Unitarity techniques in 2D

based on 1304.1798 with B. Hoare, V. Forini, 1405.**** and work in progress

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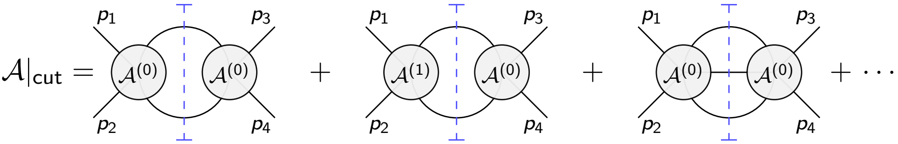
Outline

Motivation

- Combine a powerful technique with the special properties of 1+1 dimensions.
- Improve perturbative computations in integrable non-linear sigma models.
- Understand the connection between cut constructibility and integrability.
- Perform non-trivial checks of quantum integrability for classically integrable string backgrounds.
- Compute overall scalar functions for symmetry-determined S-matrices.
- Moving towards the perturbative computation of off-shell quantities.

The method

Standard unitarity in 4d [Bern, Dixon, Dunbar, Kosower, 1994]



Glue together the two amplitudes and uplift the integral with

$$i\pi\delta^+(p^2 - m^2) \to \frac{1}{p^2 - m^2 - i\epsilon}$$

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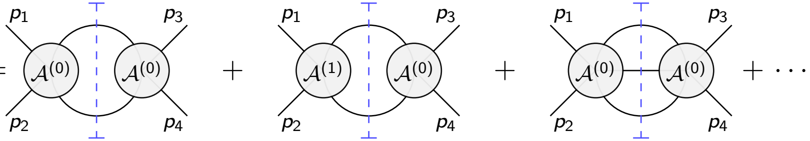

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$$\mathcal{A}^L = \sum_i c_i \, \mathcal{I}_i^{(L)}$$

$\longrightarrow$ Known basis of L-loop scalar integrals

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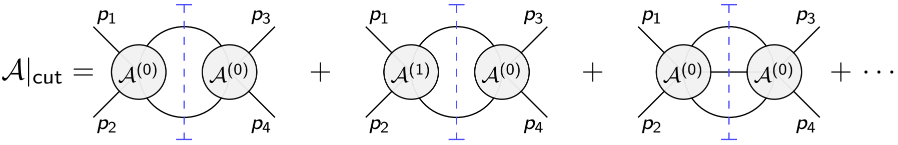

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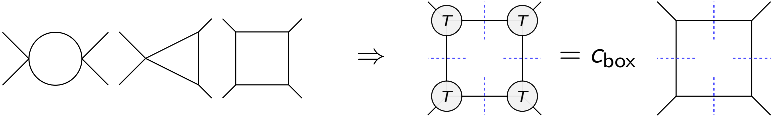

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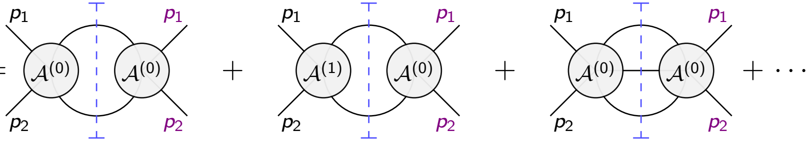

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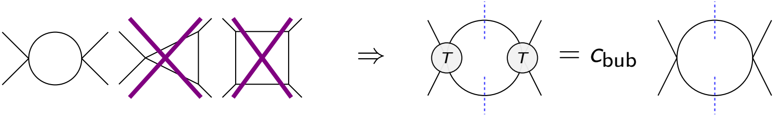
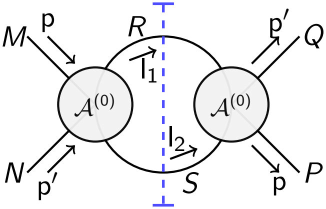
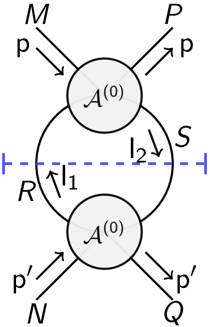


s-channel



$T_{MN}^{RS}(p,p')\,T_{RS}^{PQ}(p,p')$

t-channel



$\frac{1}{2}\,T_{MR}^{SP}(p,p)\,T_{SN}^{RQ}(p,p')$

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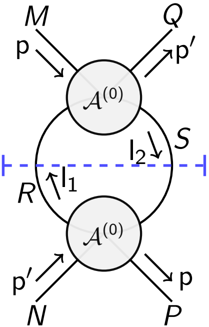
u-channel



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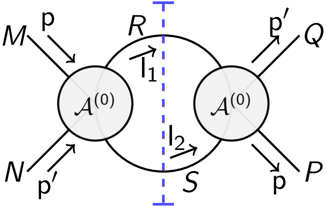

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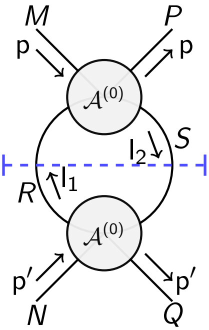

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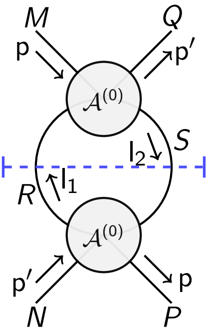

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The result

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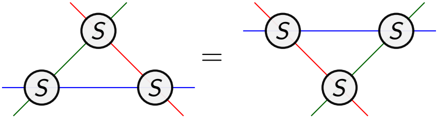
Yang-Baxter equation



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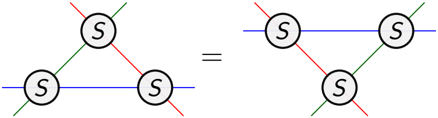
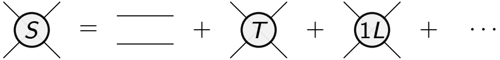


Expanding the S-matrix perturbatively



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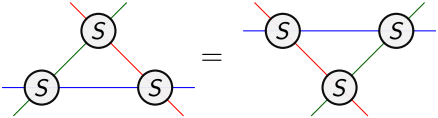

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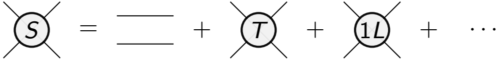
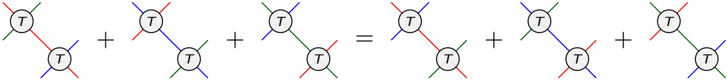


Tree-level YB



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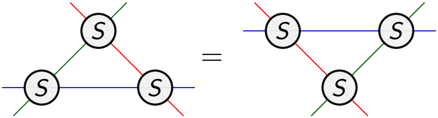

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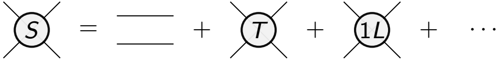

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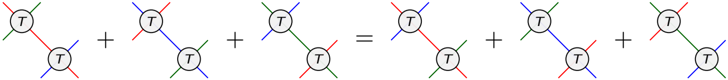

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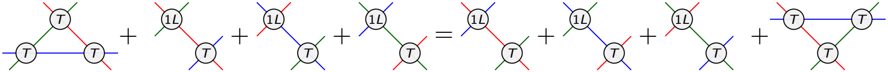

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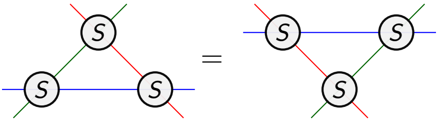

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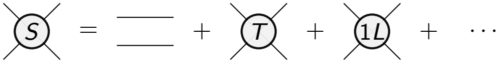

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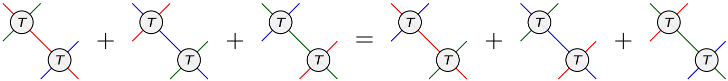
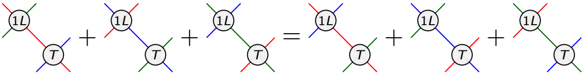


One loop YB - Homogeneous part



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↓

- Satisfies the inhomogeneous YB equation for any TL S-matrix (proven).
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- For the full result to satisfy YB equation it should be a solution to the homogeneous part of YB.
- It is again $\propto \mathbb{1}$ in all the cases but one: $AdS_3 \times S^3 \times S^3 \times S^1$. Why?

External leg correction

"Unwanted" contribution

$$\frac{1}{4\pi}\left(\frac{p^2}{m}+\frac{p'^2}{m'}\right)\ T^{(0)}\ .$$

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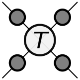
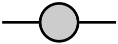


with

$$= \frac{i(1+h^{-1}\Sigma_1(p))}{\mathbf{p}^2-m^2-h^{-1}\Sigma_0(p)}+\dots$$

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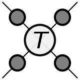

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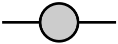
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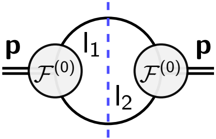

$$= \Sigma_0(p) + \Sigma_1(p)(\mathbf{p}^2 - m^2) + \mathcal{O}((\mathbf{p}^2 - m^2)^2)$$

Applications

The technique has been applied to several two-dimensional integrable models:

- Bosonic relativistic models (generalized Sine-Gordon); [Hollowood, Miramontes, Park, 1994; Bakas, Park, Shin, 1995]
- Fermionic relativistic models (Pohlmeyer reduced theories for GS string in $AdS_5 \times S_5$ and truncations) ; [Grigoriev, Tseytlin, 2007; Mikhailov, Schafer-Nameki, 2007]
- Non-relativistic models (worldsheet scattering for non-linear sigma model in $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times M^4$).[Metsaev, Tseytlin, 1998; Pesando, 1998; Rahmfeld, Rajaraman, 1998]

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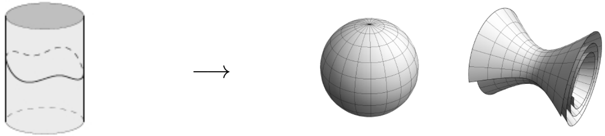
Non-linear sigma model



$$\mathbb{R} \times S^1 \longrightarrow \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$$

Worldsheet scattering in $AdS_5 \times S^5$

Lagrangian of the non-linear sigma model [Metsaev, Tseytlin, 1998]

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- Decompactification limit to define asymptotic states

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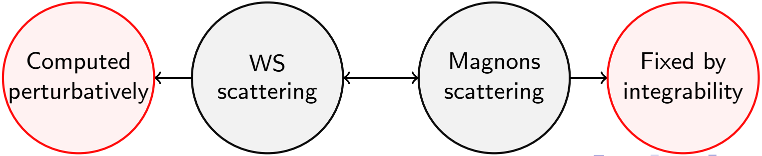

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Future directions

- We found hints of a relation with YB equation which deserves further analysis.
- Rational terms beyond one loop.
- Study of off-shell objects via unitarity.