New Frontiers in Theoretical Physics
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Off-Critical Interfaces in Two Dimensions

Exact Results from Field Theory

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Outline

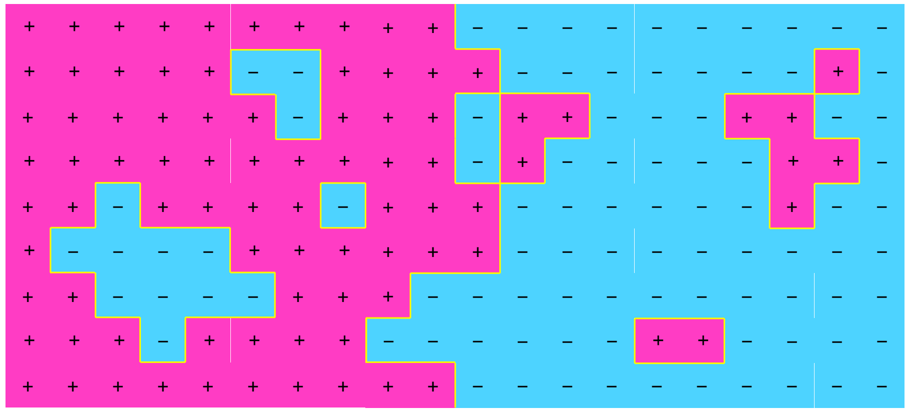
- Introduction
  - field-theoretic description of phase separation and off-critical interfaces
- Simple interfaces
  - order parameter, passage probability, interface structure
  - specific models: Ising & q-Potts
- Double interfaces
  - o.p., passage probability
  - tricritical q-Potts interfaces
  - bulk wetting transition & Ashkin-Teller
- Interfaces at boundaries
  - wedge geometry, filling transition and wedge covariance
- Conclusions

Based on:

- Gesualdo Delfino, J. Viti, Phase separation and interface structure in two dimensions from field theory, *J. Stat. Mech. (2012) P10009, [arXiv:1206.4959]*
- Gesualdo Delfino, AS, Interfaces and wetting transition on the half plane. Exact results from field theory, *J. Stat. Mech. (2013) P05010, [arXiv:1303.1938]*
- Gesualdo Delfino, AS, Exact theory of intermediate phases in two dimensions, *Annals of Physics 342 (2014) 171, [arXiv:1310.4425]*
- Gesualdo Delfino, AS, Phase separation in a wedge. Exact results, *[arXiv:1403.1138]*

Interfaces in two dimensions

Several approaches by: rigorous, numerical and exact statistical mechanics (this talk)



From LATTICE: exact transfer matrix results are available ONLY for Ising. So far no substantial progresses beyond Ising universality class.

…and from FIELD THEORY?

$T=T_C$: Interfaces are conformally invariant random curves described by Schramm-Loewner evolution (SLE). Connection with 2d CFT applied at criticality but very few is known about massive deformations.

THE ISSUES

How to avoid lattice calculations and work directly in the continuum?

Beyond Ising? How to find the order parameter of the q-Potts?

How to use field-theory for off-critical interfaces?

Ising solutions are the only available, is this due to integrability?

Does integrability play any role?

$T<T_C$ (this talk): we propose a NEW and EXACT approach to phase separation based on local fields. The role of integrability is elucidated. Field theory yields exact solutions for a larger variety of models with a simpler language.

Field-theoretic description

Scaling limit of a ferromagnetic (discrete) spin model below T$_c$

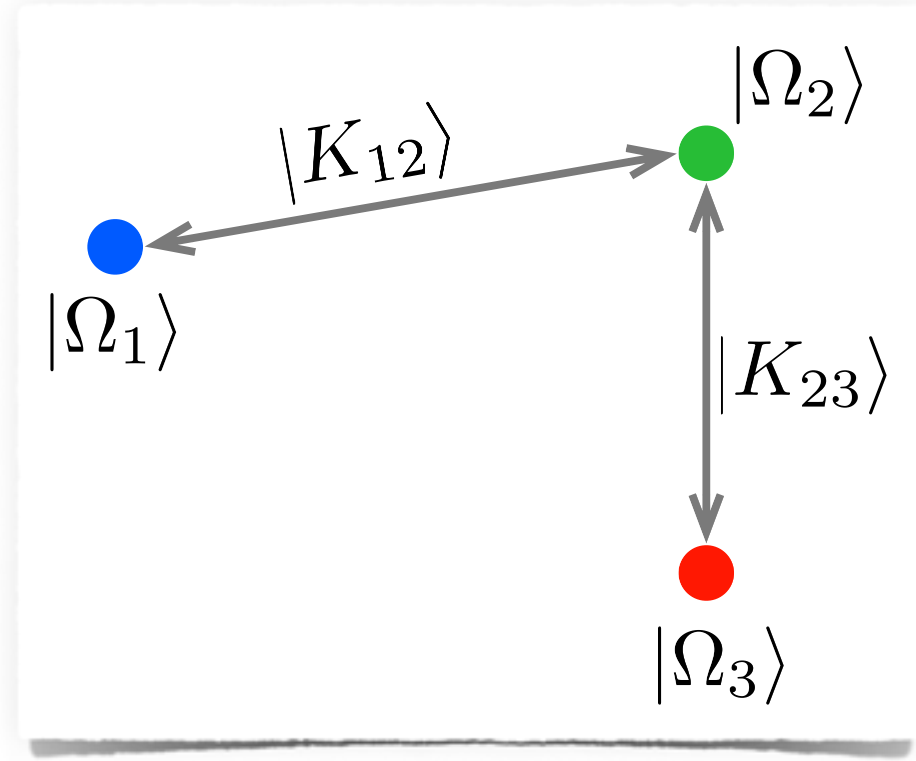
How? Analytic continuation of (1+1)-relativistic field theory to a 2d Euclidean field theory in the plane (y=-it).

coexisting phases $\longleftrightarrow$ *degenerate vacua set* $\{\Omega_{a_i}\}_{i=1,\ldots,n}$

domain walls $\longleftrightarrow$ *kinks (elementary excitations in 2D)*

$|K_{a_i,a_j}(\theta)\rangle$ *interpolating between* $|\Omega_{a_i}\rangle$, $|\Omega_{a_j}\rangle$

connectivity of vacuum structures



Excitations of the theory

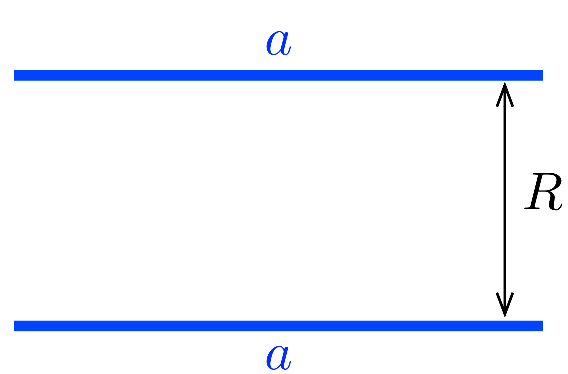
single kink state: $|K_{a_1,a_2}(\theta)\rangle$

n-kink state: $|K_{a,c_1}(\theta_1)K_{c_1,c_2}(\theta_2)\ldots K_{c_{n-1},b}(\theta_{n+1})\rangle$

adjacency rule: $c_i \neq c_{i+1}$

Pure phases

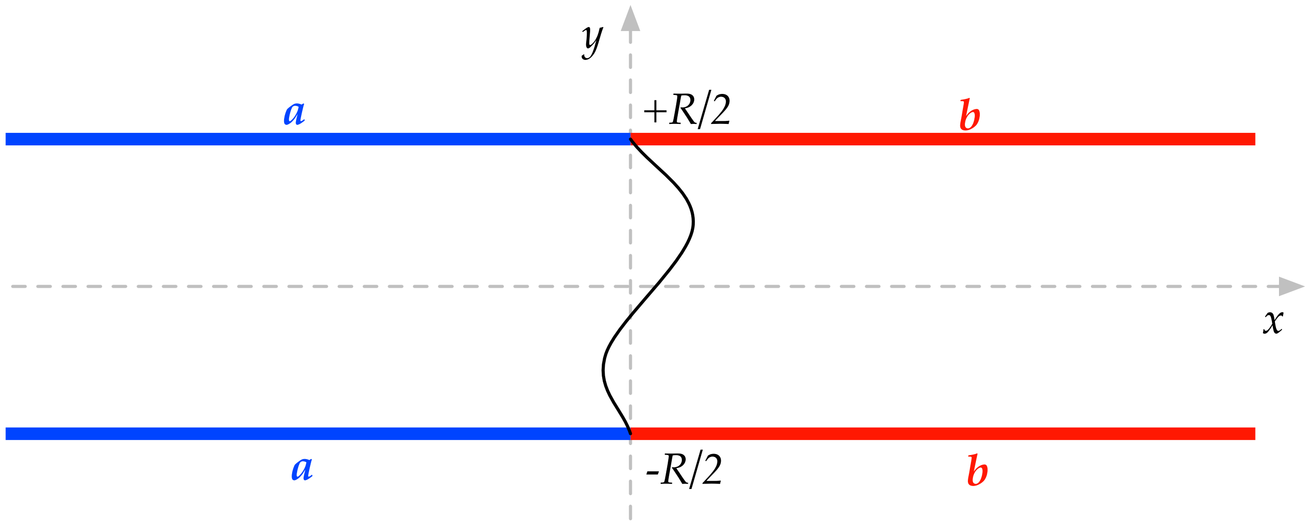
no phase separation



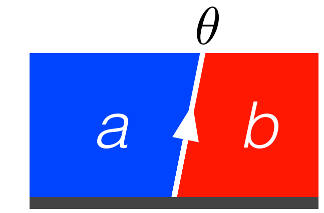
$R \longrightarrow \infty$

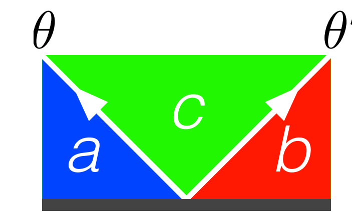
$\sigma_a \equiv \langle\Omega_a|\sigma(x,y)|\Omega_a\rangle$

Phase separation for adjacent phases



$$\left\{ \begin{aligned} f_{ab}(\theta) &= \\ f_{acb}(\theta,\theta') &= \end{aligned} \right.$$

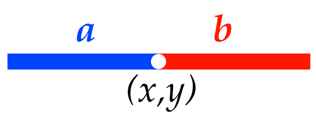




emission amplitudes

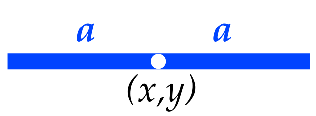
Boundary states:

$$B_{ab}(x,y) =$$



$$= \mathrm{e}^{yH+ixP}\left[\int \frac{\mathrm{d}\theta}{2\pi} f_{ab}(\theta)|K_{ab}(\theta)\rangle + \sum_{c\neq a,b} \iint \frac{\mathrm{d}\theta \mathrm{d}\theta'}{(2\pi)^2} f_{acb}(\theta,\theta')|K_{ac}(\theta)K_{cb}(\theta')\rangle + \dots\right]$$

$$B_a(x,y) =$$



$$= \mathrm{e}^{yH+ixP}\left[|\Omega_a\rangle + \sum_{c\neq a} \iint \frac{\mathrm{d}\theta \mathrm{d}\theta'}{(2\pi)^2} f_{aca}(\theta,\theta')|K_{ac}(\theta)K_{ca}(\theta')\rangle + \dots\right]$$

higher mass

Partition functions (leading order)

$$\mathcal{Z}_a(R) = \langle B_a(x,R/2)|B_a(x,-R/2)\rangle \sim \langle \Omega_a|\Omega_a\rangle = 1$$

$$\mathcal{Z}_{ab}(R) = \langle B_{ab}(x,R/2)|B_{ab}(x,-R/2)\rangle \sim \frac{|f(0)|^2}{\sqrt{2\pi m_{ab} R}} \mathrm{e}^{-m_{ab}R}$$

Interfacial tension

$$\Sigma_{ab} = -\lim_{R\to\infty} \ln \frac{Z_{ab}(R)}{Z_a(R)}$$

$$\Sigma_{ab} = m_{ab}$$

VERY IMPORTANT: *Scaling limit:* $mR \gg 1 \quad (R \gg \xi) \implies$ *projection to low energy physics:* $\theta, \theta' \ll 1$

Order parameter profile

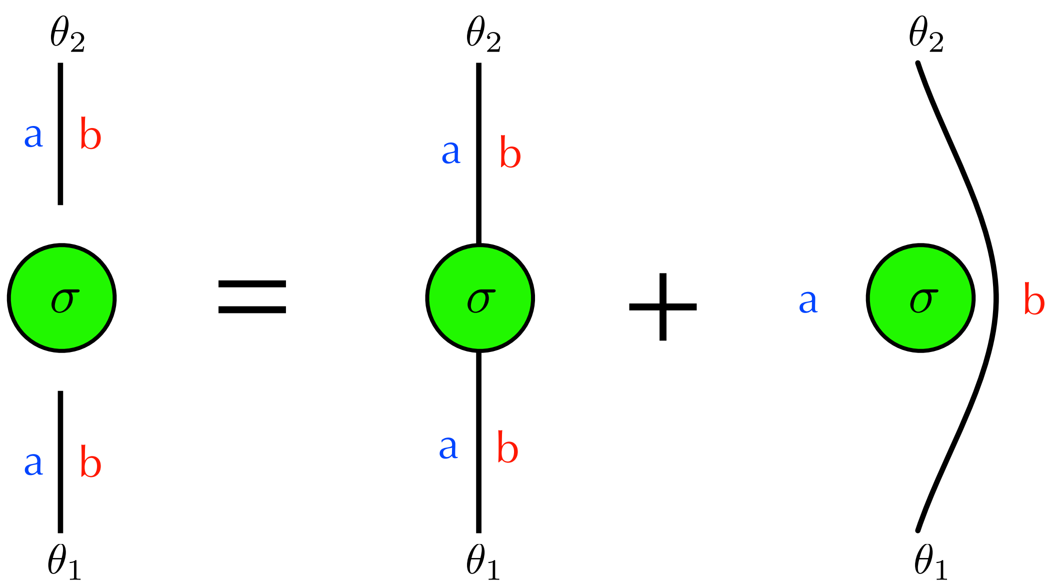
Exp. value of the spin field along the horizontal axis.

$$\langle\sigma(x,0)\rangle_{ab} = \frac{1}{Z_{ab}}\langle B_{ab}(0,iR/2)|\sigma(x,0)|B_{ab}(0,-iR/2)\rangle$$

$$\simeq \frac{|f_{ab}(0)|^2}{Z_{ab}}\iint\frac{\mathrm{d}\theta_1\mathrm{d}\theta_2}{(2\pi)^2}\mathcal{M}^\sigma(\theta_1,\theta_2)\,\mathrm{e}^{-mR\left(1+\frac{\theta_1^2}{4}+\frac{\theta_2^2}{4}\right)-imx\theta_{12}} \qquad (mR\gg 1)$$

Matrix element decomposition

$$\mathcal{M}^\sigma_{ab}(\theta_1,\theta_2) = F^\sigma_{aba}(\theta_{12}+i\pi) + 2\pi\delta(\theta_{12})\langle\sigma\rangle_a$$



Two-kink form factor

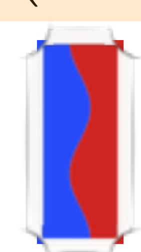
$$F^\sigma_{aba}(\theta_{12}+i\pi) = \frac{i\left(\langle\sigma\rangle_a - \langle\sigma\rangle_b\right)}{\theta_{12}} + \sum_{k=1}^{\infty} c^{(k)}_{ab}\theta^k_{12}$$

crossing symmetry: the residue **does not depend on integrability** (annihilation of two kinks)

[Berg, Karowski, Weisz, '78;
Smirnov, 80's;
Delfino, Cardy, '98]

$$\therefore\quad \langle\sigma(x,0)\rangle_{ab} = \frac{\langle\sigma\rangle_a + \langle\sigma\rangle_b}{2} - \frac{\langle\sigma\rangle_a - \langle\sigma\rangle_b}{2}\,\mathrm{erf}\left(\sqrt{\frac{2m}{R}}\,x\right) + c^{(0)}_{ab}\sqrt{\frac{2}{\pi m R}}\,\mathrm{e}^{-\frac{2mx^2}{R}} + \dots$$

non-locality of kinks w.r.t. spin field: sharp phase separation



sharp interface

local: interface structure

bifurcation

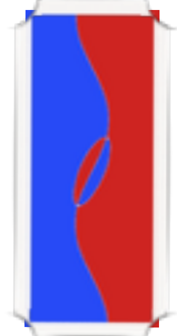
Ising model (broken $\mathbb{Z}_2$)

$$\langle\sigma\rangle_+ = -\langle\sigma\rangle_+$$

$$\langle\sigma(x,0)\rangle_{\mp} = \langle\sigma\rangle_{\pm}\mathrm{erf}\left(\sqrt{\frac{2m}{R}}x\right)$$

perfect match with lattice derivation [Abraham '81]

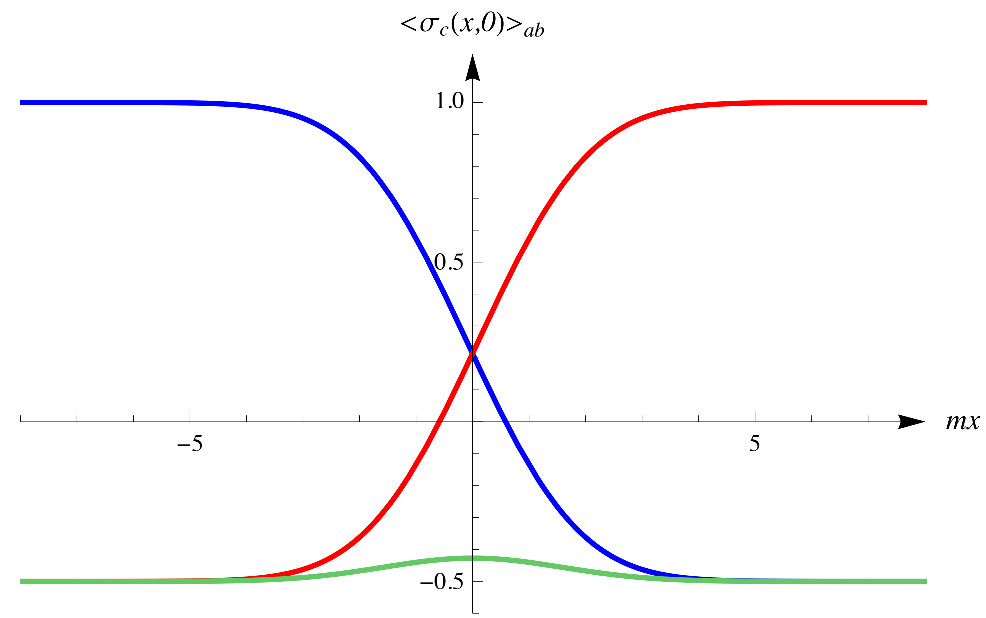
$c_{2k} = 0$



next correction: 3-furcation

q-state Potts (broken S_q)

$$\sigma_c(x) = \delta_{s(x),c} - \frac{1}{q} \qquad \langle\sigma_c\rangle_a = \frac{q\delta_{ac}-1}{q-1}M$$



the island is suppressed for $mR \gg 1$

$$\langle\sigma_3(0,0)\rangle_{12} \sim (mR)^{-1/2}$$

$$c_{ab}^{(0)} = \left[2 - q(\delta_{ac}+\delta_{bc})\right]B(q) \qquad \begin{cases} B(3) = \dfrac{M}{4\sqrt{3}} \\ B(4) = \dfrac{M}{3\sqrt{3}} \end{cases}$$

Magnetization in the strip

it amounts in the rescaling

$$x \longrightarrow \chi \equiv \frac{x}{\kappa}$$

$$\begin{cases} \chi \equiv \sqrt{\dfrac{2m}{R}}\dfrac{x}{\kappa} \\ \kappa \equiv \sqrt{1-4y^2/R^2} \end{cases}$$

contour lines are arcs of ellipses

very important: *Branching is a general phenomenon not due to integrability. But for integrable theories we can compute the amplitude of the island.*

Passage probability and interface structure

The interface will cross the horizontal axis (y=0) in u, with passage probability p(u)du, how the magnetization in x is affected?

$$\langle \sigma(x,0)\rangle_{ab} = \int_{\mathbb{R}} \mathrm{d}u\, \sigma_{ab}(x|u) p(u)$$

$$\sigma_{ab}(x|u) = \theta(u-x)\langle\sigma\rangle_a + \theta(x-u)\langle\sigma\rangle_b + A_{ab}^{(0)}\delta(x-u) + A_{ab}^{(1)}\delta'(x-u) + \dots$$

local terms: branching & recombination

matching with field theory

$$\therefore \begin{cases} p(u) = \sqrt{\dfrac{2m}{\pi R}} \mathrm{e}^{-\frac{2mx^2}{R}} \\ A_{ab}^{(0)} = c_{ab}^{(0)}/m \end{cases}$$

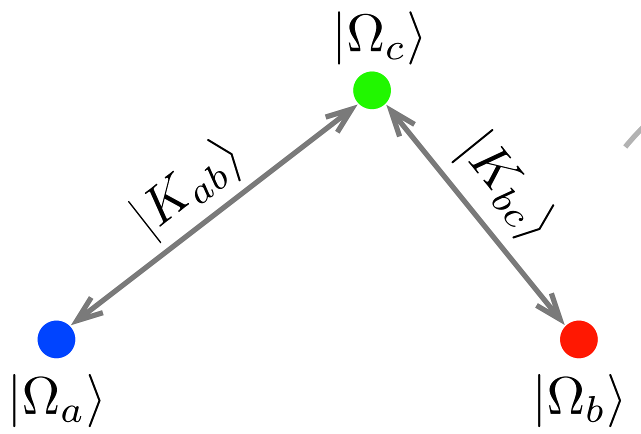
gaussian pdf → ***brownian bribge***

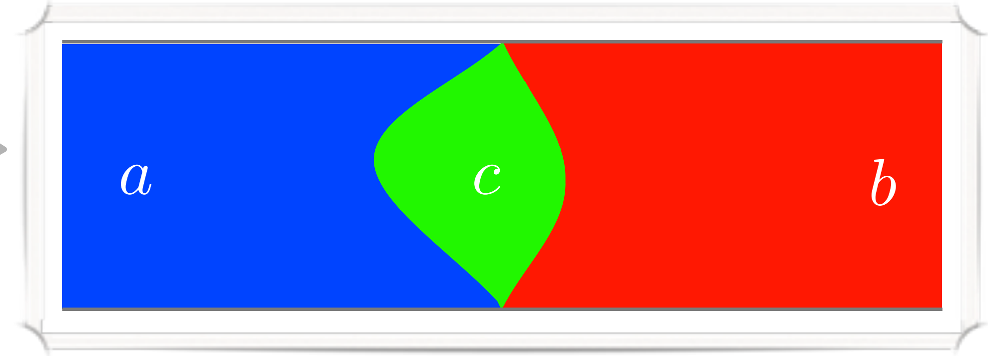
Renormalization Group perspective: large R/ξ expansion

- $R/\xi = \infty \longrightarrow$ *sharp interface picture: two clusters*
- $R/\xi \gg 1 \longrightarrow$ *proliferation of inclusions: bubbles of different phases*

Double interfaces

The vacua $|\Omega\rangle_a$ *and* $|\Omega\rangle_b$ *cannot be connected by a single kink*

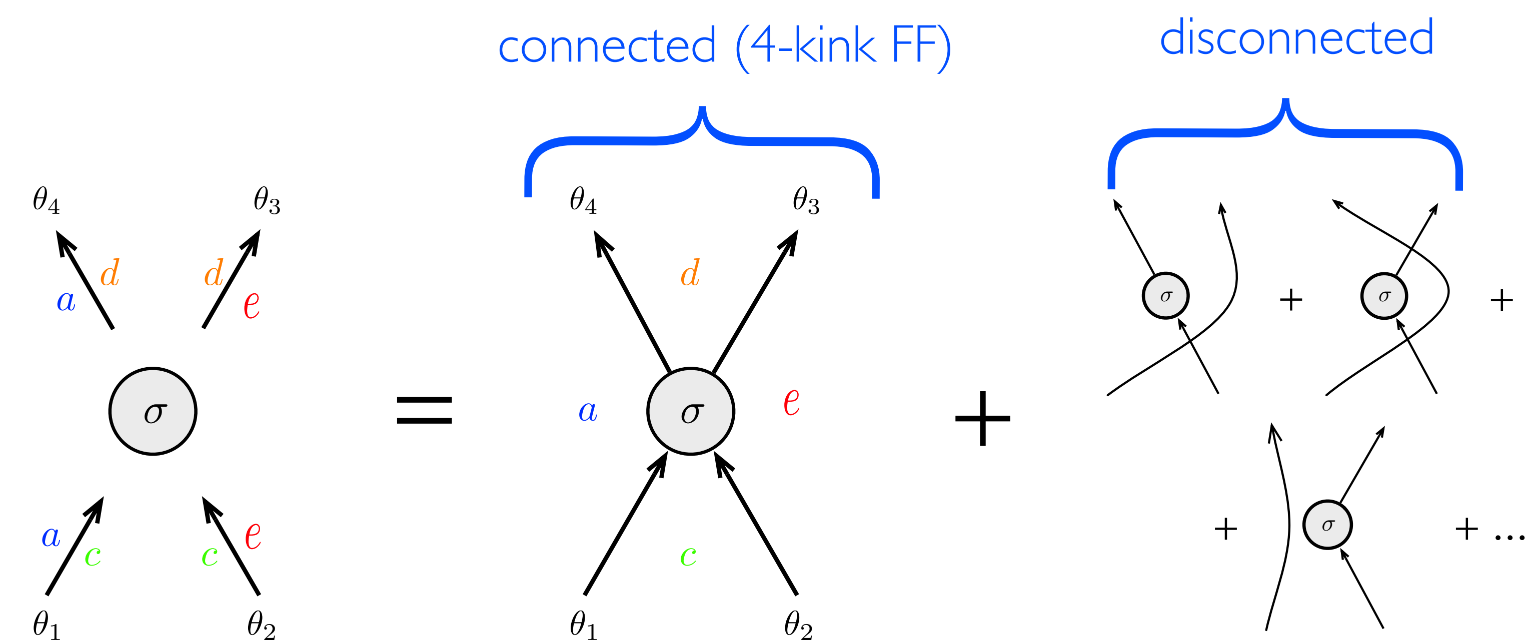




$$|B_{ab}(0,t)\rangle = \mathrm{e}^{-itH}\left[\iint \frac{\mathrm{d}\theta_1}{2\pi}\frac{\mathrm{d}\theta_2}{2\pi} f_{acb}(\theta_1,\theta_2)|K_{ac}(\theta_1)K_{cb}(\theta_2)\rangle + \dots\right]$$

an insight…

$$\langle K_{bd}(\theta_3)K_{da}(\theta_4)|\sigma|K_{ac}(\theta_1)K_{cb}(\theta_2)\rangle =$$



Order parameter profile

$$\therefore\ \langle\sigma(x,y)\rangle_{ab} = \frac{\langle\sigma\rangle_a + \langle\sigma\rangle_b - 2\langle\sigma\rangle_c}{4}\mathcal{G}(\chi) + \frac{\langle\sigma\rangle_a - \langle\sigma\rangle_b}{2}\mathcal{L}(\chi) + \frac{\langle\sigma\rangle_a + \langle\sigma\rangle_b - 2\langle\sigma\rangle_c}{4}$$

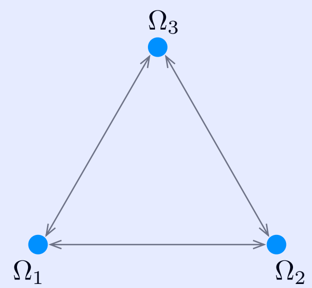
$$\mathcal{G}(\chi) = -\frac{2}{\pi}\mathrm{e}^{-2\chi^2} - \frac{2}{\sqrt{\pi}}\chi\,\mathrm{erf}(\chi)\,\mathrm{e}^{-\chi^2} + \mathrm{erf}^2(\chi) \qquad \mathcal{L}(\chi) = \frac{\chi}{\sqrt{\pi}}\mathrm{e}^{-\chi^2} - \mathrm{erf}(\chi)$$

Tricritical q-state Potts

Annealed vacancies are allowed. If no vacancies: pure q-state Potts

continuous transitions

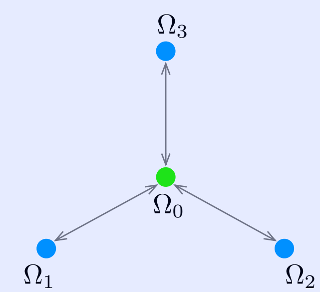
$T < T_c,\ \rho = 0,\ q \leqslant 4$

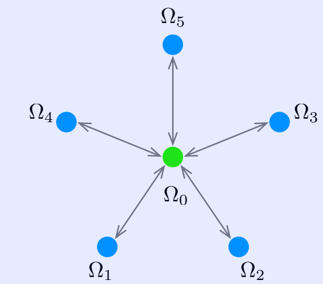


q deg. vacua

first-order transitions

$T = T_c,\ \rho > \rho_c,\ q \leqslant 4$





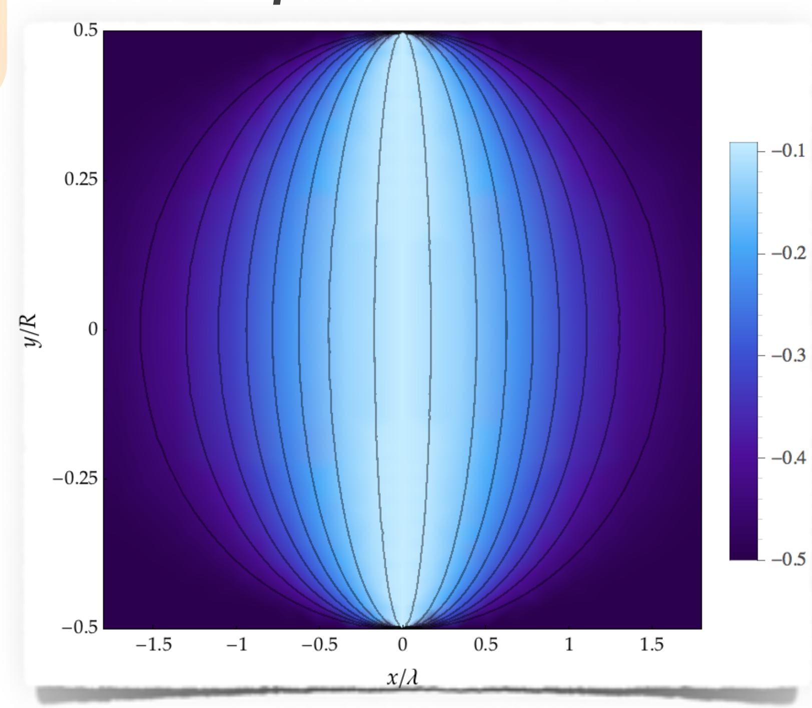
q+1 deg. vacua

$T = T_c, q > 4$ *(q-4 not large:* $q = 10, \xi \approx 10$*)*

$$\therefore \begin{cases} \langle \sigma_1(x,y)\rangle_{12} = \dfrac{\langle\sigma_1\rangle_1}{2}\left[\dfrac{q-2}{2(q-1)}\left(1-\dfrac{2}{\pi}e^{-2\chi^2}-\dfrac{2}{\sqrt{\pi}}\chi\,\mathrm{erf}(\chi)e^{-\chi^2}+\mathrm{erf}^2(\chi)\right)+\dfrac{q}{q-1}\left(\dfrac{\chi}{\sqrt{\pi}}e^{-\chi^2}-\mathrm{erf}(\chi)\right)\right] \\ \langle \sigma_3(x,y)\rangle_{12} = -\dfrac{\langle\sigma_1\rangle_1}{2(q-1)}\left(1-\dfrac{2}{\pi}e^{-2\chi^2}-\dfrac{2}{\sqrt{\pi}}\chi\,\mathrm{erf}(\chi)e^{-\chi^2}+\mathrm{erf}^2(\chi)\right) \end{cases}$$

$\mathcal{G}(\chi)$ { asymptotic of Ising <**σσσ**> [McCoy,Wu '78]
Ising bubbles [M Abraham, Upton '93]

q=3, dilute



Passage probability matches field theory with

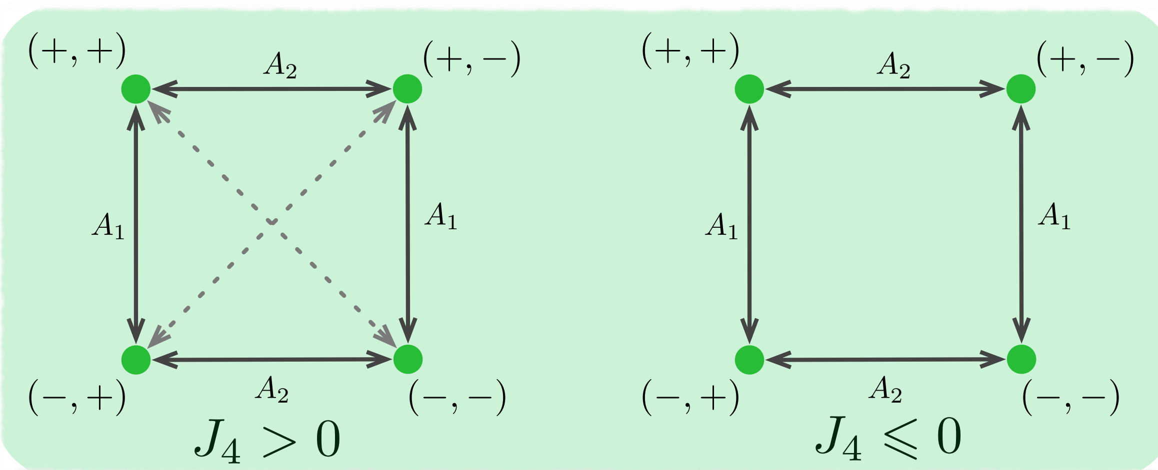
$$p(x_1,x_2) = \frac{2m}{\pi R}(\eta_1-\eta_2)^2\,e^{-(\eta_1^2+\eta_2^2)}$$

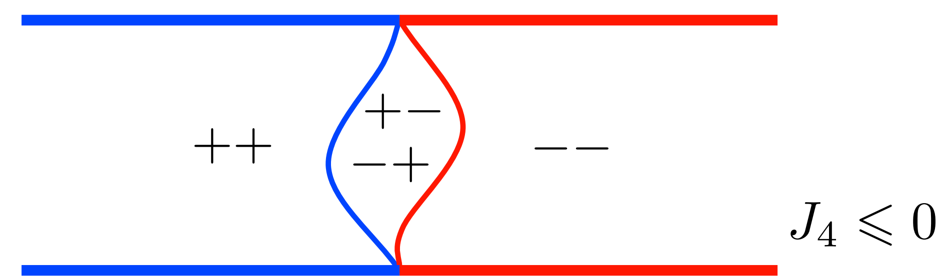
mutually avoiding interfaces

Bulk wetting transition: Ashkin Teller

$$\mathcal{H}_{AT} = -\sum_{\langle x_1, x_2 \rangle} \left\{ \left[J\sigma_1(x_1)\sigma_1(x_2) + \sigma_2(x_1)\sigma_2(x_2) \right] + J_4 \sigma_1(x_1)\sigma_1(x_2)\sigma_2(x_1)\sigma_2(x_2) \right\}$$

A-T renormalizes into Sine-Gordon → *solitons, antisolitons* $\frac{4\pi}{\beta^2} = 1 - \frac{2}{\pi}\sin^{-1}\left(\frac{\tanh 2J_4}{\tanh 2J_4 - 1}\right)$ *on square lattice*





Surface tensions

$$\Sigma_{(++)(+-)} = m$$

$$\Sigma_{(++)(--)} = \begin{cases} 2m \sin \frac{\pi\beta^2}{2(8\pi - \beta^2)}, & J_4 > 0 \\ 2m, & J_4 \leqslant 0 \end{cases}$$

Decoupling point $J_4 = 0$

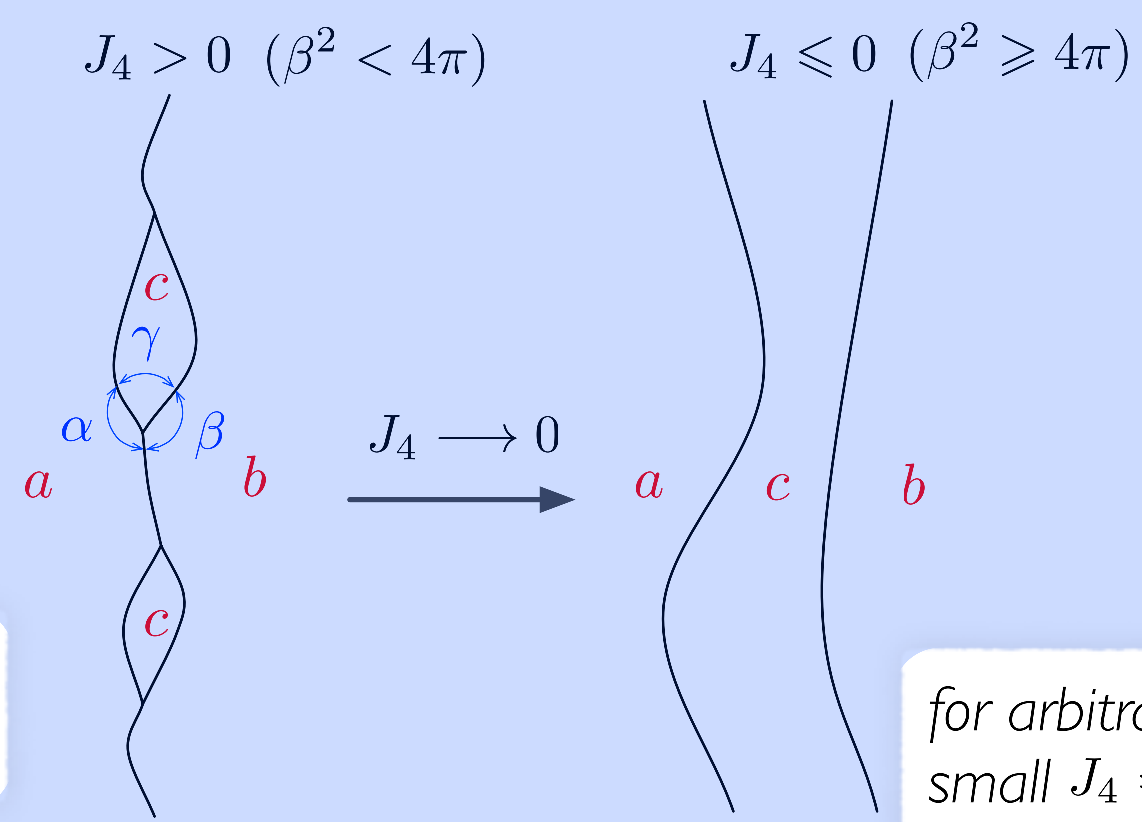
Ising result is recovered

4-Potts $(\beta^2 = 2\pi)$

$\alpha = \beta = \gamma = 2\pi/3$

$$\gamma = 2\pi \frac{4\pi - \beta^2}{8\pi - \beta^2}$$

Transition



for arbitrary small $J_4 \leqslant 0$

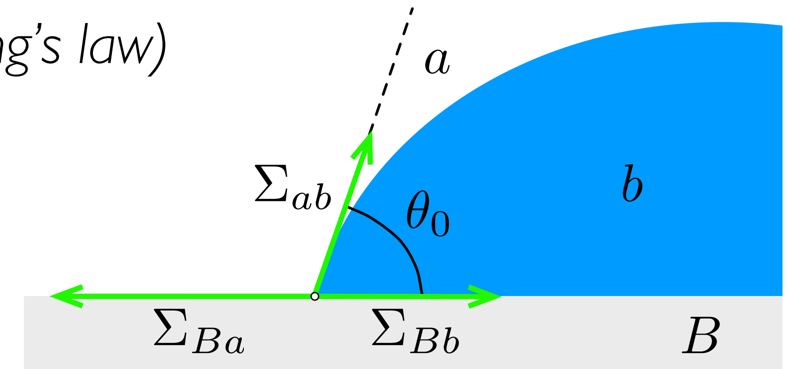
Interfaces at boundaries & boundary wetting transition

Phenomenological description in terms of contact angle and surface tensions

equilibrium condition at contact points (Young's law)

$$\Sigma_{Ba} = \Sigma_{Bb} + \Sigma_{ab} \cos\theta_0$$

wetting transition: $\theta_0 \to 0$

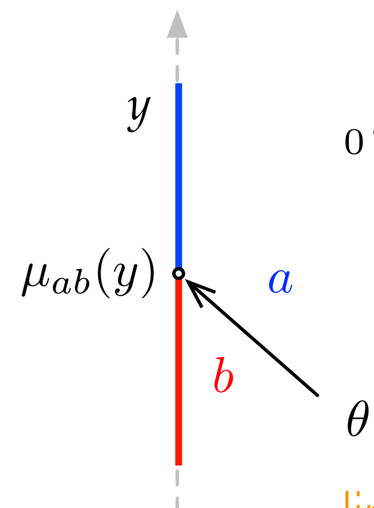


Field theory with a boundary

Pure phase $\langle\sigma\rangle_a$ *selected with the boundary condition*

Pinned interface: selected with a boundary condition B_a
changing field, $\mu_{ab}(y)$ *switches from* B_a *to* B_b

Vacuum $|\Omega_a\rangle_0$ *, with energy* E_0



$${}_0\langle\Omega_a|\mu_{ab}(y)|K_{ba}(\theta)\rangle_0 = \mathrm{e}^{-my\cosh\theta}\mathcal{F}_0^{\mu}(\theta)$$

no particle rest on the boundary

$$\mathcal{F}_0^{\mu}(\theta) = c\,\theta + \mathcal{O}(\theta^2)$$

linear behavior for $\theta \simeq 0$, known feature [Ghoshal, A.Zamolodchikov' 94]

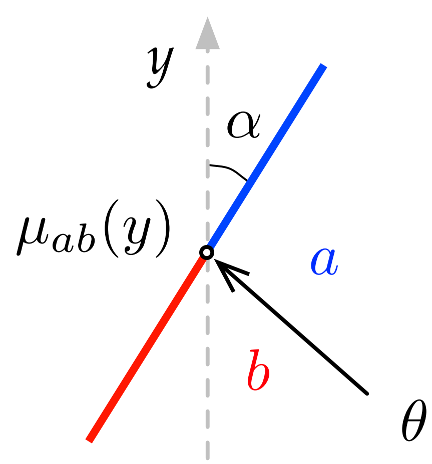
Tilted boundary

Lorentz boost

$\mathcal{B}_\Lambda : \theta \to \theta + \Lambda$

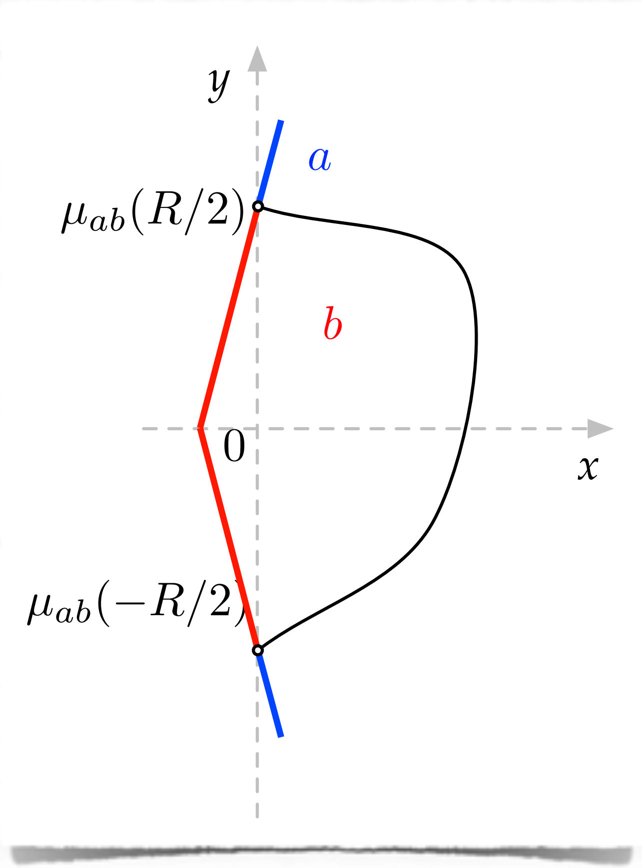
$\mathcal{B}_{-i\alpha} : \mathcal{F}_0^{\mu}(\theta) \to \mathcal{F}_\alpha(\theta) = \mathcal{F}_0(\theta + i\alpha)$

rotates the boundary



$$\mathcal{F}_\alpha^{\mu}(\theta) = c\,(\theta + i\alpha) + \mathcal{O}(\theta^2)$$

Interface in a shallow wedge



$$\langle\sigma(x,y)\rangle_{W_{aba}} = \frac{{}_{\alpha}\langle\Omega_a|\mu_{ab}(0,R/2)\sigma(x,y)\mu_{ba}(0,-R/2)|\Omega_a\rangle_{\alpha}}{{}_{\alpha}\langle\Omega_a|\mu_{ab}(0,R/2)\mu_{ba}(0,-R/2)|\Omega_a\rangle_{\alpha}}$$

order parameter in the wedge

$$\langle\sigma(x,y)\rangle_{W_{aba}} = \langle\sigma\rangle_b + (\langle\sigma\rangle_a - \langle\sigma\rangle_b)\left[\mathrm{erf}(\chi) - \frac{2}{\sqrt{\pi}}\frac{\chi + \sqrt{2mR}\frac{\alpha}{\kappa}}{1+mR\alpha^2}\,\mathrm{e}^{-\chi^2}\right]$$

$\alpha \ll 1$

matches lattice result for Ising with $\alpha = 0$
Abraham ['80]

Drop: *for any R*

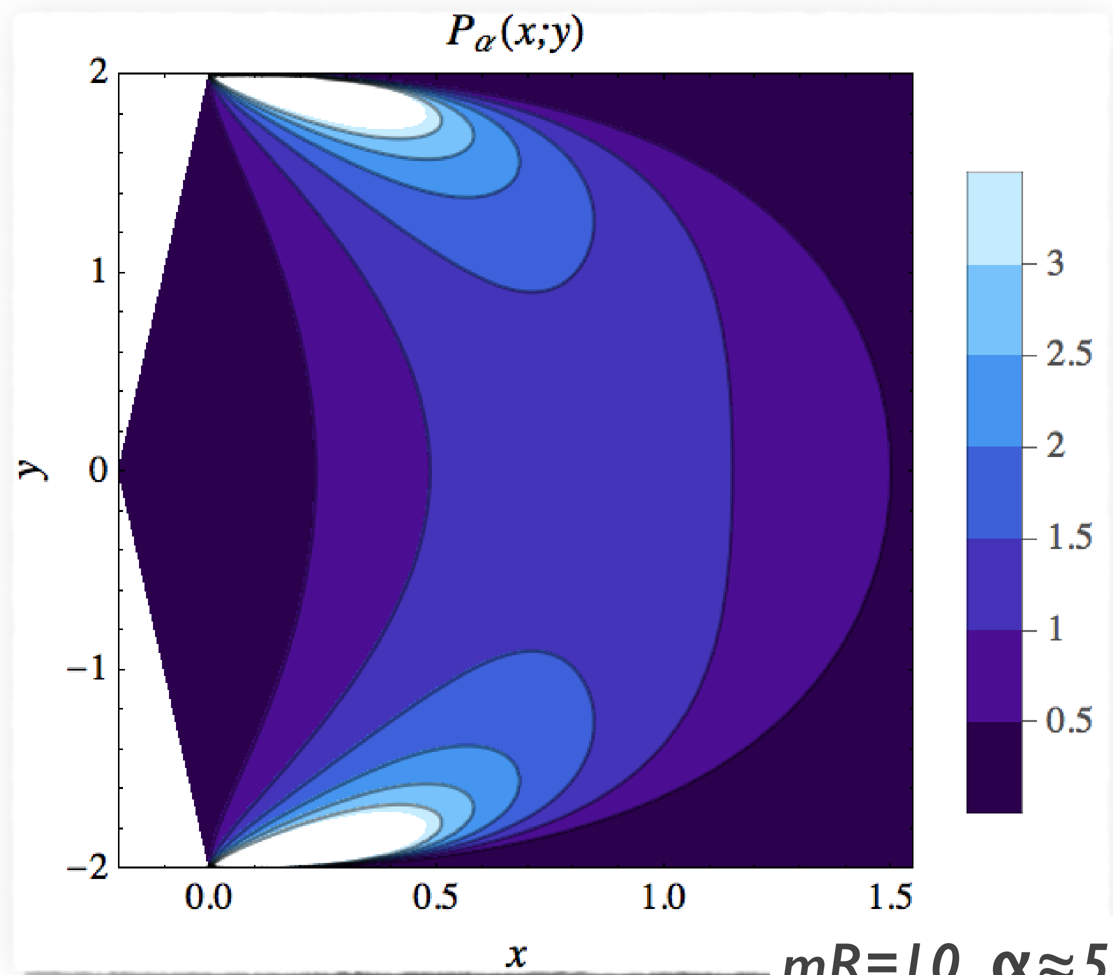
$$\lim_{x\to\infty}\langle\sigma(x,0)\rangle_{W_{aba,\alpha}} = \langle\sigma\rangle_a$$

Passage probability density

$$p(x;y) = \frac{8\sqrt{2}}{\sqrt{\pi}\kappa^3}\left(\frac{m}{R}\right)^{3/2}\frac{(x+\alpha R/2)^2-(\alpha y)^2}{1+mR\alpha^2}\,\mathrm{e}^{-\chi^2}$$

vanishes on the boundary

midpoint fluctuations $\sim\sqrt{R}$



mR=10, α≈5.5°

Boundary wetting & filling transitions

on the half plane

Energy of the system: Boundary+Kink $E_0' = E_0 + m\cosh\theta$

in presence of a boundary bound state $|\Omega_a'\rangle_0$ *with energy*

$$E_0' = E_0 + m\cos\theta_0 \quad \text{(Young's law)}$$

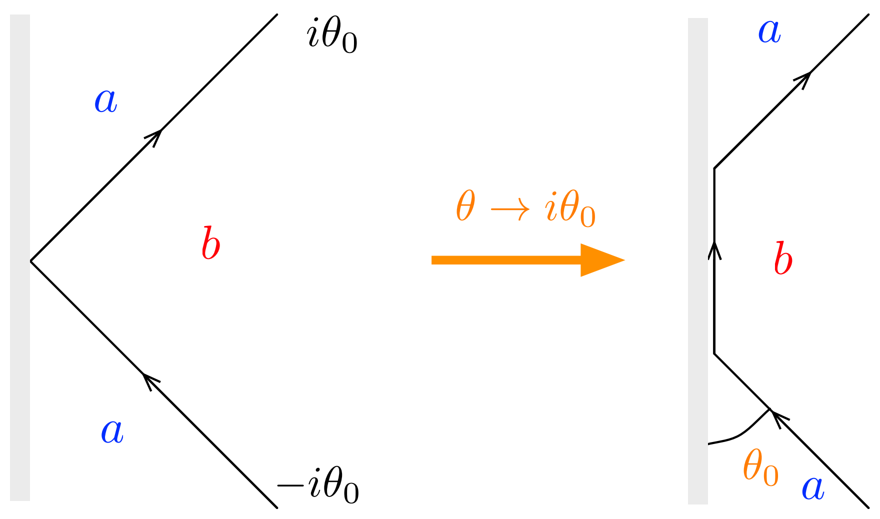
dictionary

resonant angle ⟷ *contact angle*

kink unbinding ⟷ *boundary wetting transition*

$$\theta_0(T_0) = 0, \quad T_0 < T_c$$

$(\alpha = 0)$



Wetting: *at large mx, R large enough*

$$\langle\sigma(x,0)\rangle_{W_{aba,\alpha}} \to \langle\sigma\rangle_b$$

for a wedge

Lorenz invariance is at the origin of wedge covariance

$$\theta_0 \longrightarrow \theta_0 - \alpha$$

condition encountered in effective hamiltonian theories, but not fully justified

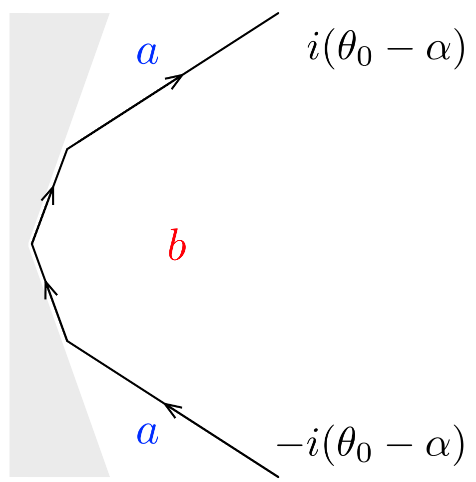
binding energy $E_\alpha' = E_\alpha + m\cos(\theta_0 - \alpha)$

kink unbinding: filling condition

$$\theta_0(T_\alpha) = \alpha$$

condition known from macroscopic thermodynamic arguments [Hauge, '92]

$(\alpha \neq 0)$



Conclusions

- *A new method:* exact field-theoretic formulation of phase separation and related issues: passage probabilities, interface structure (branching & recombination), interfaces at boundaries, wetting & filling,...
- the known solutions (from lattice) for Ising are obtained as a particular case. Phase separation is investigated also for other models for the first time directly from field theory
- Lorentz invariance is the deep origin of wedge covariance
- extended observables (interfaces) captured by local fields
- the validity of the whole technique does not relies on integrability but rather on the fact that domain walls are particle trajectories
- although $mR \gg 1$ projects to low energies, relativistic particles are essential for kinematical poles and contact angle