q}^{\bar{L}_0 - \frac{c}{24}}$$

  where $q = e^{2\pi i \tau}$ and $z = e^{2\pi i \alpha}$.

  - L_0 and $\bar{L}_0$ are the left/right conformal dimensions.
  - J_0 measures left-moving R-charge.
  - c is the central charge of the CFT.
  - Trace is in the Ramonnd sector.

- There is also a path integral definition. Quantum fields on the torus, with twisted periodic boundary conditions, determined by the R-charge of the fields.

- It is a right-moving Witten index.

$$\chi(\tau, \alpha) = \mathrm{Tr}_{\mathcal{H}} \left\{ \left[(-1)^{F_L} q^{L_0 - \frac{c}{24}} z^{J_0} \right] \left[(-1)^{F_R} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right] \right\}$$

Elliptic genus as a Jacobi form

- **Holomorphic** because it is a right-moving index.
- It has good **modular** properties (seen from the path integral):

$$\chi(-\frac{1}{\tau}, \frac{\alpha}{\tau}) = e^{\frac{\pi i c}{3} \frac{\alpha^2}{\tau}} \chi(\tau, \alpha) .$$

- Spectral flow symmetry of $\mathcal{N} = 2$ theory guarantees that χ has good **elliptic** properties:

$$\chi(\tau, \alpha + m\tau + n) = (-1)^{\frac{c}{3}(m+n)} q^{-\frac{c}{6} m^2} z^{-\frac{c}{3} m} \chi(\tau, \alpha) .$$

- The modular and elliptic properties means that the elliptic genus of a CFT is a holomorphic Jacobi form of weight zero with an index proportional to the central charge c.

Overview of earlier work

- There are cases when these general conclusions are incorrect. [Troost '10; Eguchi-Sugawara, '10, SA, Troost '11]

- **Lesson 1:** When the CFT has
  - a continuum of states and
  - if there is spectral asymmetry in the continuum sector: i.e. $\rho_B - \rho_F \neq 0$, where $\rho_{B/F}$ is the bosonic/fermionic density of states,

  then, the "long multiplets" contribute to the elliptic genus [SA, Troost '11]. It is no longer holomorphic.

- Well known for the Witten index (weighted trace) [Cecotti-Fendley-Intriligator-Vafa, Comtet-Akhoury]

$$Tr[(-1)^F e^{-\beta H}] = N_B - N_F + \int dE \, e^{-\beta E} \left(\rho_B(E) - \rho_F(E)\right)$$

Elliptic genus as mock-Jacobi forms

- **Lesson 2:** The elliptic genus is a very special kind of real analytic Jacobi form: the modular completion of a mock-Jacobi form.
- The holomorphic part is a sum over discrete states while the non-holomorphic part is an integral over the radial momentum.

- **Caveat:** There are examples where the elliptic genus is not strictly mock but nevertheless shows the same type of behaviour. [i.e. holomorphic part + remainder]. But these higher dimensional models will not be discussed here. [SA, Doroud, Troost; Murthy]
- The shadow does not satisfy the properties required for it to be a mock-Jacobi form.

Case study: $\mathcal{N} = 2$ cigar

- The elliptic genus of the cigar is calculated using the path integral for the (axially) gauged WZW model $SL(2,\mathbb{R})/U(1)$ at level k. We will present the answer for a twisted elliptic genus, namely:

$$\chi(\tau,\alpha,\beta) = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} z^{J_0} \; y^P \, .$$

where $q = e^{2\pi i\tau}$, $z = e^{2\pi i\alpha}$ and $y = e^{2\pi i\beta}$.

- We find

$$\chi(\tau,\alpha,\beta) = k \int_{\mathbb{C}} \frac{d^2u}{2\tau_2} \left[\frac{\theta_{11}(\tau, u - \alpha - \frac{\alpha}{k} + \beta)}{\theta_{11}(\tau, u - \frac{\alpha}{k} + \beta)} \right] e^{-\frac{k\pi}{\tau_2}|u|^2} e^{-2\pi i \alpha_2 u} \, .$$

u is the (complex) holonomy of the gauge field around the cycles of the torus T^2.

- Non-holomorphic in τ and β, holomorphic in α.

Properties of the elliptic genus

Modular and elliptic properties:

$$
\begin{aligned}
\chi(\tau+1,\alpha,\beta) &= \chi(\tau,\alpha,\beta) \\
\chi(-\frac{1}{\tau},\frac{\alpha}{\tau},\frac{\beta}{\tau}) &= e^{\pi i \frac{c}{3}\frac{\alpha^2}{\tau}-2\pi i\frac{\alpha\beta}{\tau}}\chi(\tau,\alpha) \\
\chi(\tau,\alpha+k,\beta) &= (-1)^{\frac{c}{3}k}\chi(\tau,\alpha,\beta) \\
\chi(\tau,\alpha+k\tau,\beta) &= (-1)^{\frac{c}{3}k}e^{-\pi i\frac{c}{3}(k^2\tau+2k\alpha)}e^{2\pi i\beta k}\chi(\tau,\alpha) \\
\chi(\tau,\alpha,\beta+1) &= \chi(\tau,\alpha,\beta) \\
\chi(\tau,\alpha,\beta+\tau) &= e^{2\pi i\alpha}\chi(\tau,\alpha,\beta)\,.
\end{aligned}
$$

Here, $c = 3 + \frac{6}{k}$, the central charge of the cigar CFT.

- Jacobi form in three variables, with weight zero and a matrix index that can be obtained from the above transformation rules.

The split

In order to make contact with the math literature, it is easiest to work with a $\mathbb{Z}_k$ orbifold of this elliptic genus.

$$\chi_L = \sum_{n,m} \int \frac{d^2u}{2\tau_2} \frac{\theta_{11}(\tau, u-\alpha)}{\theta_{11}(\tau, u)} e^{2\pi i \alpha \frac{n}{k}} e^{-\frac{k\pi}{\tau_2}|(u+\frac{\alpha}{k}-\beta+\frac{n}{k}\tau+\frac{m}{k}|^2} \times e^{-2\pi i \alpha_2 (u+\frac{\alpha}{k}-\beta\frac{n}{k}\tau+\frac{m}{k})} .$$

A non-trivial calculation shows that it is a sum of two terms:

$$\chi_{L,hol} = \frac{i\theta_{11}(\tau, -\alpha)}{\eta^3(q)} z^{\frac{[k]\beta_2}{k}} \sum_{m\in\mathbb{Z}} \frac{(z^{-2}y^k q^{-[k\beta_2]})^m q^{km^2}}{1 - z^{-\frac{1}{k}} q^m}$$

- This is a contribution from discrete states of the theory.
- It is a sum over spectral flowed (extended) $\mathcal{N}=2$ superconformal characters.
- It is holomorphic, elliptic but **not** modular.

The remainder

The second piece is called the remainder term and is denoted $\chi_{L,rem}$:

$$\frac{i\theta_{11}(\tau,-\alpha)}{\pi\eta^3(\tau)}\sum_{v,w}\int_{\mathbb{R}}\frac{ds}{2is+v-k\beta_2}z^{\frac{v}{k}-2w}y^{kw}q^{kw^2-vw}(q\bar{q})^{\frac{s^2}{k}+\frac{(v-k\beta_2)^2}{4k}}.$$

- This is the contribition from the continuum states in the CFT.
- The measure factor is precisely the spectral asymmetry, as can be checked independently using reflection coefficients.
- One can read off the R-charge and conformal dimensions from the above expressions.

Effects of the twist β

- Consider now the remainder piece stripped of the oscillators:

$$\sum_{v,w} \int_{\mathbb{R}} \frac{ds}{2is + v - k\beta_2} z^{\frac{v}{k} - 2w} y^{kw} q^{kw^2 - vw} (q\bar{q})^{\frac{s^2}{k} + \frac{(v - k\beta_2)^2}{4k}} .$$

  Here, $v = n + kw$ and is the (usual) right moving momentum. We can read off

$$\bar{L}_0 - \frac{c}{24} = \frac{s^2}{k} + \frac{(n + kw - k\beta_2)^2}{4k}$$

$$L_0 - \bar{L}_0 = -nw \qquad J_0 = \frac{n - kw}{k} .$$

- The right-moving momentum shifts due to the y^P insertion in the trace. This shifts the right-moving Hamiltonian as well as the measure (spectral asymmetry).

Effects of the twist β: crossing phenomena

$$\chi_{L,hol} = z^{\frac{[k\beta_2]}{k}} \frac{i\theta_{11}(\tau,-\alpha)}{\eta^3(q)} \sum_{m\in\mathbb{Z}} \frac{(z^{-2}y^k q^{-[k\beta_2]})^m q^{km^2}}{1 - z^{-\frac{1}{k}} q^m}$$

- Whenever $[k\beta_2]$ crosses an integer value, terms are subtracted and add to the holomorphic sector; corresponding terms are subtracted and added, respectively, to the remainder so that the sum is left invariant.
- The latter fact is especially clear from the original path integral expression.
- Therefore there are jumps in the bound state spectra as a function of $k\beta_2$; in particular, to the R-charges of the Ramond ground states that contribute to the holomorphic part of the elliptic genus.

Relation to completed Appell-Lerch sums

The holomorphic Appell-Lerch sum is given by

$$A_{1,k}(\tau, u, v) = a^k \sum_{n\in\mathbb{Z}} \frac{q^{kn(n+1)} b^n}{1 - aq^n}$$

Here $a = e^{2\pi i u}$, $b = e^{2\pi i v}$ and $q = e^{2\pi i \tau}$. It was shown by Zwegers that, this can be completed to a Jacobi form by adding the following remainder term

$$\mathcal{R}_{1,k}(\tau, u, v) = \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} \left(\mathrm{sgn}(\nu) - \mathrm{Erf}\left[\sqrt{2\pi\tau_2}\left(\nu + \frac{\mathrm{Im}(u)}{\tau_2}\right)\right]\right) (-1)^{\nu-\frac{1}{2}} a^{-\nu} q^{-\frac{\nu^2}{2}}$$

We show explicitly in [arXiv 1404:7396] that

$$\boxed{\chi_L(\tau, \alpha, \beta) = \frac{i\theta_{11}(\tau, \alpha)}{\eta^3(\tau)}\ \widehat{A}_{1,k}(\tau, \frac{\alpha}{k}, 2\alpha - k\beta)}$$

Higher pole Appell-Lerch sums

- Dabholkar, Murthy and Zagier [DMZ] define an infinite number of higher pole Appell-Lerch sums; for intance,

$$A_{2,k} = \sum_{w} \frac{q^{kw^2+w} y^{kw} z^{-\frac{1}{k}-2w}}{(1 - z^{-\frac{1}{k}} q^w)^2}$$

  This can also be completed by adding a remainder and is (technically) a mock-Jacobi form.

- Can we give an interpretation to this $A_{2,k}$ in the CFT?

- What about its remainder? Is there a sum over states interpretation?

Modular derivatives: new modular forms from old

The main observation is that $A_{2,k}$ can be obtained from $A_{1,k}$ via a derivative w.r.t the chemical potentials; defining

$$\mathcal{D} = \frac{1}{2\pi i}\left(2\frac{\partial}{\partial\beta} + k\frac{\partial}{\partial\alpha}\right),$$

$$\mathcal{D}\cdot A_{1,k} = A_{2,k}\,.$$

On the modular completion, one can check that (with $\beta = \beta_1 + \tau\beta_2$)

$$\widehat{A}_{2,k} = (\mathcal{D} - k\beta_2)\widehat{A}_{1,k}$$

is a modular form of weight one and the same index as $A_{1,k}$.

Strategy: use our path integral representation of the elliptic genus (and hence $\widehat{A}_{1,k}$) to obtain a representation for the remainder.

Interpretation in CFT

If we write the remainder of $A_{1,k}$ in the form

$$\mathcal{R}_{1,k} = \sum_{v} S(\tau, \alpha, v)$$

where $v = n + kw \equiv$ right-moving momentum, then we find that if we set $\beta = 0$ after differentiation,

$$\mathcal{R}_{2,k}(\tau, \alpha) = \boxed{\sum_{v} v \; S(\tau, \alpha, v)} + Y(\tau, \alpha, v)\,.$$

i.e. we find an insertion of right-moving momentum plus an additional term Y, which is an ordinary partition sum.

Operator insertions from differentiation

Recall that

$$\chi(\tau,\alpha,\beta) = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{L_0}\bar{q}^{\bar{L}_0} z^{J_0} y^P$$

The modular covariant derivative $\mathcal{D}$ acting on χ leads to

$$\mathcal{D}\chi(\tau,\alpha,\beta) = \mathrm{Tr}_{\mathcal{H}}\left[(kJ_0 + 2P)\,(-1)^F q^{L_0}\bar{q}^{\bar{L}_0} z^{J_0} y^P\right]$$

In the cigar CFT, the R current is related to the angular momentum and fermion number as follows:

$$J_0 = -\frac{2}{k}P_L + F_L \qquad P = P_L + P_R\,.$$

$$\mathcal{D}\chi(\tau,\alpha,\beta) = \mathrm{Tr}_{\mathcal{H}}\left[(kF_L + \boxed{2P_R})\,(-1)^F q^{L_0}\bar{q}^{\bar{L}_0} z^{J_0} y^P\right]$$

The extra contribution to the completion Y can be explained studying the quantum mechanics for right-movers.
Proceed similarly for all $A_{n,k}$ and completions.

Summary of results

- Using the path integral formulation of the cigar CFT (as a gauged WZW model), we obtained the elliptic genus.
- We checked modularity, ellipticity and obtained a sum over states interpretation.
- The elliptic genus is the modular completion of a mock-Jacobi form. There is a holomorphic piece (discrete) and a non-holomorphic remainder (continuum).
- The remainder arises because of spectral asymmetry in the continuum sector.
- The elliptic genus can be identified with the completed Appell-Lerch sums studied by Zwegers in 2002.

Summary of results

- In fact, a twisted elliptic genus was computed, with the insertion of $e^{2\pi i\beta P}$ in the trace.
- One can observe crossing phenomena in this 2d CFT, where the contribution to the discrete spectrum jumps every time $k\beta_2$ crosses an integer. But the full elliptic genus is continuous in β. (Similar phenomena in $d = 4$.)
- All higher pole Appell-Lerch sums introduced by DMZ can be understood within the CFT as operator insertions of (powers of) right-moving momenta, augmented by extra terms corresponding to ordinary parition sums.

Some interesting questions

- Using a GLSM description, we obtained the elliptic genus of a two dimensional σ-model with target space

$$ds^2 = \frac{g_N(Y)}{2} dY^2 + \frac{2}{N^2 g_N(Y)}(d\psi + NA_{FS})^2 + 2Y\ ds^2_{\mathbb{CP}^{N-1}}\,,$$

$$\Phi = -\frac{NY}{k}\,.$$

These were first studied by Kiritsis-Kounnas-Lúst (KKL).

- These are **not** mock-Jacobi forms, according to the DMZ definition. i.e. the shadow

$$\chi_{\mathrm{shad}} = \partial_{\bar{\tau}} \chi$$

does not have good modular properties.

- What is the mathemtatical characterization of these real analytic Jacobi forms?

More questions to think about

- Is there a geometric criterion that determines when the elliptic genus shows mock behaviour? For instance, the Taub-NUT metric has the same form as the $d = 4$ KKL metric.

$$ds^2 = g(r)dr^2 + \frac{1}{g(r)}(d\psi + A_{FS})^2 + f(r)ds^2_{\mathbb{CP}^1}$$

  What is its elliptic genus? Does it show this type of mock behaviour? Or is it a holomorphic Jacobi form?

- Are there other observables that also show mock behaviour in the noncompact CFT?