Correlators of Chiral Primaries and 1/8 Wilson Loops from Perturbation Theory

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Outline

$\mathcal{N} = 4$ Super Yang-Mills

- The maximally supersymmetric gauge theory in d=4;
- Gauge supermultiplet $(A_\mu, \lambda^a_\alpha, \Phi^I)$ transform under $SU(4)_R$ $(\mathbf{1},\mathbf{4},\mathbf{6})$;
- Only 2 free parameters N and g_{YM};
- Superconformal symmetry $PSU(2,2|4)$;
- SUSY transformation

$$\delta\Phi = \lambda, \qquad \delta A_\mu = \bar{\lambda},$$
$$\delta\bar{\lambda} = D\Phi, \qquad \delta\lambda = F + [\Phi, \Phi].$$

- Three class of observables
  - Scattering amplitudes,
  - Correlation functions of local operators,
  - Wilson loops.

Localization

Obtained by deforming the action of the theory by Q-exact term

$$S_{YM} \to S(t) = S_{YM} + tQV$$

with $V = (\Psi, \overline{Q\Psi})$ and setting t to infinity.

- Q squares to a symmetry of the theory, and the action and the Wilson loop observable must be Q-closed;
- At the limit $t \to \infty$ we shall integrate in the path integral over the configurations solving $Q\Psi = 0$;
- The partition function and the expectation value of observables do not depend on the t-deformation (compact space of fields);
- Localize usually means computing a classical action (fluctuation around classical solution) $\to$ Matrix model.

Outline

Chiral Primary Operator (CPO)

Chiral primary operators (CPO) $\mathcal{O}(x)$ are local operator annihilated by the SUSY generators S and a number of Q generators.

- Conformal dimension pretected by quantum correction;
- Supersymmetric objects (BPS).

In general

$$\mathcal{O}(x) = tr\left(\Phi^i, D_\mu, F_{\mu\nu}, \lambda_a\right)(x)$$

For example

$$\mathcal{O}^{ij} = tr\left(\Phi^i\Phi^j - \frac{1}{6}\delta^{ij}\Phi^2\right)$$

Another class of CPO operator

$$\mathcal{O}_J(x) = tr[u_I(x)\Phi^I(x)]^J$$

[N.Drukker and J.Plefka '09]

On the 2-sphere

$$\mathcal{O}_J(x) = tr[x_i\Phi^i + i\Phi^4]^J$$

Wilson loop

Non-local gauge-invariant observable

$$W(C) = \frac{1}{N} tr\mathcal{P} \exp \oint_C dx \; iA_\mu(x)$$

QCD$\rightarrow$ phase factor related to the propagation of a massive quark in the fundamental representation of the gauge group.
$\mathcal{N}$**=4 SYM**$\rightarrow$ scalar coupling

$$W(C) = \frac{1}{N} tr\mathcal{P} \exp \oint_C d\tau \; iA_\mu(x)\dot{x}^\mu(\tau) + \dot{y}_I(\tau)\Phi^I$$

Can be made supersymmetric by a suitable choice of scalar couplings

- Zarembo $\dot{y}_I = M_I^\mu \dot{x}_\mu \rightarrow \langle W \rangle = 1$ [K.Zarembo '02]
- DGRT $\dot{y}_I = -\dot{x}_\mu \sigma_i^{\mu\nu} x_\nu M_I^i \rightarrow$ loop on S^3 [N.Drukker, S.Giombi, R.Ricci, D.Trancanelli '07]

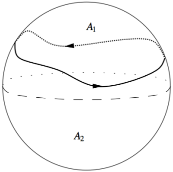
The number of preserved supercharges depends on the path $x_\mu(\tau)$

From $\mathcal{N}$=4 SYM in $d = 4$ to pure YM in $d = 2$ (I)

If we restrict the DGRT contour on a two sphere the situation is more interesting

$$W = \frac{1}{N} tr\mathcal{P} \exp \oint d\tau (iA_\mu \dot{x}^\mu + \epsilon_{\mu\nu\rho} \dot{x}^\mu x^\nu \Phi^\rho)$$

Indeed at leading order the expectation value of these WL on S^2 is



$$\langle W \rangle = 1 + g_{4d}^2 N \frac{A_1 A_2}{2A^2} + \mathcal{O}(g_{4d}^4)$$

The VEV resembles a similar result for pure 2d Yang-Mills on S^2.

From $\mathcal{N}$=4 SYM in $d = 4$ to pure YM in $d = 2$ (II)

The pure YM theory in 2d is almost topological (invariant under area preserving diffeomorphism) and exactly soluble. An exact compact expression for the VEV on Wilson Loop on S^2 is known

$$\langle W \rangle = \frac{1}{N} L^1_{N-1}\left(g^2_{2d} \frac{A_1 A_2}{A} \right) \exp\left[- g^2_{2d} \frac{A_1 A_2}{2A} \right]$$

[A.Bassetto and L.Griguolo '98]

DGRT conjecture ($g^2_{2d} = -g^2_{4d}/A$)

VEV of Wilson loop on S^2 in $\mathcal{N} = 4$ SYM in 4d $\Longleftrightarrow$ **VEV of Wilson loop on S^2 in pure YM in 2d**

Is true for correlation function with local BPS operator too?

[N.Drukker, S.Giombi, R.Ricci, D.Trancanelli '07]

The matrix model

$\mathcal{N}=4$ SYM on $S^2 \xrightarrow{localization}$ Zero instanton sector of pure 2d YM on $S^2 \rightarrow$ Gaussian multi-matrix model [S.Giombi and V. Pestun '09]

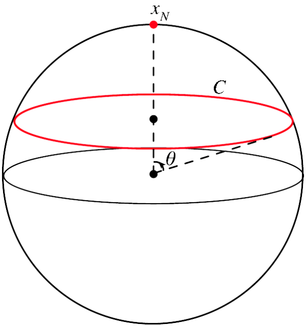
$$\langle W_{\mathcal{R}_1}[\mathcal{C}_1] W_{\mathcal{R}_2}[\mathcal{C}_2] \ldots \mathcal{O}_{J_1}(x_1) \mathcal{O}_{J_2}(x_2) \ldots \rangle_{4d} = \frac{1}{\mathcal{Z}} \int [dX][dY] \mathrm{Tr}_{\mathcal{R}_1} e^{X_1} \mathrm{Tr}_{\mathcal{R}_2} e^{X_2} \ldots \mathrm{Tr}\, Y_1^{J_1} \mathrm{Tr}\, Y_2^{J_2} \ldots e^{S[X,Y]}$$

- $S[X,Y]$ is a quadratic form in X_i, Y_i;
- Coefficients depend on the areas singled out by the WLs;

2-points function

$$\langle W_{\mathcal{R}}[\mathcal{C}] \mathcal{O}_J(x_1) \rangle = \frac{1}{\mathcal{Z}} \int [dX][dY] \mathrm{Tr}_{\mathcal{R}} e^X \mathrm{Tr}\, Y^J e^{-\frac{A^2}{2g_{YM}} \mathrm{Tr}\left(\frac{A_1}{A_2} Y^2 - \frac{2i}{A_2} XY\right)}$$

Outline



Operators and propagators

- Latitude on S^2

$$W[\mathcal{C}] = \frac{1}{N} \mathrm{Tr}\mathcal{P} \exp \oint d\tau \, \mathcal{A}(x(\tau)),$$

where

$$\mathcal{A}(x(\tau)) = (iA_\mu \dot{x}^\mu + \sin^2\theta \, \Phi^3 - \sin\theta\cos\theta(\sin\tau \, \Phi^2 + \cos\tau \, \Phi^1));$$

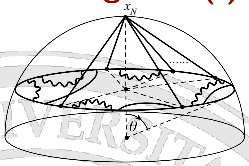
- CPO operator $\mathcal{O}_2(x_N) = \left(\frac{2\pi}{\sqrt{\lambda}}\right)^2 \frac{1}{\sqrt{2}} \mathrm{Tr}\Big(\Phi^3(x_N) + i\Phi^4(x_N)\Big)^2;$
- Effective propagators

$$\langle \mathcal{A}_i^{ab} \, \mathcal{A}_j^{cd} \rangle = \frac{\lambda'}{16\pi^2} \frac{\delta^{ad}\delta^{bc}}{N},$$

$$\langle \mathcal{A}_i^{ab} \, \Phi^{I\,cd}(x_N) \rangle = \frac{\lambda'}{16\pi^2} \frac{\delta_{I3}}{1-\cos\theta} \frac{\delta^{ad}\delta^{bc}}{N},$$

where $\lambda' = 4\lambda A_1 A_2 / A^2 = \lambda \sin^2\theta$.

Ladder diagrams (I)



$$\langle W[\mathcal{C}]\mathcal{O}_J(x_N)\rangle_{\text{ladder}} = \frac{1}{N}\sum_{n=0}^{\infty}\int_0^{2\pi} d\tau_1 \ldots \int_0^{\tau_{2n+J-1}} d\tau_{2n+J}\,\langle \mathrm{Tr}(\mathcal{A}_1 \ldots \mathcal{A}_{2n+J})\,\mathcal{O}_J(x_N)\rangle,$$

$$\mathcal{A}_1\,\overbrace{\mathcal{A}_2\ldots\mathcal{A}_{2s_1+1}}^{2s_1}\,\mathcal{A}_{2s_1+2}\,\overbrace{\mathcal{A}_{2s_1+3}\ldots\mathcal{A}_{2s_1+2s_2+2}}^{2s_2}\ldots//\ldots\overbrace{\mathcal{A}_{2n+J-2s_{J-1}}\ldots\mathcal{A}_{2n+J-1}}^{2s_{J-1}}\,\mathcal{A}_{2n+J}\quad\overbrace{\Phi^3_N\,\Phi^3_N\ldots//\ldots\Phi^3_N}^{J}$$

Using effective propagators

$$\langle \mathrm{Tr}(\mathcal{A}_1 \ldots \mathcal{A}_{2n+J})\,\mathcal{O}_J(x_N)\rangle = \left(\frac{\lambda'}{16\pi^2}\right)^{n+J}\left(\frac{1}{1-\cos\theta}\right)^{J} N_{\text{tot}},$$

$$N_{\text{tot}} = (2n+J)\sum_{s_1=0}^{n} N_{s_1}\sum_{s_2=0}^{n-s_1} N_{s_2}\sum_{s_3=0}^{n-s_1-s_2}\ldots//\ldots\sum_{s_{J-1}=0}^{n-\sum_{i=1}^{J-2}s_i} N_{s_{J-1}}\,N_{n-\sum_{i=1}^{J-2}s_i-s_{J-1}},$$

N_{tot} is the number of total planar graph and

$$N_s = \frac{(2s)!}{(s+1)!\,s!}.$$

Ladder diagrams (II)

Using the recurrence relation

$$N_{n+1} = \sum_{k=0}^{n} N_{n-k}\, N_k , \qquad \text{with} \quad N_0 = 1$$

[J.K.Erickson, G.W.Semenoff and K.Zarembo '00]

Performing $(J-1)$ sums, we get

$$N_{\text{tot}} = \frac{J(2n+J)!}{n!(n+J)!} .$$

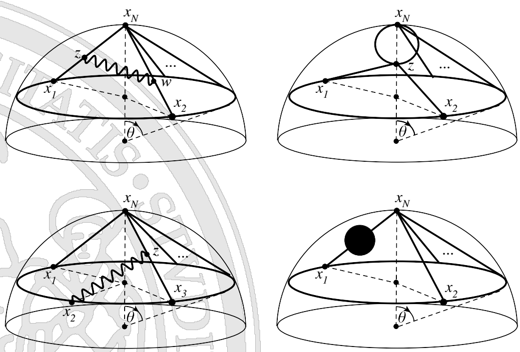
Finally

$$\langle W[\mathcal{C}]\, \mathcal{O}_J(x_N) \rangle_{\text{ladder}} = \frac{1}{N} \frac{\sqrt{J}}{2^J} \left(\frac{A_2}{A_1} \right)^{J/2} I_J(\sqrt{\lambda'}) .$$

NB: the sum of all ladder contribution reproduces the localization result!! [S.Giombi, V.Pestun '12]

Interaction diagrams (I)

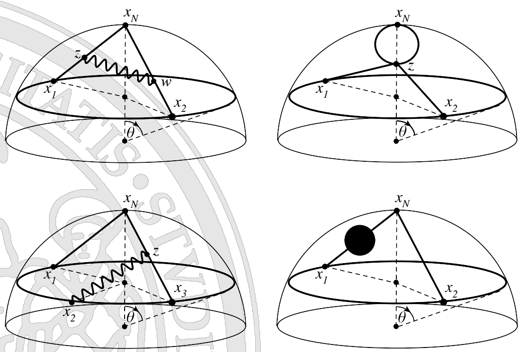
Diagrams with interaction vertices $\langle W[\mathcal{C}]\mathcal{O}_J \rangle_{int}$ at order λ^2



- Interactions involve at most 2 adjacent legs
- Every diagram is divergent;
- Particular combinations of graphs are finite.

Interaction diagrams (I)

Diagrams with interaction vertices $\langle W[\mathcal{C}]\mathcal{O}_2\rangle_{int}$ at order λ^2



- Interactions involve at most 2 adjacent legs
- Every diagram is divergent;
- Particular combinations of graphs are finite.

Interaction diagrams (II)

Adding up the contributions of all the interacting diagrams we get

$$\langle W[\mathcal{C}]\mathcal{O}_2(x_N)\rangle_{int} = \mathbf{H}+\mathbf{O}+\mathbf{X}+\mathbf{IY} =$$

$$-\frac{\lambda'^2}{2^3\sqrt{2}N}\frac{1}{(1-\cos\theta)^2}\oint d\tau_1\, d\tau_2\;\; \mathcal{I}_1(x_1-x_2, x_N-x_2)\,(x_1-x_2)^2 \qquad \to P_1$$

$$+\frac{\lambda'^2}{2\sqrt{2}N}\frac{1}{1-\cos\theta}\oint d\tau_1\, d\tau_2\left[\mathcal{I}_1(x_1-x_N, x_2-x_N)+\mathcal{I}_2(x_2-x_N, x_1-x_N)\right] \qquad \to P_2$$

$$-\frac{\lambda'^2}{2\sqrt{2}N}\frac{1}{1-\cos\theta}\oint d\tau_1\, d\tau_2\left[\mathcal{I}_2(x_2-x_N, x_2-x_N)+\mathcal{I}_1(0, x_2-x_N)\right] \qquad \to P_3$$

$$-\frac{\lambda'^2}{2^8\pi^4\sqrt{2}N}\frac{1}{(1-\cos\theta)^2}\oint d\tau_1 d\tau_2\left[\mathrm{Li}_2\left(1-\frac{\sin^2\theta}{1-\cos\theta}(1-\cos\tau_{21})\right)-\frac{\pi^2}{6}\right] \qquad \to P_4$$

with

$$\mathcal{I}_n(x,y) = (-1)^{n+1}\frac{\Gamma(2\omega-3)}{64\pi^{2\omega}(\omega-1)}\int_0^1 d\alpha\,\frac{\alpha^{\omega-3+n}(1-\alpha)^{\omega-2}}{[\alpha(1-\alpha)x^2+(y-\alpha x)^2]^{2\omega-3}}$$

$$\times\,{}_2F_1\left(1, 2\omega-3, \omega, \frac{(y-\alpha x)^2}{(y-\alpha x)^2+\alpha(1-\alpha)x^2}\right).$$

Interaction diagrams (III)

Performing the previous integrals

$$P_1 = \frac{1}{8\pi^2}\left[\frac{\pi^2}{6} - \mathrm{Li}_2\left(\sin^2\frac{\theta}{2}\right)\right];$$

$$P_2 = \frac{1}{96\pi^2}\frac{1}{1-\cos\theta}\left[\pi^2 - 3\,\mathrm{Li}_2\left(\sin^2\frac{\theta}{2}\right) - 6\left(\log^2\left(\cos\frac{\theta}{2}\right) + \arctan^2\left(\sqrt{1+2\cos\theta}\right)\right)\right];$$

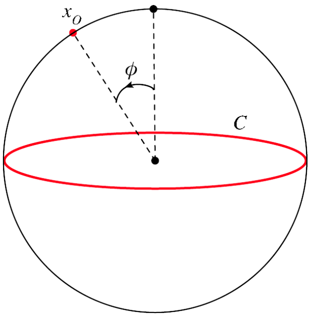
$$P_3 = \frac{1}{192}\frac{1}{1-\cos\theta};$$

$$P_4 = \frac{2}{3}\pi^4 - 8\pi^2\left[\log^2\left(\cos\frac{\theta}{2}\right) + \arctan^2\left(\sqrt{1+2\cos\theta}\right)\right].$$

Thus

$$\langle W[\mathcal{C}]\mathcal{O}_2(x_N)\rangle_{int} = -\frac{\lambda'^2}{2^3\sqrt{2}N}\frac{1}{(1-\cos\theta)^2}\left[P_1 - \frac{1-\cos\theta}{4}(P_2 - P_3) + \frac{P_4}{2^5\pi^4} - \frac{1}{3\cdot 2^4}\right] = 0$$

Outline



Operators and propagators

- Equator on S^2

$$W[\mathcal{C}] = \frac{1}{N} \mathrm{Tr}\mathcal{P} \exp \oint d\tau \, (iA_\mu \dot{x}^\mu + \Phi^3);$$

- CPO operator

$$\mathcal{O}_2(x_\mathcal{O}) = \left(\frac{2\pi}{\sqrt{\lambda}}\right)^2 \frac{1}{\sqrt{2}} \mathrm{Tr}\Big(\sin\phi \, \Phi^1(x_\mathcal{O}) + \cos\phi \, \Phi^3(x_\mathcal{O}) + i\Phi^4(x_\mathcal{O})\Big)^2;$$

- Effective propagators

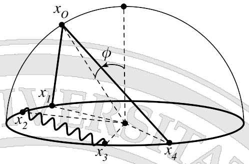
$$\langle \mathcal{A}_i^{ab} \; \mathcal{A}_j^{cd} \rangle = \frac{\lambda}{16\pi^2} \frac{\delta^{ad}\delta^{bc}}{N},$$

$$\langle \mathcal{A}_i^{ab} \; \Phi^{I\,cd}(x_\mathcal{O}) \rangle = \frac{\lambda}{16\pi^2} f(\tau_i) \cos\phi \frac{\delta^{ad}\delta^{bc}}{N} \delta_{I3},$$

where

$$f(\tau_i) = \frac{1}{1 - \sin\phi \cos\tau_i}.$$

Ladder diagram



$$\langle W[\mathcal{C}]\mathcal{O}_2(x_\mathcal{O})\rangle_{\text{ladder}} = \frac{1}{N}\int_0^{2\pi} d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 \, \langle \text{Tr}(\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4)\, \mathcal{O}_2(x_\mathcal{O})\rangle.$$

The effective propagators depend on $\tau \rightarrow$ ladder diagram can't reproduce the matrix model result

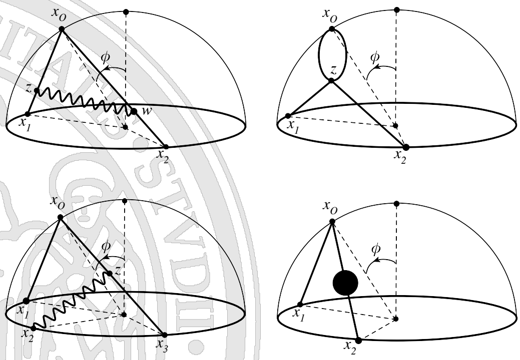
$$\langle W[\mathcal{C}]\mathcal{O}_2(x_\mathcal{O})\rangle_{\text{ladder}} = \frac{\lambda^2}{192\sqrt{2}N} + \mathbf{L}(\sigma)$$

where $\sigma = \sqrt{\frac{1+\sin\phi}{1-\sin\phi}}$

$$\mathbf{L}(\sigma) = -\frac{\lambda^2}{2^6\sqrt{2}\pi^2}\left[\log\left(\frac{2\sigma}{1+\sigma}\right)^2 + \log\left(\frac{1+\sigma}{2}\right)^2 + 2\text{Li}_2\left(\frac{1-\sigma}{2}\right) + 2\text{Li}_2\left(\frac{\sigma-1}{2\sigma}\right)\right]$$

Interaction diagrams (I)

We want to compute the contribution of diagrams with interaction vertices $\langle W[\mathcal{C}]\mathcal{O}_2 \rangle$ at order λ^2



- Every diagram is divergent;
- Particular combinations of graphs are finite.

Interaction diagrams (II)

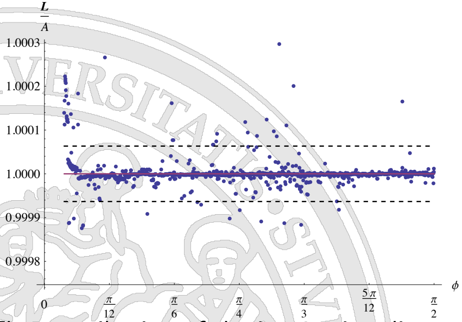
Adding up the contributions of all the interacting diagrams we get

$$
\begin{aligned}
&\langle W[\mathcal{C}]\mathcal{O}_2(x_\mathcal{O})\rangle_{\text{int}} = \\
&\left.\begin{aligned}
&-\frac{\lambda^2\cos^2\phi}{2^3\sqrt{2}N}\oint d\tau_1 d\tau_2\, f(\tau_1)\, f(\tau_2)\;\; \mathcal{I}_1(x_1-x_2, x_\mathcal{O}-x_2)\,(x_1-x_2)^2\\
&+\frac{\lambda^2\cos^2\phi}{2\sqrt{2}N}\oint d\tau_1 d\tau_2\, f(\tau_1)\left[\mathcal{I}_1(x_1-x_\mathcal{O}, x_2-x_\mathcal{O})+\mathcal{I}_2(x_2-x_\mathcal{O}, x_1-x_\mathcal{O})\right]\\
&-\frac{\lambda^2\cos^2\phi}{2^8\pi^4\sqrt{2}N}\oint d\tau_1 d\tau_2\, f(\tau_1)\, f(\tau_2)\left[\mathrm{Li}_2\Big(1-(1-\cos\tau_{21})f(\tau_1)\Big)\right]
\end{aligned}\right\}\ \mathrm{A}\\
&+\frac{\lambda^2\cos^2\phi}{2^9\pi^4\sqrt{2}N}\oint d\tau_1 d\tau_2 d\tau_3\,\epsilon(\tau_1,\tau_2,\tau_3)\, f(\tau_1)\, f(\tau_3)\,\cot\left(\frac{\tau_{32}}{2}\right)\log\left(\frac{f(\tau_3)}{f(\tau_2)}\right). \qquad \mathrm{B}
\end{aligned}
$$

- B is computed analytically and $B = -2\mathbf{L}$
- A is computed numerically

Interaction diagrams (III)

We plot the ratio $\frac{\mathbf{L}}{A}$ as a function of angle ϕ



$$A = \mathbf{L}$$

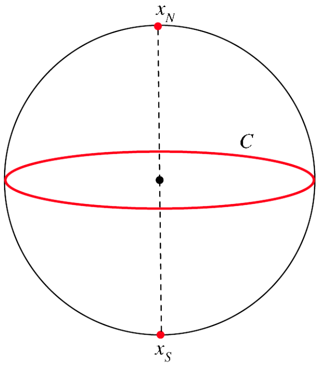
The contribution of the interacting diagrams is therefore

$$\langle\, W[\mathcal{C}]\mathcal{O}_2(x_{\mathcal{O}})\,\rangle_{\text{int}} = -\mathbf{L}(\sigma).$$

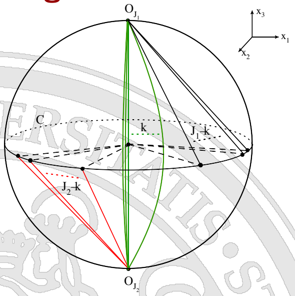
Thus

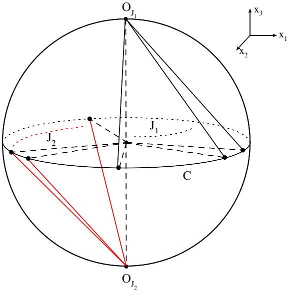
$$\langle\, W[\mathcal{C}]\mathcal{O}_2(x_{\mathcal{O}})\,\rangle = \langle\, W[\mathcal{C}]\mathcal{O}_2(x_{\mathcal{O}})\,\rangle_{\text{int}} + \langle\, W[\mathcal{C}]\mathcal{O}_2(x_{\mathcal{O}})\,\rangle_{\text{ladder}} = \frac{\lambda^2}{192\sqrt{2}N}$$

Outline



Ladder diagrams



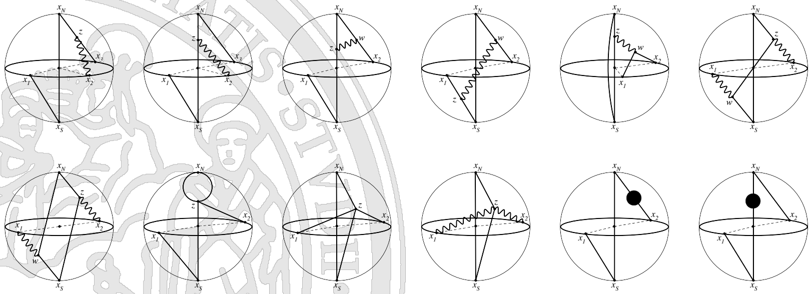


$$\langle W[\mathcal{C}]\,\mathcal{O}_{J_1}(x_N)\,\mathcal{O}_{J_2}(x_S)\rangle = \frac{1}{N} J_1 J_2 \left(\frac{i\sqrt{\lambda}}{4\pi}\right)^{J_1} \left(\frac{-i\sqrt{\lambda}}{4\pi}\right)^{J_2}$$

$$\times \begin{cases} \frac{\sqrt{\lambda}}{2} I_1(\sqrt{\lambda}) - \sum_{k=1}^{\frac{|J_1-J_2|-2}{2}} (2k) I_{2k}(\sqrt{\lambda}) & \text{if } J_1+J_2 \text{ even} \\ \frac{\sqrt{\lambda}}{2} I_2(\sqrt{\lambda}) - \sum_{k=1}^{\frac{|J_1-J_2|-3}{2}} (2k+1) I_{2k+1}(\sqrt{\lambda}) & \text{if } J_1+J_2 \text{ odd} \end{cases}$$

Interaction diagrams (Work in progress)

The ladder diagram resummation confirm the localization result $\rightarrow$ all the interaction diagrams must sum to zero.



Outline

Conclusions and outlook

Conclusions

- 2-points functions match the localization results
  - Latitude-north pole ladder diagrams reproduce all the matrix model and we have checked the interaction cancellation at order λ^2 (for any J);
  - Equator-arbitrary point ladder diagrams involve an extra term: at order λ^2 it is cancel by the interaction diagrams;
- 3-point function $\langle W\mathcal{O}\mathcal{O} \rangle$ ladder diagrams match the localization result;

Outlook

- Check the cancellation of $\langle W\mathcal{O}\mathcal{O} \rangle_{int}$;
- Compute $\langle WW\mathcal{O} \rangle$

Thank you!