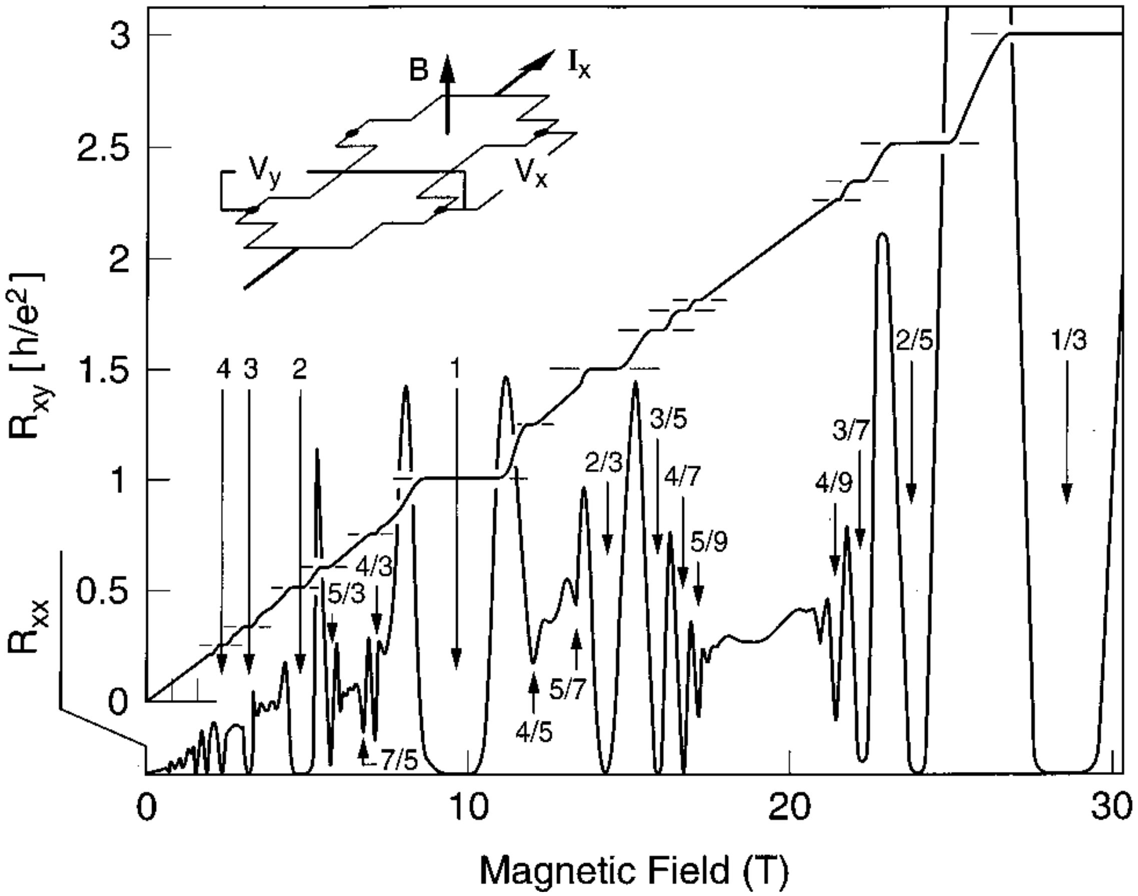
Partition functions and stability criteria of topological insulators

Enrico Randellini
(INFN & dipartimento di Fisica, Firenze)
with A.Cappelli, (arXiv: 1309.2155)

Cortona, Maggio 2014

The Quantum Hall Effect

- Two dimensional electron systems in a strong magnetic field B at low temperature



$$R_{xx} = \frac{V_x}{I_x} = 0 \quad \longrightarrow \quad \text{insulating bulk}$$

$$\sigma_H = \frac{I_x}{V_y} = \frac{e^2}{h}\nu \quad \longrightarrow \quad \text{quantized Hall conductance}$$

$$\nu = 1, 2, 3, \cdots \quad \mathrm{IQHE}$$ **(K. Von Klitzing et al.1980)**

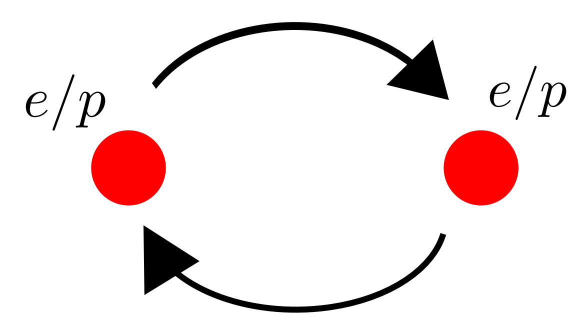
$$\nu = \frac{1}{p}, \quad p = 3, 5, \cdots \quad \mathrm{FQHE}$$ **(D.C.Tsui et al. 1982)**

- the electrons fill Landau levels $\longrightarrow$ $\nu = \dfrac{N_e}{N_s} = N_e \dfrac{\Phi}{\Phi_0}\,, \quad \Phi_0 = \dfrac{hc}{e}$

- incompressible quantum fluid **(Laughlin 1983, Wilczek 1990)**

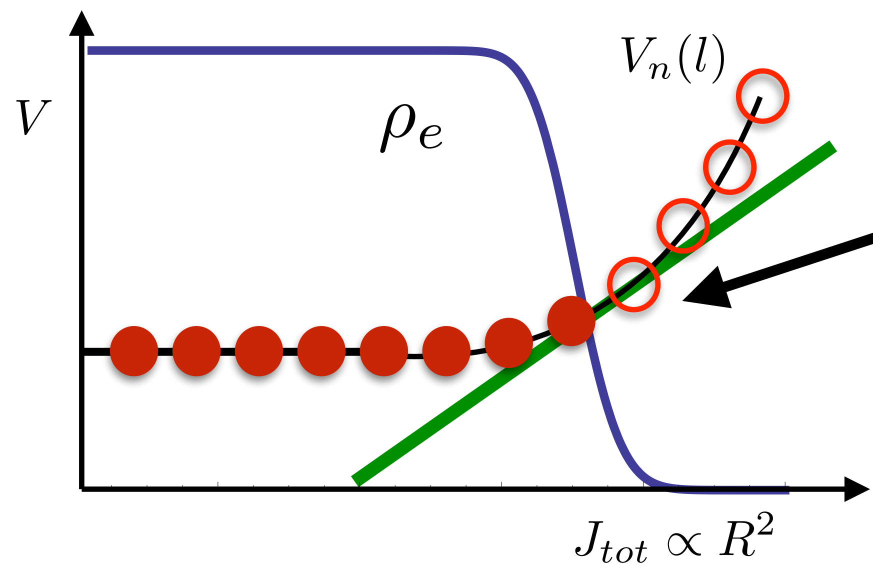
- for FQHE

$$Q = \frac{ek}{p}, \;\; k = \pm 1, \cdots, \pm p$$



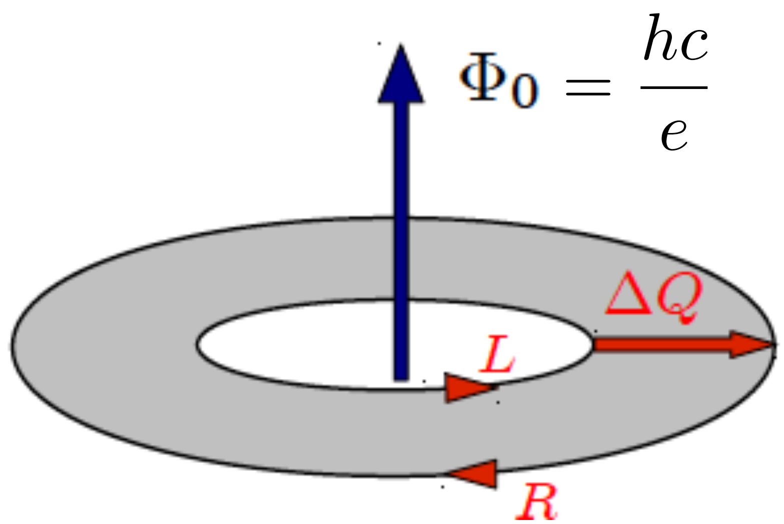
$$\frac{\theta}{\pi} = \frac{1}{p}$$

Chiral edge states and Laughlin's flux argument



$$\epsilon_l \simeq \frac{v}{R}(l - L - \mu), \quad O(1/R)$$

$(1+1)$ chiral CFT effective description



- Laughlin's argument for $\nu = \frac{1}{3}$ **(Laughlin 1983)**

$$\Phi \to \Phi + \Phi_0, \qquad H[\Phi + \Phi_0] = H[\Phi]$$

$$Q_R \to Q_R + \Delta Q = \nu, \quad \Delta Q = \int dt dx \partial_t J_R^0 = \frac{e\nu}{2\pi} \int F = \nu e n \longrightarrow U(1)_Q \text{ chiral anomaly}$$

- $\Phi \to \Phi + n\Phi_0$ spectral flow $\{0\} \to \{\frac{1}{3}\} \to \{\frac{2}{3}\} \to \{1\}$

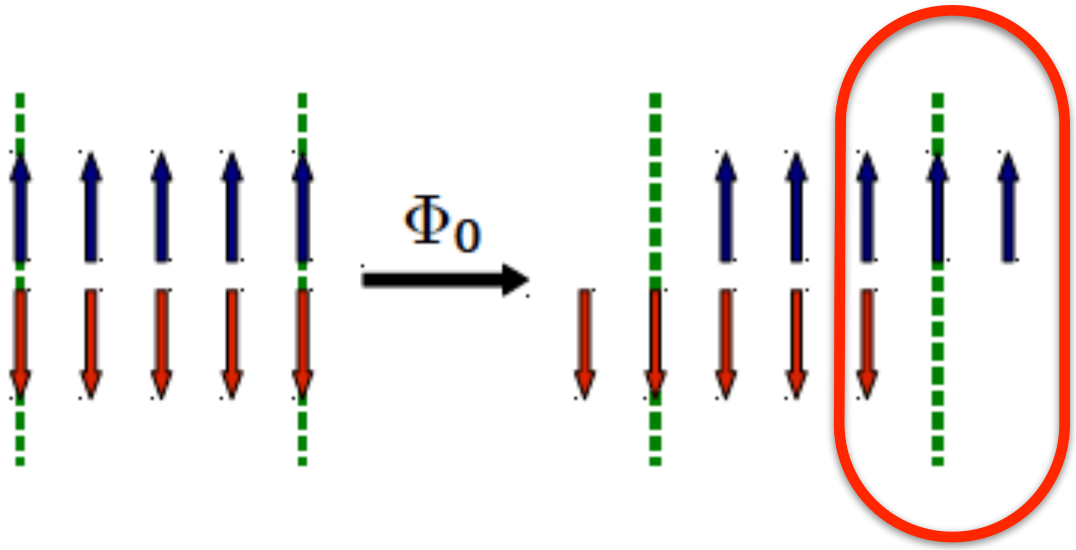
- chiral edge state cannot be gapped $\longleftrightarrow$ stable topological phase

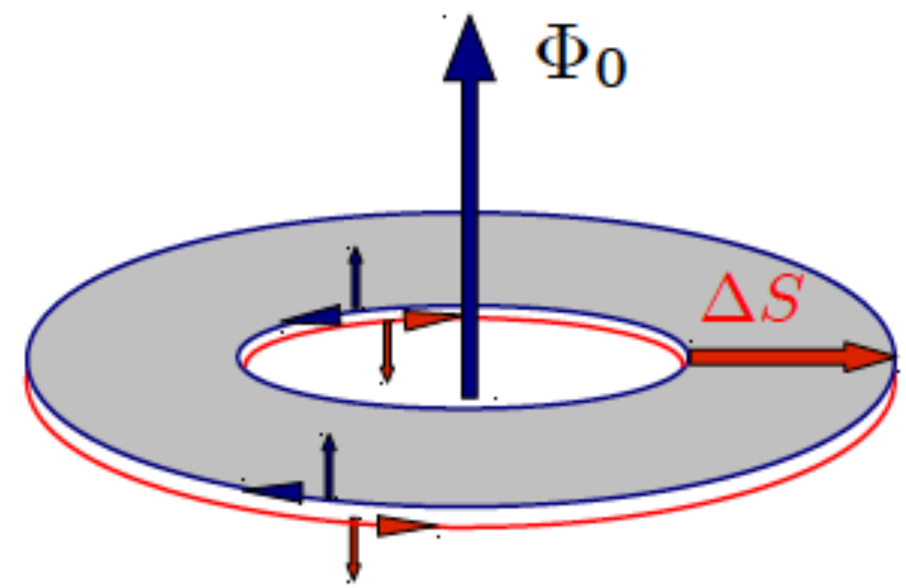
- no Time Reversal symmetry

Time Reversal topological insulators

Quantum Spin Hall Effect (**Bernevig & al. '06, Molenkamp & al. '07**)

- two copies of $\nu = 1$ QHE with opposite chiralities and spins $\longrightarrow$ $U(1)_Q \times U(1)_S$





$$\Delta Q = \Delta Q^{\uparrow} - \Delta Q^{\downarrow} = 1 - 1 = 0$$

$$\Delta S = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1 \qquad \longrightarrow \qquad U(1)_S \text{ anomaly}$$

$$\Delta S = \Delta Q^{\uparrow} = \nu^{\uparrow}$$

- spin orbit coupling and other relativistic effect $\longrightarrow$ $\cancel{U(1)_S}$

- no spin current and spin Hall conductivity $\longrightarrow$ Time Reversal Topological Insulator (TI)

Symmetry protected topological (SPT) phases

- Stability of T.I. $\longleftrightarrow$ stability of non chiral edge states

Example for free fermion system $\nu = 1$

- TR symmetry forbids a mass term in a CFT with odd number of free fermions

$$\mathcal{T} : \psi_{\uparrow k} \to \psi_{\downarrow -k}, \qquad \mathcal{T} : \psi_{\downarrow k} \to -\psi_{\uparrow -k}$$

$$\mathcal{T} : H_{int} \propto m \int dk \; \psi^{\dagger}_{\uparrow k} \; \psi_{\downarrow k} + \mathrm{h.c.} \to -H_{int}$$

$\mathbb{Z}_2$ classification (free fermions)

Fu-Kane-Mele flux argument: spin parity anomaly

- TR symmetry plus gauge symmetry $\nu = 1$

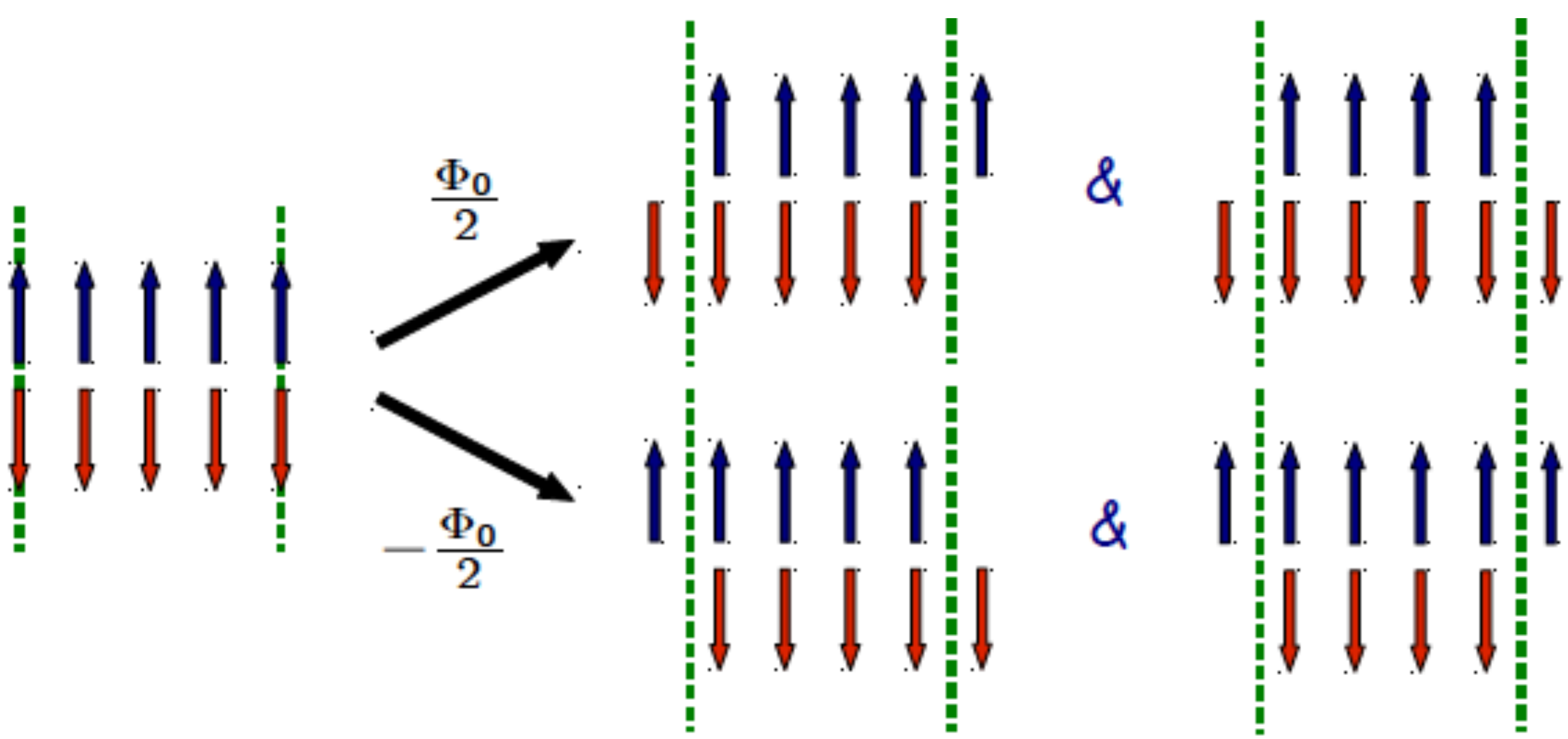
$$\mathcal{T} H[\Phi] \mathcal{T}^{-1} = H[-\Phi] \qquad\qquad H[\Phi + \Phi_0] = H[\Phi]$$

- TR invariant points $\Phi = 0, \dfrac{\Phi_0}{2}, \Phi_0, \dfrac{3\Phi_0}{2}, \cdots$

- TR invariant bulk polarization $(-1)^{P_\theta}$ **(Kane & Mele, '05; Fu & Kane, '06)**

i) topological and conserved by TR symmetry

ii) at the edge $\longrightarrow$ $(-1)^{P_\theta} = (-1)^{N_\uparrow + N_\downarrow} = (-1)^{2S}$ spin parity



$(-1)^{2S} = -1$

pairs of degenerate edge states

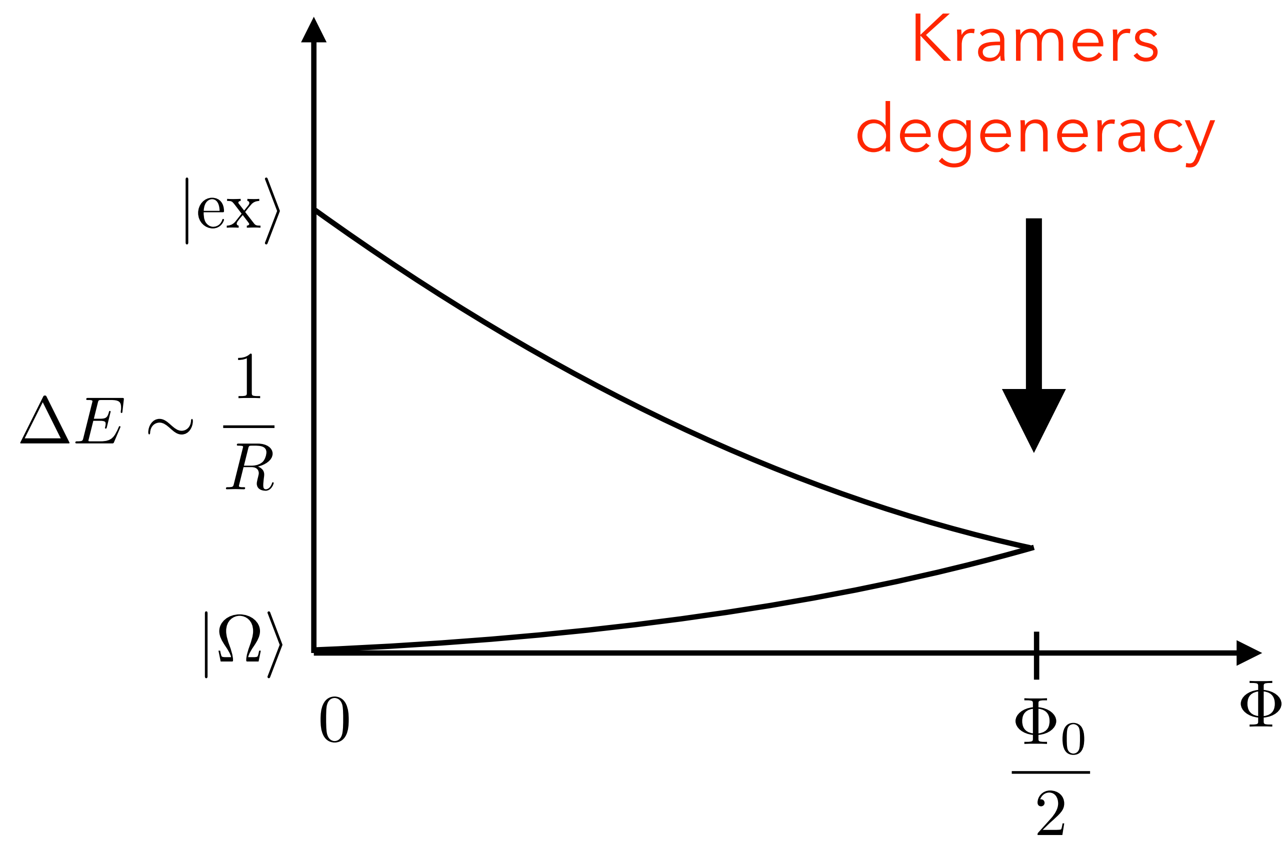
(Levin & Stern, '09)

- energy change of the edge ground state

$$\Phi = 0 \;:\quad (-1)^{2S} = 1$$

$$\Phi = \frac{\Phi_0}{2} \;:\quad (-1)^{2S} = -1$$

$|\mathrm{ex}\rangle$ gapless edge state



- <u>conclusion for free fermion system</u>

i) topological phase is protected by TR symmetry if exist edge Kramers pair

N_{pairs} odd N_{pairs} even

ii) $(-1)^{2S}$ spin parity is anomalus for stable topological insulators

iii) $\mathbb{Z}_2$ anomaly $\longleftrightarrow$ $\mathbb{Z}_2$ classification of free fermion topological insulators

Our answers for interacting T.I.

$\boxed{\text{partition functions}}$ $\longrightarrow$ stability of interacting T.I. ($\nu^{\uparrow}$ fractional)

Flux insertion $\longrightarrow$ stability and electromagnetic background responses

Modular transformation $\longrightarrow$ gravitational background responses

- Main results

$$\boxed{2\Delta S = \frac{\sigma_{SH}}{e^*} = \frac{\nu^{\uparrow}}{e^*}}$$

(Levin & Stern, '09)

$(-1)^{2\Delta S} = 1$ ~~$\mathbb{Z}_2$ anomaly~~, unstable, Z modular invariant

$= -1$ $\mathbb{Z}_2$ anomaly, stable, Z not modular invariant

(Cappelli & E.R., '13)

QHE partition functions

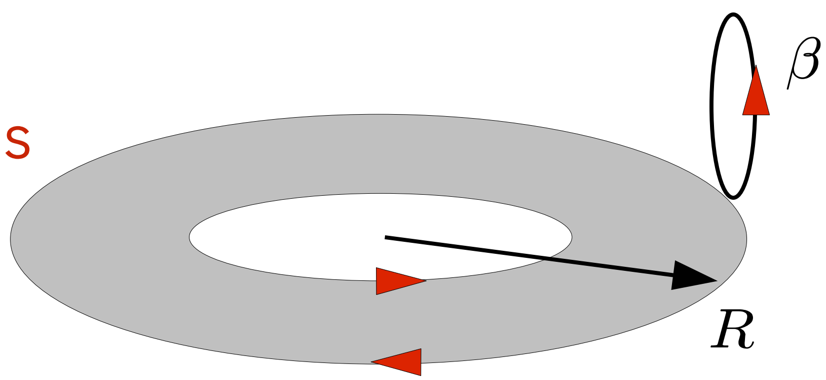
- Partition function for QHE are completely understood

(Cappelli & Zemba '97;
Cappelli, Georgiev & Todorov '01;
Cappelli & Viola '11)

- states of outer edge for $\nu = \frac{1}{p}, \quad p = 3, 5, \cdots$ described by a CFT with $c = 1$

$E \sim P \sim v\frac{L_0}{R}$ energy and momentum of chiral excitations

τ, ζ modular parameter and "coordinate"



- decomposition into orthogonal sectors $\mathcal{H}^{(\lambda)}$, corresponding to $Q = \frac{\lambda}{p} + n, n \in \mathbb{Z}$

- given charge sector Q_λ, the partition function is a sum of characters of the representation of $\widehat{U(1)}$ current algebra

$$K_\lambda(\tau, \zeta; p) = \mathrm{Tr}_{\mathcal{H}^{(\lambda)}}\left[\exp\bigl(2i\pi\tau L_0 + 2i\pi\zeta Q\bigr)\right] = \frac{\text{theta function}}{\text{Dedekind function}}$$

$$\lambda = 1, \cdots, p$$

Gravitational and electromagnetic background for QHE

- modular transformations: discrete coordinate changes respecting double periodicity

$T : \tau \to \tau + 1$ adds a twist

$S : \tau \to -1/\tau$ exchanges periods

- electromagnetic background changes: (periodicity in the "coordinate")

$U : \zeta \to \zeta + 1$ adds a weight $\exp(2i\pi Q)$

$V : \zeta \to \zeta + \tau$ adds a flux quantum $\Phi \to \Phi + \Phi_0$

$$T^2 : K_\lambda(\tau + 2, \zeta) = \exp(i4\pi h_\lambda) K_\lambda(\tau, \zeta)$$

odd-integer electron statistics

$$S : K_\lambda\left(-\frac{1}{\tau}, -\frac{\zeta}{\tau}\right) = \sum_{\mu=1}^{p} S_{\lambda\mu} K_\mu(\tau, \zeta)$$

unitary S matrix, completeness

$$U : K_\lambda(\tau, \zeta + 1) = \exp(2i\pi\lambda/p) K_\lambda(\tau, \zeta)$$

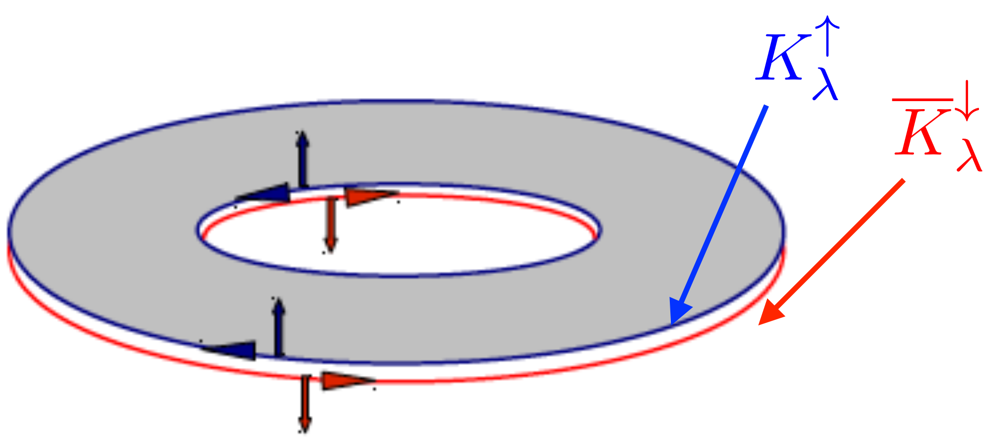
integer electron charge

$$V : K_\lambda(\tau, \zeta + \tau) \sim K_{\lambda+1}(\tau, \zeta)$$

spectral flow and chiral anomaly

Partition functions of topological insulators

- Partition function for a single edge $\nu^{\uparrow} = \nu^{\downarrow} = 1/p$



$$\boxed{Z^{NS}(\tau,\zeta) = \sum_{\lambda=1}^{p} K^{\uparrow}_{\lambda} \overline{K}^{\downarrow}_{-\lambda}, \qquad T^2, S, V, U \quad \text{invariant}}$$

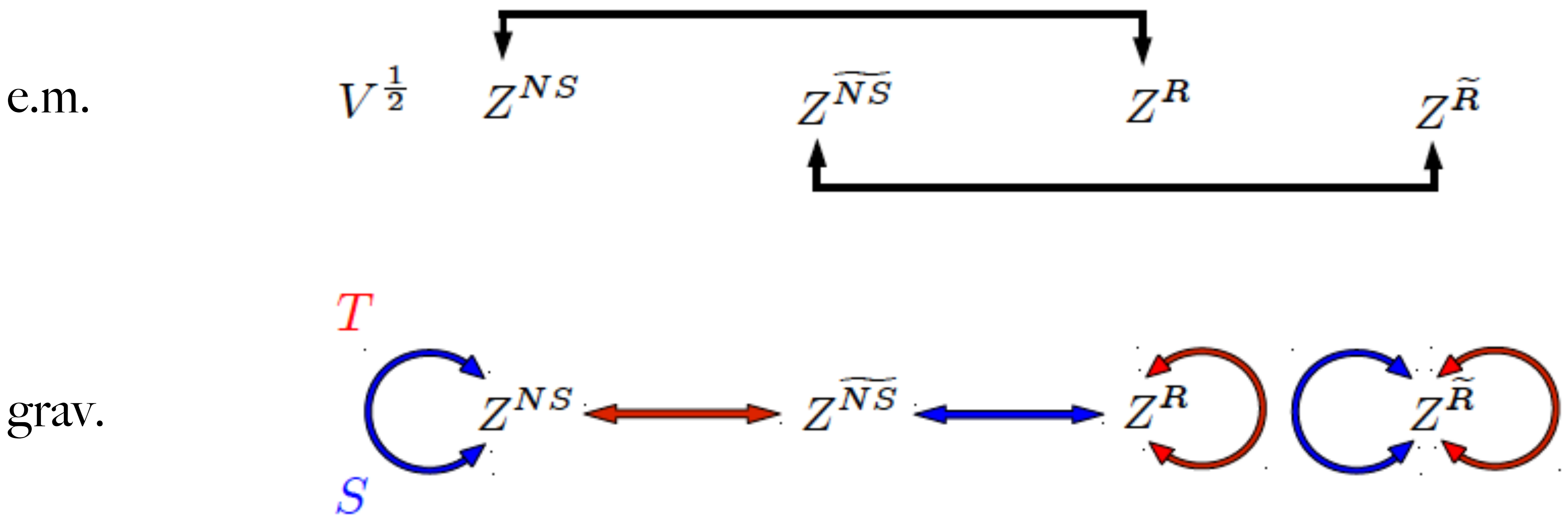
- Spin structures in torus geometry $\longrightarrow$ boundary conditions of fermion fields in (space,time)

$$NS, \quad \widetilde{NS}, \quad R, \quad \widetilde{R} \qquad \text{risp.} \qquad (A,A), \quad (A,P), \quad (P,A), \quad (P,P)$$

- Ramond sector describes half-flux insertions $\Phi \to \Phi + \dfrac{\Phi_0}{2}$

$$V^{\frac{1}{2}} : Z^{NS}(\tau,\zeta) \to Z^{NS}(\tau,\zeta + \frac{\tau}{2}) \sim Z^{R}(\tau,\zeta)$$

Gravitational and electromagnetic background for T.I.



- Modular invariant partition function of single TI edge can always be found

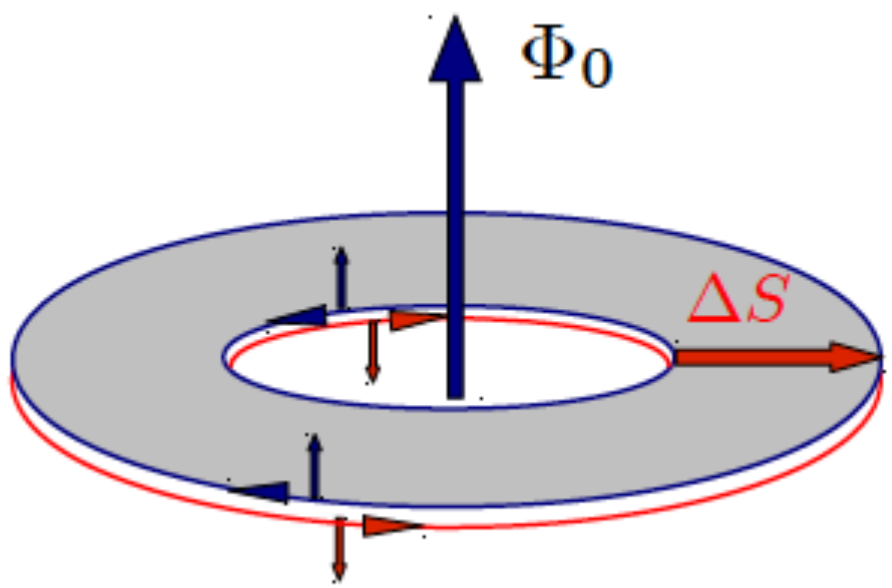
$$Z_{\text{Ising}} = Z^{NS} + Z^{\widetilde{NS}} + Z^{R} + Z^{\widetilde{R}}, \qquad S, T, U, V^{\frac{1}{2}} \text{ invariant}$$

Question: is Z_{Ising} consistent with TR symmetry?

Stability, modular (non)-invariance, SPT phases

- $p/2$ fluxes $\longrightarrow$ neutral spin one-half edge excitation

$$V^{p/2} : K_0^{\uparrow} \;\rightarrow\; K_{p/2}^{\uparrow} \sim K_0^{R\uparrow}, \quad |\Omega\rangle_{NS} \rightarrow |\Omega\rangle_R\,, \qquad \Delta Q^{\uparrow} = \Delta S = \frac{1}{2}$$



- from the partition functions you read the g.s. spin parity

$$(-1)^{2\Delta S}\,|\Omega\rangle_{NS} = |\Omega\rangle_{NS} \quad \rightarrow \quad (-1)^{2\Delta S}\,|\Omega\rangle_R = -\,|\Omega\rangle_R$$

$\longrightarrow$ $\mathbb{Z}_2$ spin-parity anomaly

stable T.I.

spin parity of Ramond g.s. is different from that of Neveu - Schwartz g.s.

- Agreement with Levin-Stern index

$$2\Delta S = \frac{\sigma_{SH}}{e^*} = \frac{\nu^{\uparrow}}{e^*}, \quad \nu^{\uparrow} = e^* = \frac{1}{p}, \quad (-1)^{2\Delta S} = -1$$

- What happens in Ising partition function?

$$\begin{array}{lllll} Z_{\text{Ising}} = Z^{NS} & + Z^{\widetilde{NS}} & + Z^{R} & + Z^{\widetilde{R}} \\ (-1)^{2S} = 1 & 1 & -1 & -1 \end{array}$$

not consistence with TR symmetry

(also violates spin-statistic)

- For Laughlin $\nu^{\uparrow} = \nu^{\downarrow} = 1/p$ T.I., the partition function shows that

TR symmetry + anomaly $\rightarrow$ no modular invariance $\rightarrow$ topological insulators

TR symmetry + modular invariance $\rightarrow$ no anomaly $\rightarrow$ trivial insulator

Conclusion and extensions

- Partition functions for Topological Insulators give

$\longrightarrow$ response to e.m. background and discrete diffeomorphisms

$\longrightarrow$ $\mathbb{Z}_2$ spin-parity anomaly

$\longrightarrow$ recover and generalize stability analysis of T.I. with TR symmetry protection

for all Abelian states and for non Abelian states (Read-Rezayi and NASS)

$$\boxed{\mathbb{Z}_2 \text{ classification and index } (-1)^{2\Delta S}, \qquad \Delta S = \frac{\nu^\uparrow}{e^*}}$$

- For non stable T.I. we have to write down the interaction that gap out the edge modes

Abelian states $\longrightarrow$ ok **(Levin & Stern, '09, '1; Chamon & al.'12)**

non Abelian states $\longrightarrow$ **(in preparation Cappelli & E.R.)**

- Relation between Topological Insulators and Topological Superconductors

(Ryu & Zhang, '12)

Thanks