Schouten identities
and the (diff. eq.s of the) two-loop sunrise graph

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New Frontiers in Theoretical Physics, Cortona 27.05 - 01.06.2014

Based on [arXiv:1311.3342], in collaboration with *Ettore Remiddi*

Why studying the two-loop massive sunrise?

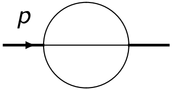
- **Phenomenologically:** needed for any two-loop calculation in realistic theories with *massive particles*

  1. Electro-weak sector of SM
  2. Massive QCD (*see* $gg \to t\bar{t}$ at two loops!)

- **Mathematically:** it is the first ($\approx$ *easiest*) diagram that escapes our understanding through *multiple-polylogs*.

  1. Functions: Elliptic functions, iterated integrals, **concept of transcendental weight**...?
  2. Differential Equation method: it is possible to find a **canonical basis**?

The two-loop massive sunrise graph

$\mathcal{I}(n_1, n_2, n_3, n_4, n_5) \quad =$ 

$$= \int \mathfrak{D}^d k \, \mathfrak{D}^d l \, \frac{(k \cdot p)^{n_4} \, (l \cdot p)^{n_5}}{(k^2 + m_1^2)^{n_1} (l^2 + m_2^2)^{n_2} ((k - l - p)^2 + m_3^2)^{n_3}}$$

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How do we compute all these integrals *(in dimensional regularisation!!)* ?

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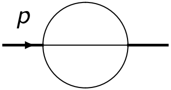
$\mathcal{I}(n_1, n_2, n_3, n_4, n_5) \quad = $ 

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Standard way to proceed:

- Use Integration-by-parts id.s (IBPs) to reduce them to **Master Integrals**

$$\int \mathfrak{D}^d k \, \mathfrak{D}^d l \, \left(v_j^\mu \frac{\partial}{\partial k_i^\mu} \frac{(k \cdot p)^{n_4} (l \cdot p)^{n_5}}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} \right) = 0$$

- Use IBPs to derive **differential equations** (DE) in the external invariants for the MIs.

$$p^2 = p_\mu \, p^\mu \quad \rightarrow \quad \frac{\partial}{\partial p^2} = \frac{1}{2p^2} \left(p^\mu \frac{\partial}{\partial p^\mu} \right)$$

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One finds for the Sunrise:

- Sunrise graph reduced to **4 MIs**

$$M_1(d;p^2)\,,\quad M_2(d;p^2)\,,\quad M_3(d;p^2)\,,\quad M_4(d;p^2)\,,$$

- Deriving DE we find 4 **coupled** differential equations:

$$\frac{\partial}{\partial p^2}\begin{pmatrix} M_1 \\ \dots \\ M_4 \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{14} \\ \dots & \dots & \dots \\ c_{41} & \dots & c_{44} \end{pmatrix}\begin{pmatrix} M_1 \\ \dots \\ M_4 \end{pmatrix} + \text{sub-topologies}$$

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- How to solve a system of differential equations?

Different **choices** of the basis of MIs can **simplify** DE.

- Try to triangularise the system as $d \to 4$
- We can try and solve it oder by order in $(d-4)$

    1. Is this *always* possible?
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- This concept has been **generalised** by J.Henn's Canonical Basis.

- Find a basis $m_j(d; p^2)$ such that the system takes canonical form

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  Where now c_{ij} is function ONLY of p^2!!

  $\rightarrow$ also now some criteria start to be known!
  [arXiv:1304.1806; arXiv:1401.2979; arXiv:1404.2922; arXiv:1404.4853]

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$\rightarrow$ 4 **coupled** DE in $d = 4$ and in $d = 2$ (and in *any even number of dimensions...*).

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$$\widetilde{S} = \int \mathfrak{D}^d k \mathfrak{D}^d l \frac{1}{D_1\, D_2\, D_3}\,, \quad \widetilde{S}_1 = \int \mathfrak{D}^d k \mathfrak{D}^d l \frac{1}{D_1^2\, D_2\, D_3}\,,$$
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Can we generalise this? $\longrightarrow$ Schouten (Pseudo-)Identities!

- Valid only in **integer** number of dimensions $\rightarrow$ *pseudo-identities*!
- in $d = n \in \mathbb{N}$ dimensions only n vectors can be **linearly independent**.

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   a^μ, b^μ cannot be independent $\rightarrow$ $a^\mu b^\nu \epsilon_{\mu\nu} = 0$.
2. $d = 2$ **dimensions**:
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"good basis" of MIs as $d \approx n \in \mathbb{N}$?

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Squaring relations above we obtain **Schouten polynomials**:

- For Example take in 3 dimensions

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  Squaring it we get

$$\rightarrow \; P_3(a,b,c) = a^2 b^2 c^2 - a^2 (b\cdot c)^2 - b^2 (a\cdot c)^2 - c^2 (a\cdot b)^2 + 2(a\cdot b)(b\cdot c)(a\cdot c)$$

(**Gram determinants** of the n vectors!)

The polynomial can now be considered as d dimensional!
(d never appears on r.h.s!)

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Sunrise depends on **3** vectors $k, l, p \to$ Schouten in 2 dimensions!!

$\to$ “natural” number of dimensions for the Sunrise is **2**!

The Sunrise satisfies a **second-order differential equation** in $d = 2$!!

[arXiv:1112.4360] by S.Müller-Stach, S.Weinzierl, R.Zayadeh

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$$P_3(d; k, l, p) = k^2 l^2 p^2 - k^2 (l \cdot p)^2 - l^2 (k \cdot p)^2 - p^2 (k \cdot l)^2 + 2(k \cdot l)(l \cdot p)(k \cdot p)$$

Note that as $k, l \to \infty$ then $P_3(d; k, l, p) \approx k^2 l^2$

Consider now:

$$Z(d; n_1, n_2, n_3) = \int \mathfrak{D}^d k \mathfrak{D}^d l \, \frac{P_3(d; k, l, p)}{D_1^{n_1} D_2^{n_2} D_3^{n_3}}$$

Power-counting in $d = 2$

$Z(d \to 2; 1, 1, 1)$ UV div, $Z(d \to 2; 2, 1, 1)$ UV div, (with permutations)

$Z(d \to 2; 2, 2, 1)$, $Z(d \to 2; 2, 1, 2)$, $Z(d \to 2; 1, 2, 2)$, UV finite!

and all higher powers of denominators!

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For all Zs that are finite we then also have at once:

$$Z(d \to 2; n_1, n_2, n_3) \to 0\,.$$

- We can the **reduce** them to the **4 MIs**

$$Z(d; n_1, n_2, n_3) = \sum_{i=0}^{3} C_i(d; p^2)\, S_i(d; p^2) + \text{sub-topologies}$$

- This gives in principle a **relation** among the MIs in $d = 2$!

$$0 = \sum_{i=0}^{3} C_i(d \to 2; p^2)\, S_i(d \to 2; p^2) + \text{sub-topologies}\Big|_{d \to 2}$$

- **Question:** There are **infinite converging** Zs!
  Which ones give non-trivial information (if any...)?

Apparently **only those** with the minimal powers required for **convergence**!

For all Zs that are finite we then also have at once:

$$Z(d \to 2; n_1, n_2, n_3) \to 0 \,.$$

- We can the **reduce** them to the **4 MIs**

$$Z(d; n_1, n_2, n_3) = \sum_{i=0}^{3} C_i(d; p^2)\, S_i(d; p^2) + \text{sub-topologies}$$

- This gives in principle a **relation** among the MIs in $d = 2$!

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Some examples of Zs:

- If the powers are too high **no information** as $d \to 2$

$$Z(d;2,2,2) = -\frac{(d-1)(d-2)}{4} \times \left[(d-3)S(d;p^2) + m_1^2 S_1(d;p^2) + m_2^2 S_2(d;p^2) + m_3^2 S_3(d;p^2)\right]$$

- On the contrary:

$$\begin{aligned} Z_2(d;p^2) &= Z(d;2,1,2,p^2) \\ &= \frac{(d-1)}{12}\left[-(d-2)p^2 + (d-3)(m_1^2 - 2m_2^2 + m_3^2)\right] S(d;p^2) \\ &+ \frac{(d-1)}{12}(p^2 + m_1^2 - 3m_2^2 + 3m_3^2)\ m_1^2 S_1(d,p^2) \\ &- \frac{(d-1)}{6}(p^2 + m_2^2)\ m_2^2 S_2(d;p^2) \\ &+ \frac{(d-1)}{12}(p^2 + 3m_1^2 - 3m_2^2 + m_3^2)\ m_3^2 S_3(d;p^2) \\ &+ \frac{(d-1)(d-2)}{24}\left[T(d;m_1,m_2) - 2\,T(d;m_1,m_3) + T(d;m_2,m_3)\right] \end{aligned}$$

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Which in $d = 2$ becomes:

$$\begin{aligned} Z_2(2; p^2) = 0 = & -\frac{1}{12}(m_1^2 - 2m_2^2 + m_3^2) S(2; p^2) \\ & + \frac{1}{12}(p^2 + m_1^2 - 3m_2^2 + 3m_3^2)\ m_1^2 S_1(2, p^2) \\ & - \frac{1}{6}(p^2 + m_2^2)\ m_2^2 S_2(2; p^2) \\ & + \frac{1}{12}(p^2 + 3m_1^2 - 3m_2^2 + m_3^2)\ m_3^2 S_3(2; p^2) \\ & + \frac{1}{96} \ln \frac{m_2^2}{m_1 m_3} \end{aligned}$$

In the very same way we can obtain similar relations for:

$$Z_1(d;p^2) = Z(d;1,2,2)\,, \qquad Z_3(d;p^2) = Z(d;2,2,1)\,,$$

- One **two** of them are really independent in $d = 2$:

$$Z_1(d;p^2) + Z_2(d;p^2) + Z_3(d;p^2) = -\frac{(d-1)(d-2)}{4}p^2\; S(d;p^2)\;,$$

- We can choose 2 of them as **new MIs**! For example choose as basis:

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Differential equations in the new basis are in **block form** in $d = 2$

$$\frac{d}{d\,p^2}\begin{pmatrix} S \\ S_1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{pmatrix}\begin{pmatrix} S \\ S_1 \\ Z_2 \\ Z_3 \end{pmatrix} + \text{sub-topologies}$$

$$\frac{d}{d\,p^2}\begin{pmatrix} Z_2 \\ Z_3 \end{pmatrix} = (d-2)\begin{pmatrix} c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}\begin{pmatrix} S \\ S_1 \\ Z_2 \\ Z_3 \end{pmatrix} + \text{sub-topologies}$$

The second system **decouples** completely in $d = 2$!

$\rightarrow$ order by order in $(d-2)$ we can derive a **second order differential equation** satisfied by $S(d; p^2)$. [see Weinzierl et al., 2011/2013]

- What about the **alternative basis** $\tilde{S}_j$ (found *by chance*...)??

- Let us express Z_1, Z_2, Z_3 in terms of the $\widetilde{S}_j$!

$$Z_1 = -\,\frac{(d-1)(d-2)}{4}\,\widetilde{S}_2$$

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Conclusions and outlook

- Schouten identities are a **new class** of (pseudo-)identities among MIs!
- Schouten identities can be used to select a **"good" basis** of MIs as $d \approx n \in \mathbb{N}$
- They show that in $d = 2$ the Sunrise has **only** 2 **MIs**
  (the other two are linearly dependent!)

  $\rightarrow$ **Decoupling** of the system in $\underline{d = n}$ seems to be due to *"hidden relations"* between the MIs in $\underline{d = n}$???!!!

1. Can this be used in more involved cases?
2. What happens with IR divergences?

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Thanks!

Back-up slide

What about $d = 4$?

- Schouten Polynomial has also the role of **shifting the dimensions** $d \to d+2$ [R. Lee]

- Implies that:

$$Z_1(d;p^2) = \int \mathfrak{D}^d k \mathfrak{D}^d l \, \frac{P_3(d;k,l,p)}{D_1^1 \, D_2^2 \, D_3^2} \approx \int \frac{\mathfrak{D}^{d+2} k \, \mathfrak{D}^{d+2} l}{D_1^1 \, D_2^2 \, D_3^2}$$

and the same way for the others...

- Which implies that

$$\int \frac{\mathfrak{D}^d k \, \mathfrak{D}^d l}{D_1^1 \, D_2^2 \, D_3^2} \quad \text{and its permutations...}$$

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