BPS Quivers and $\mathcal{N}=2$ superconformal theories

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$\mathcal{N} = 2$ theories and the BPS bound

The infrared effective action of an $\mathcal{N} = 2$ theory is a $U(1)^r$ gauge theory. It can be written in terms of a single function $\mathcal{F}$ (prepotential), encoded in an algebraic curve. N. Seiberg, E. Witten '94.

The states of the theory have electric and magnetic charges and flavor quantum numbers

$$(e_i, m_i, s_j) \equiv \gamma \in \Gamma^{2r+f}; \quad i = 1, \dots, r \;\; j = 1, \dots, f = \text{rank}(G_F)$$

$$\langle \gamma_1, \gamma_2 \rangle = \sum_i (e_i^1 m_i^2 - m_i^1 e_i^2) \in \mathbb{Z} \quad \text{Dirac quantization condition}$$

All the states satisfy the **BPS bound**:

$$M_\gamma \geq |Z(\gamma)|; \quad Z(\gamma_1 + \gamma_2) = Z(\gamma_1) + Z(\gamma_2).$$

States saturating the bound are called BPS and preserve 4 supercharges.

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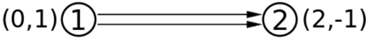
BPS quivers

Quiver property: The BPS spectrum has finitely many generators

$$\gamma = \pm \sum_{i=1}^{N} n_i \gamma_i \quad \forall \gamma \in \mathcal{H}_{BPS}; \quad n_i \in \mathbb{N}.$$

The BPS quiver of the theory is an oriented graph:

- One node for each generator γ_i.
- $\langle \gamma_j, \gamma_i \rangle$ arrows from node i to node j.



SUSY quantum mechanics and BPS states

On the worline of a BPS particle we have a $0+1$ dimensional theory with 4 supercharges!

For the state $\gamma = \sum_{i=1}^{N} n_i \gamma_i$ we consider the theory Denef '02.

$$G = \prod_{i=1}^{N} U(n_i) \quad \text{with } \langle \gamma_j, \gamma_i \rangle \text{ bifundamentals } B_{ij}^a.$$

i.e. the reduction of the theory on a bound state of D-branes.

$$\mathcal{M}_\gamma = \{\text{F-term equations}\} \Big/ \prod_i Gl(n_i, \mathbb{C}) \cdot$$

FI parameters for $U(n_i)$: $\theta_i = |Z(\gamma_i)| \left(\arg Z(\gamma_i) - \arg Z(\gamma)\right)$.

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BPS states and quiver representations

In a vacuum bifundamentals are constant and reduce to linear maps $B_{ij}^a : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j}$. We recover a **quiver representation**!

Subrepresentation S $(= \sum_i k_i \gamma_i)$: collection of subspaces $\mathbb{C}^{k_i}$ and maps b_{ij}^a s.t. all diagrams commute

$$\begin{array}{ccc} \mathbb{C}^{n_i} & \xrightarrow{B_{ij}^a} & \mathbb{C}^{n_j} \\ \uparrow & & \uparrow \\ \mathbb{C}^{k_i} & \xrightarrow{b_{ij}^a} & \mathbb{C}^{k_j} \end{array}$$

The representation γ is **stable** if

$$\arg Z(S) < \arg Z(\gamma) \quad \text{for all subrepresentations.}$$

Equivalent to D-term constraints coming from FI terms.

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The 4d/2d correspondence

Let us consider Type IIB string theory on $\mathbb{R}^4 \times CY_3$, where CY_3 is an hypersurface singularity in $\mathbb{C}^4$ ($f(x_1, x_2, x_3, x_4) = 0$).

The LG theory with superpotential f **encodes the quiver:**

- Vanishing 3-cycles $\Longleftrightarrow$ Vacua of the 2d theory.
- Intersection number of vanishing cycles $\Longleftrightarrow$ Number of solitons between the vacua.

4d/2d correspondence (Cecotti, Neitzke, Vafa '10)

For every $\mathcal{N}=2$ theory with the quiver property, there is a $\mathcal{N}=(2,2)$ 2d theory (with $\hat{c}<2$) such that $B = S^t - S$ ($B_{ij} = \langle \gamma_j, \gamma_i \rangle$ and $S = tt^*$ Stokes matrix).

If $f(X_i, Y_a) = f_1(X_i) + f_2(Y_a)$ the 2d theory is the sum of two LG models with superpotential f_1 and f_2. The Stokes matrix is $S = S_1 \otimes S_2$.

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Two simple examples

- For $SU(2)$ SYM (setting $\Lambda = 1$) Klemm, Lerche, Mayr, Vafa, Warner '96.

$$f(x,y,w,v) = e^w + e^{-w} + x^2 + y^2 + v^2.$$

The superpotential is then $\mathcal{W} = e^X + e^{-X}$. This is the $\mathbb{CP}^1$ σ-model

- For $SU(N)$ SYM:

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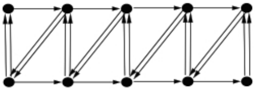


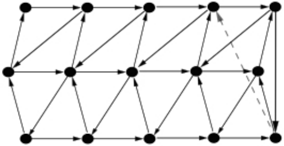
$D_p(G)$ models and their BPS quivers

Consider the local Calaby-Yau geometry (with $G = ADE$)

$$f_{p,G} = e^{pw} + \mathcal{W}_G(x,y) + v^2 + e^{-w} = 0 \qquad \left(\Omega = \frac{dx \wedge dy \wedge dv}{\partial_w f_{p,G}}\right)$$

Describes G SYM coupled to a matter sector ($D_p(G)$).

$D_p(G)$ is superconformal and has (at least) G flavor symmetry.



Scaling dimensions and R-symmetry

$$f_{p,G} = z^p + v^2 + \mathcal{W}_G(x,y) + \sum_{ijk} u_{ijk} z^i x^j y^k \quad (z = e^w).$$

From the 2d perspective $f_{p,G}$ has dimension one.

From the 4d perspective Ω has dimension one

$$D_{2d}(\Omega) = D_{2d}\left(\frac{dxdydv}{z\partial_z f_{p,G}}\right) = D_{2d}(x) + D_{2d}(y) - \frac{1}{2} = \frac{1}{h(G)}.$$

$$D_{4d}(u_{ijk}) = h(G) D_{2d}(u_{ijk}).$$

The same relation holds between 2d and 4d R-charges:

$$R_{4d}(u_{ijk}) = h(G) R_{2d}(u_{ijk}).$$

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One-loop beta function

For $\mathcal{N}=2$ SCFT's $\beta_1\delta^{ab}=-\mathrm{Tr}RT^aT^b$.

$$e^{2\pi inR}\rightarrow\Delta\theta=-2\pi n\beta_1;\quad (e,m)\rightarrow(e-2n\beta_1 m,m).$$

The 2d monodromy $M=(S^t)^{-1}S$ acts as a $U(1)_R$ rotation on the space of vacua (i.e. the 4d charge lattice)

$$e^{2\pi inR}\Gamma=M_{p,G}^{nh(G)}(=M_p^{nh(G)}\otimes M_G^{nh(G)})\Gamma=(-1)^{nh(G)}M_p^{nh(G)}\otimes 1\Gamma.$$

For $n=p$ we find

$$(-1)^{ph(G)}M_p^{ph(G)}\gamma=\gamma+\frac{p+1}{2}h(G)\delta\langle\delta,\gamma\rangle\quad \delta=\sum_{i=1}^{p+1}\gamma_i.$$

$$\beta_1=-\frac{p+1}{p}h(G)\rightarrow\beta_1(D_p(G))=\frac{p-1}{p}h(G).$$

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$$e^{2\pi i n R} \rightarrow \Delta\theta = -2\pi n \beta_1; \quad (e, m) \rightarrow (e - 2n\beta_1 m, m).$$

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Concluding remarks

- The quiver is a powerful tool to study the BPS spectrum of $\mathcal{N}=2$ theories. We can determine it studying quiver stable representations.
- The 4d/2d correspondence allows to determine the quiver for $\mathcal{N}=2$ theories with a string theory realization.
- The study of 4d theories can be related, via the 4d/2d correspondence, to problems in 2d field theories. This allows to study problems in 4d theories using 2d techniques and viceversa (e.g. classification of 2d theories with $\hat{c}=1$). Cecotti, Vafa 1103.5832.

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