Hyperbolic monopoles, JNR data and spectral curves

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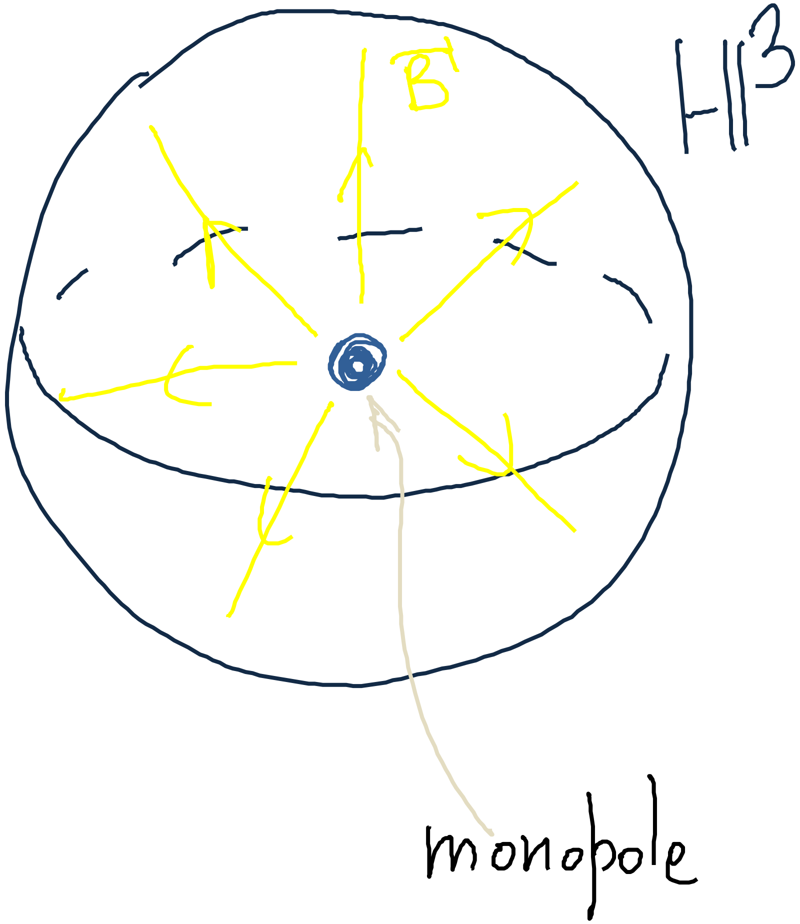
Based on arXiv:1404.1846 with A. Cockburn and P. Sutcliffe

Introduction



Bogomolny equation

$$D\Phi = *F$$

Hyperbolic space

$$ds^2(\mathbb{H}^3) = \frac{4(dX_1^2 + dX_2^2 + dX_3^2)}{(1 - R^2)^2}$$

Plan of the talk

- Relation between instantons and Hyperbolic monopoles (Atiyah)

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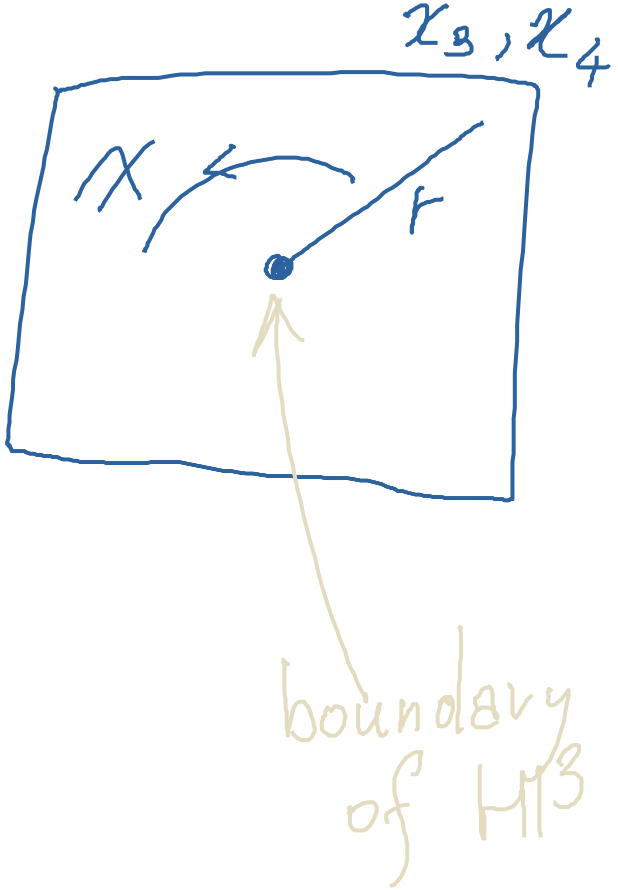
- Hyperbolic monopoles from JNR data (Manton-Sutcliffe)
- Twistor methods: spectral curve and rational map
- Examples of multi-monopole solutions (dihedral and cyclic symmetries, scattering families)

Conformalities and invariant instantons

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$
$$= r^2\left(d\chi^2 + \frac{1}{r^2}\left(dx_1^2 + dx_2^2 + dr^2\right)\right)$$



Conformalities and invariant instantons

$$
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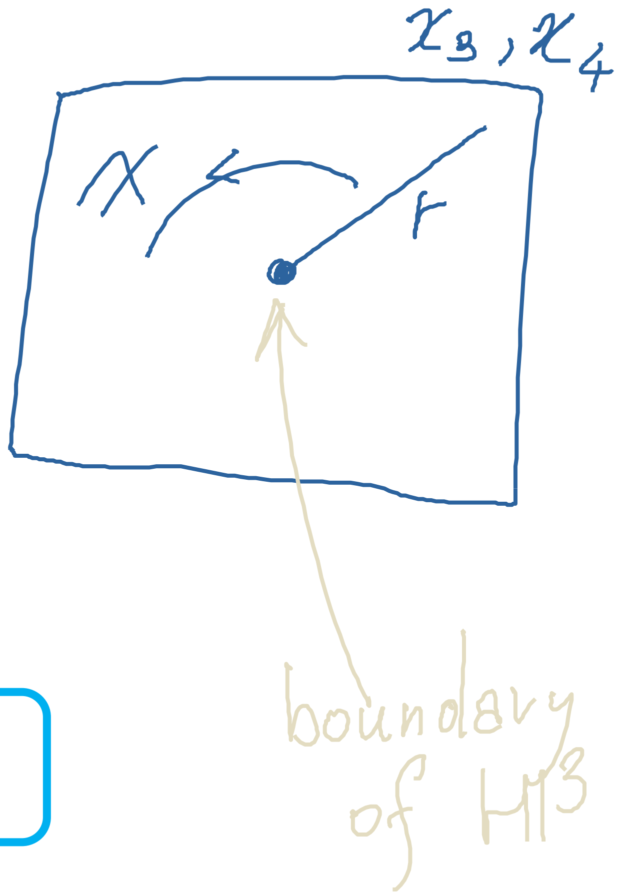
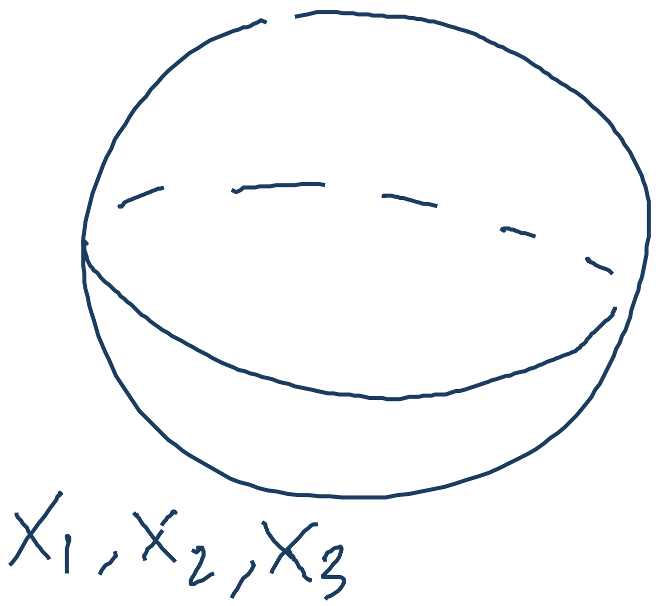
$$



$\mathbb{R}^4 - \mathbb{R}^2$ is conformal equivalent to $S^1 \times \mathbb{H}^3$.

Ball and Poincare

$$ds^2(\mathbb{H}^3) = \frac{4(dX_1^2 + dX_2^2 + dX_3^2)}{(1 - R^2)^2}$$



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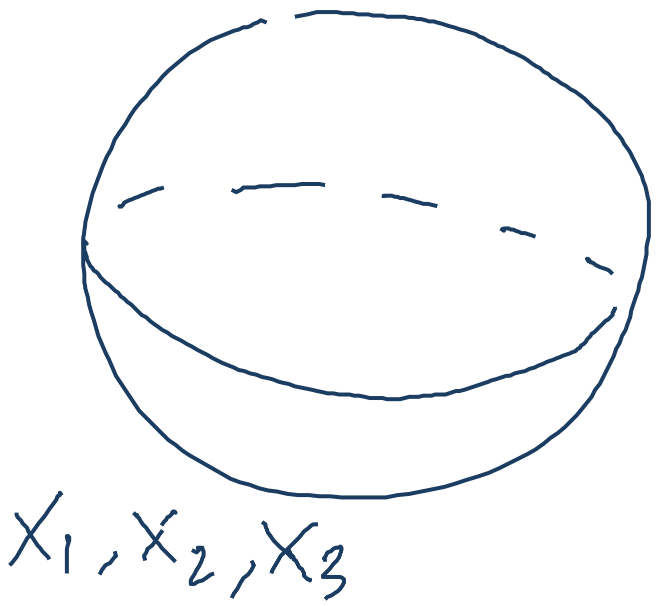

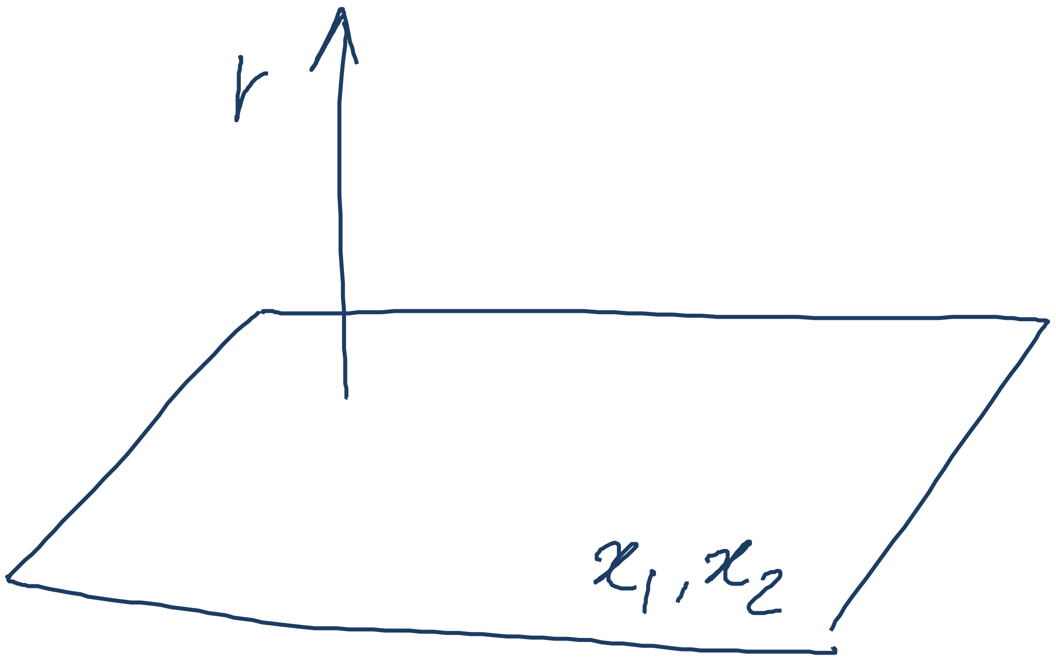

Circle invariant instantons

$$I = 2vN$$

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Instanton number

Monopole charge

Higgs vev (half integer)

' t Hooft and JNR ansatz

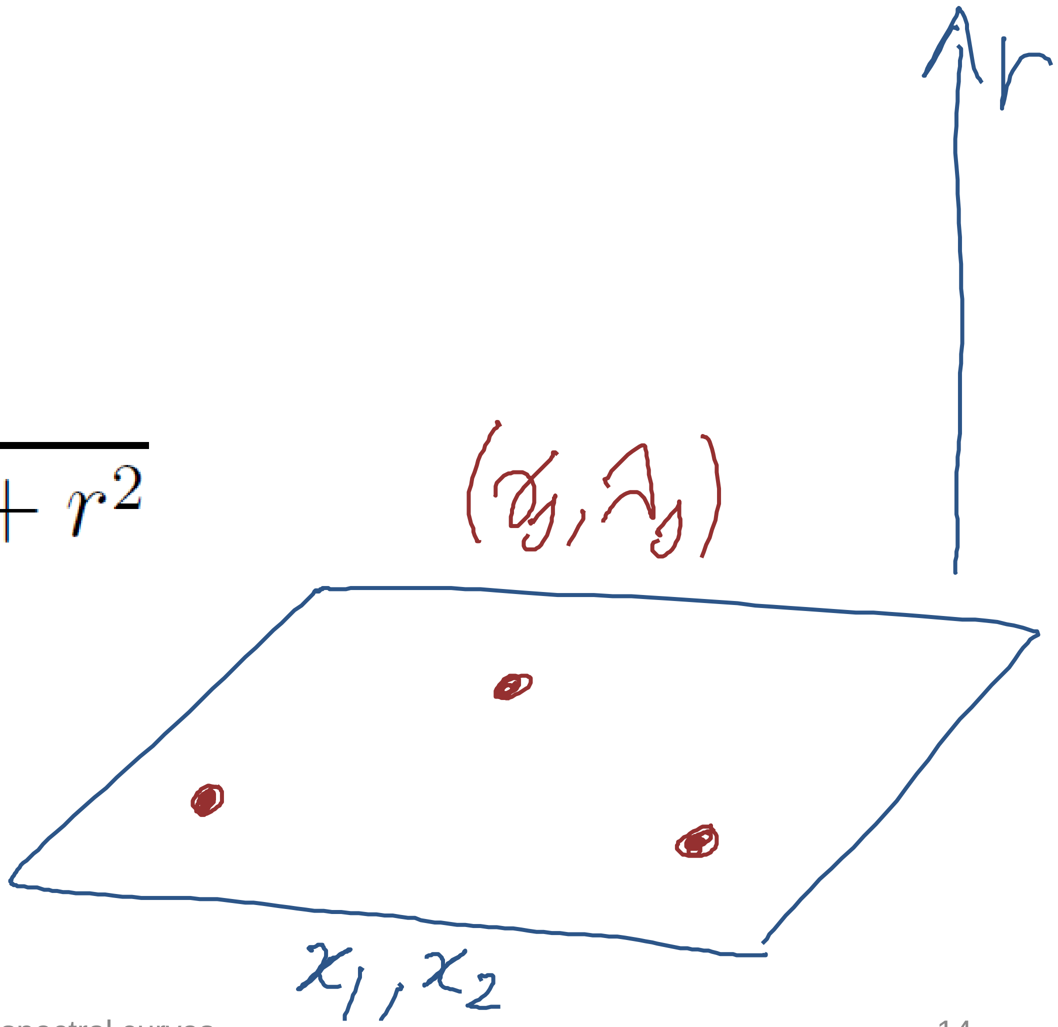
$$A_\mu = \frac{i}{2}\sigma_{\mu\nu}\partial_\nu \varrho \qquad \varrho = \log \psi$$

ψ is an arbitrary harmonic function

Circle invariance, poles and weigths

't Hooft ansatz

$$\psi = 1 + \sum_{j=1}^{N} \frac{\lambda_j^2}{|x_1 + ix_2 - \gamma_j|^2 + r^2}$$



Circle invariance poles and weights

Jackiw-Nohl-Rebbi (JNR) ansatz

$$\psi = \sum_{j=0}^{N} \frac{\lambda_j^2}{|x_1 + ix_2 - \gamma_j|^2 + r^2}$$

Reduces to 't Hooft for $\lambda_0^2 = 1 + |\gamma_0|^2 \to \infty$

Explicit solution

Higgs field

$$|\Phi|^2 = \frac{r^2}{4\psi^2}\left(\left(\frac{\partial\psi}{\partial x_1}\right)^2 + \left(\frac{\partial\psi}{\partial x_2}\right)^2 + \left(\frac{\psi}{r} + \frac{\partial\psi}{\partial r}\right)^2\right)$$

Energy density

$$\mathcal{E} = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j|\Phi|^2\right)$$

Two limitations

1) The Higgs vev is fixed by the $v = 1/2$, and so $I = N$

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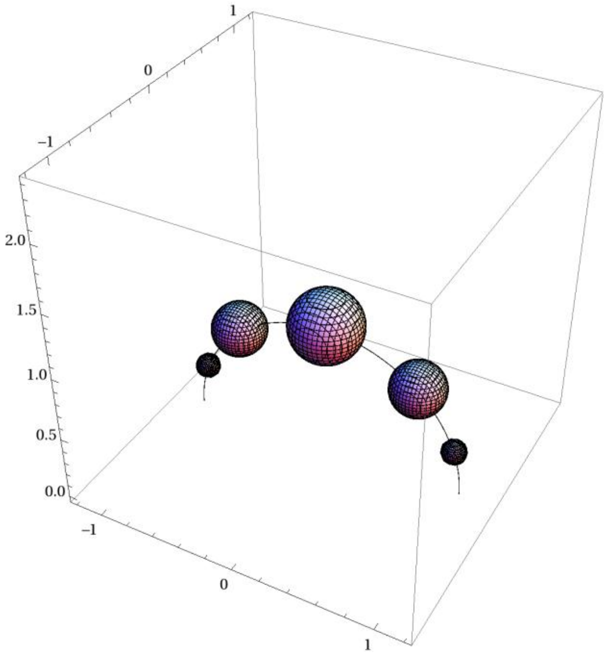
2) We can access only a subset of the full moduli

$$\dim(\mathbb{M}_N^{\mathrm{JNR}}) = 3N + 2 < 4N - 1 = \dim(\mathbb{M}_N)$$

An example: one monopole

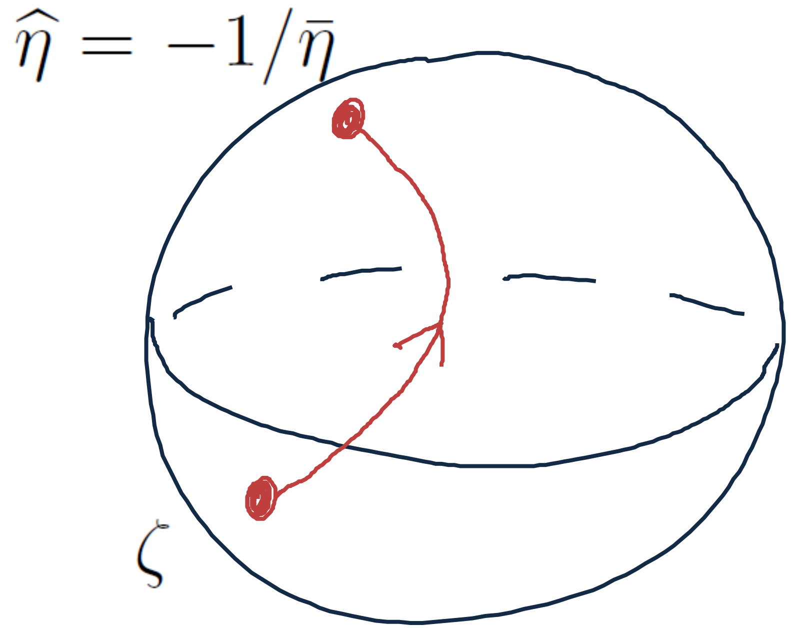
$$\zeta = \frac{\lambda_0}{|x_1 + ix_2 - \xi_0|^2 + r^2} + \frac{\lambda_1}{|x_1 + ix_2 - \xi_1|^2 + r^2}$$



geodesic connecting the two poles

Twistor space

$$(\eta, \zeta) \in \mathbb{CP}^1 \times \mathbb{CP}^1$$



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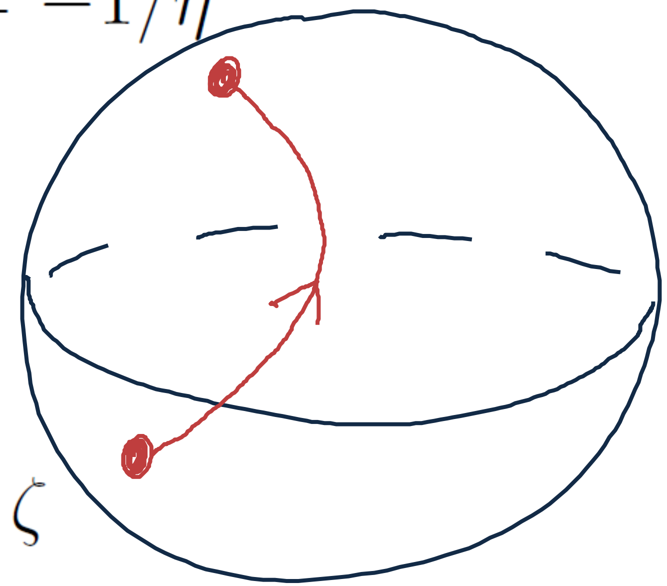


Scattering equation $(D_s - i\Phi)w = 0$

Holomorphic data

The spectral curve is a bi-holomorphic curve of degree N x N

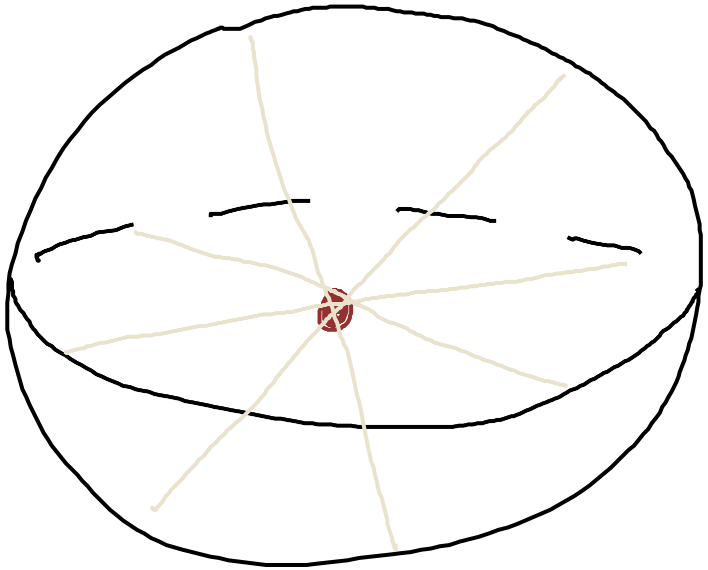
$$\sum_{i=0,j=0}^{N} c_{ij}\eta^i\zeta^j = 0$$

Corresponds to the set of geodesics where the scattering equation has normalizable solutions

Holomorphic data

For example the one-monopole has spectral curve

$$2\eta\zeta(X_1 - iX_2) + \zeta(1 + R^2 - 2X_3) - \eta(1 + R^2 + 2X_3) - 2(X_1 + iX_2) = 0$$



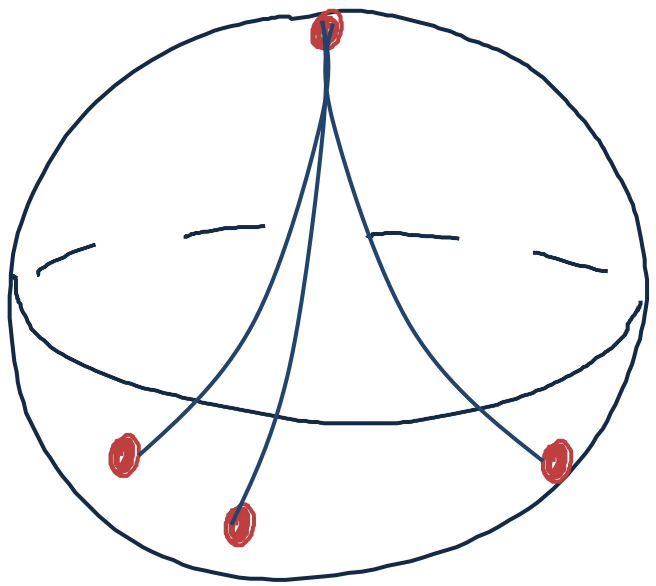
Spectral curve for generic JNR monopole

Using ADHM we can compute the explicit spectral curve for any JNR monopole:

$$\sum_{j=0}^{N} \lambda_j^2 \prod_{\substack{k=0 \\ k\neq j}}^{N} (\zeta - \gamma_k)(1 + \eta\bar{\gamma}_k) = 0$$

Rational map

Analogue of the Donaldson rational map in flat space



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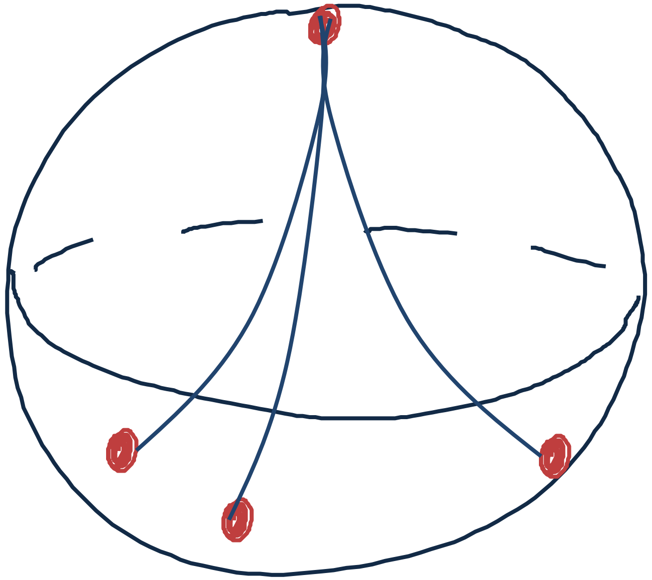


There is a very simple expression for ‘t Hooft ansatz

$$\mathcal{R} = \sum_{j=1}^{N} \frac{\lambda_j^2}{z - \gamma_j}$$

Tetrahedral monopole

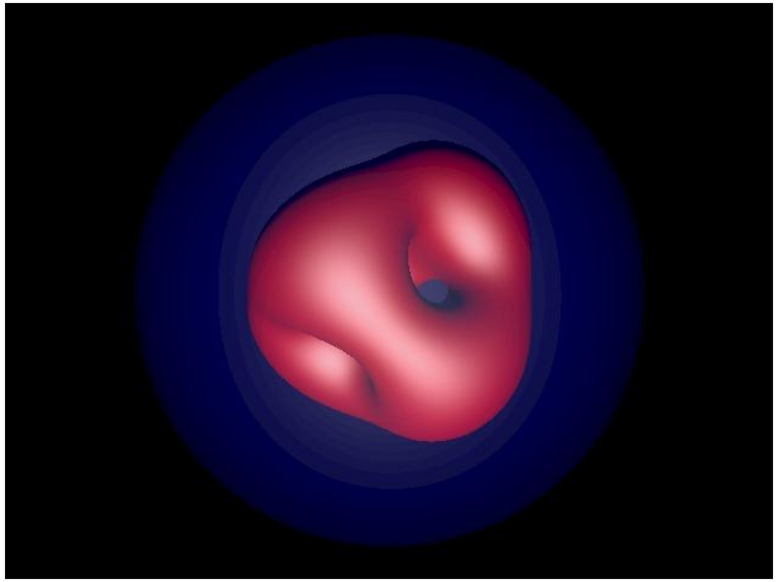
Poles are placed at the roots of the Klein polynomial

$$\mathcal{T}_v(\gamma) = \gamma^4 + 2i\sqrt{3}\gamma^2 + 1$$

with canonical weights $\lambda_j^2 = 1 + |\gamma_j|^2$

Tetrahedral monopole

Energy density level:



Spectral curve: $(\eta - \zeta)^3 + \frac{i}{\sqrt{3}}(\eta + \zeta)(\eta\zeta + 1)(\eta\zeta - 1) = 0$

Dihedral one-parameter families

D2 three monopole $\gamma_0 = \sqrt{\frac{1+a}{1-a}} e^{i\pi/4}, \ \gamma_1 = -\gamma_0, \ \gamma_2 = 1/\gamma_0, \ \gamma_3 = -1/\gamma_0$

$a \in (-1, 1)$ with canonical weights

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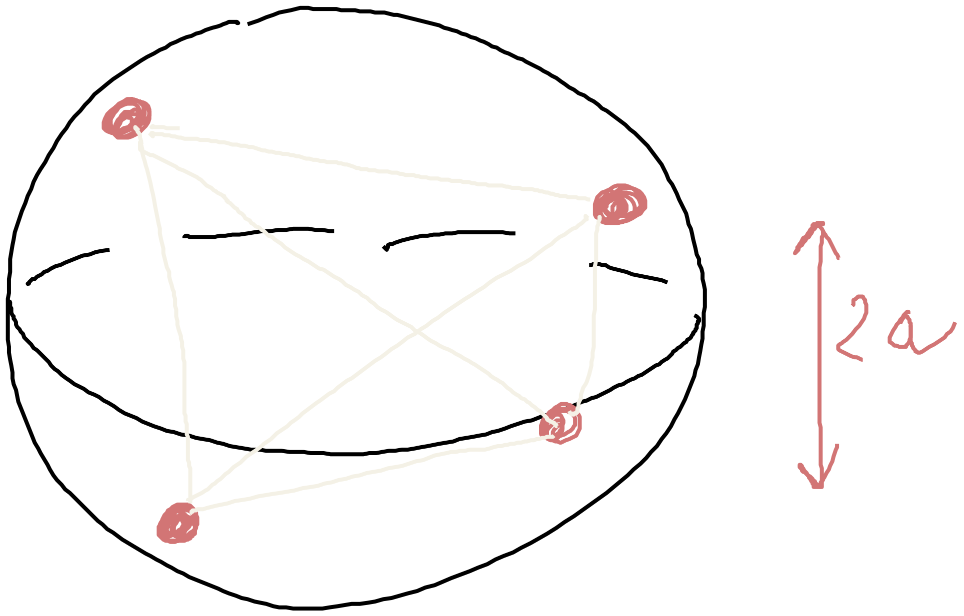

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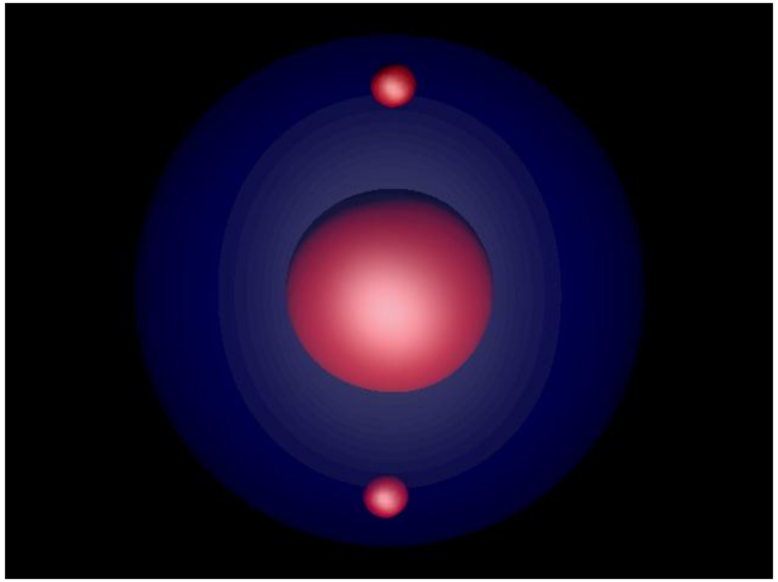

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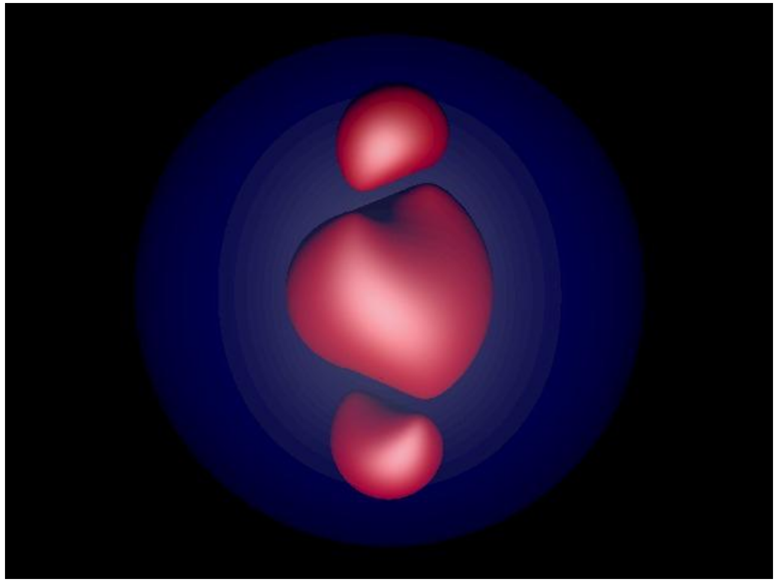

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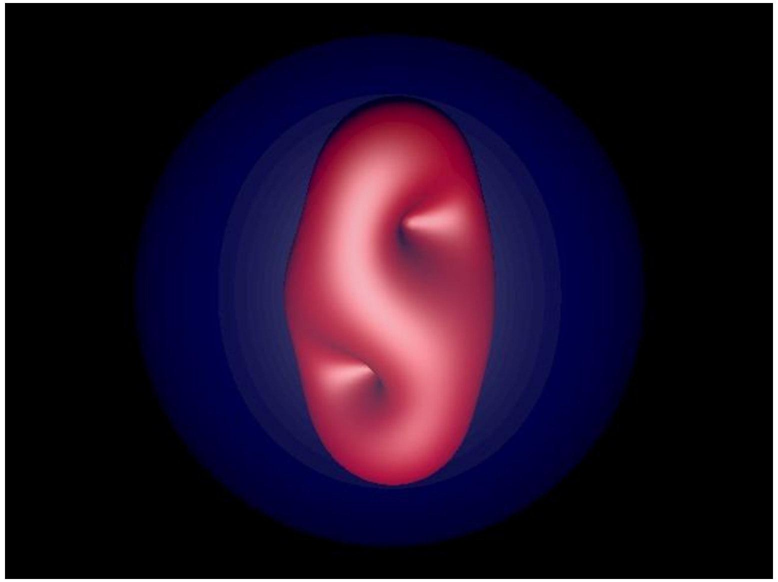

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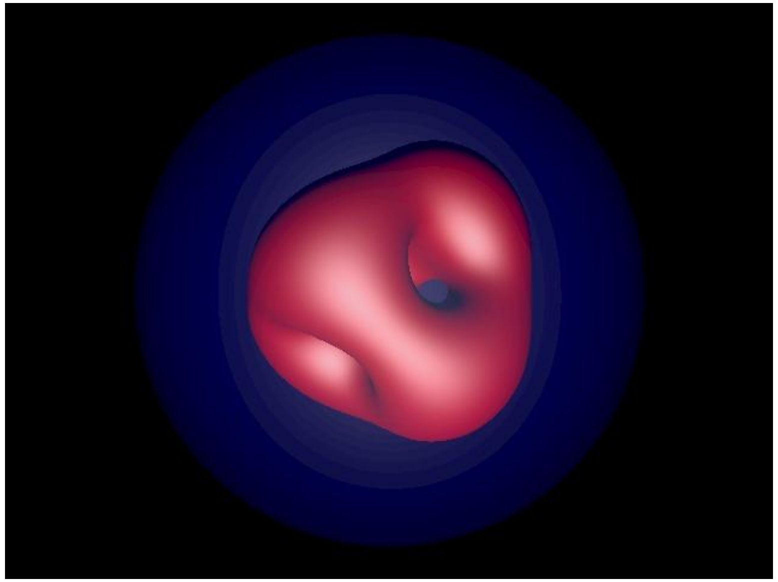

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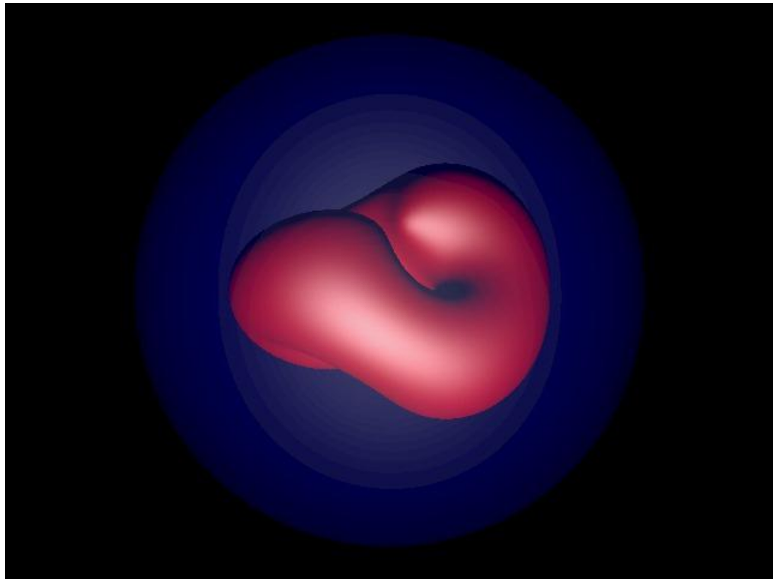

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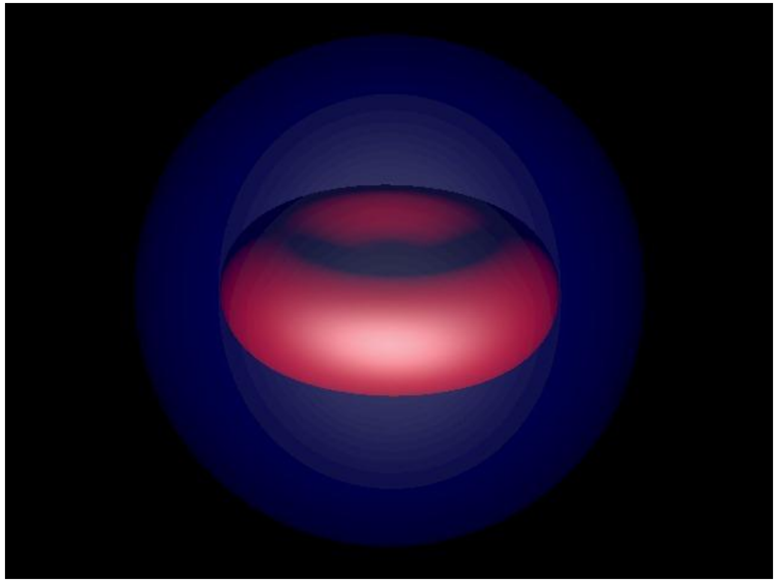
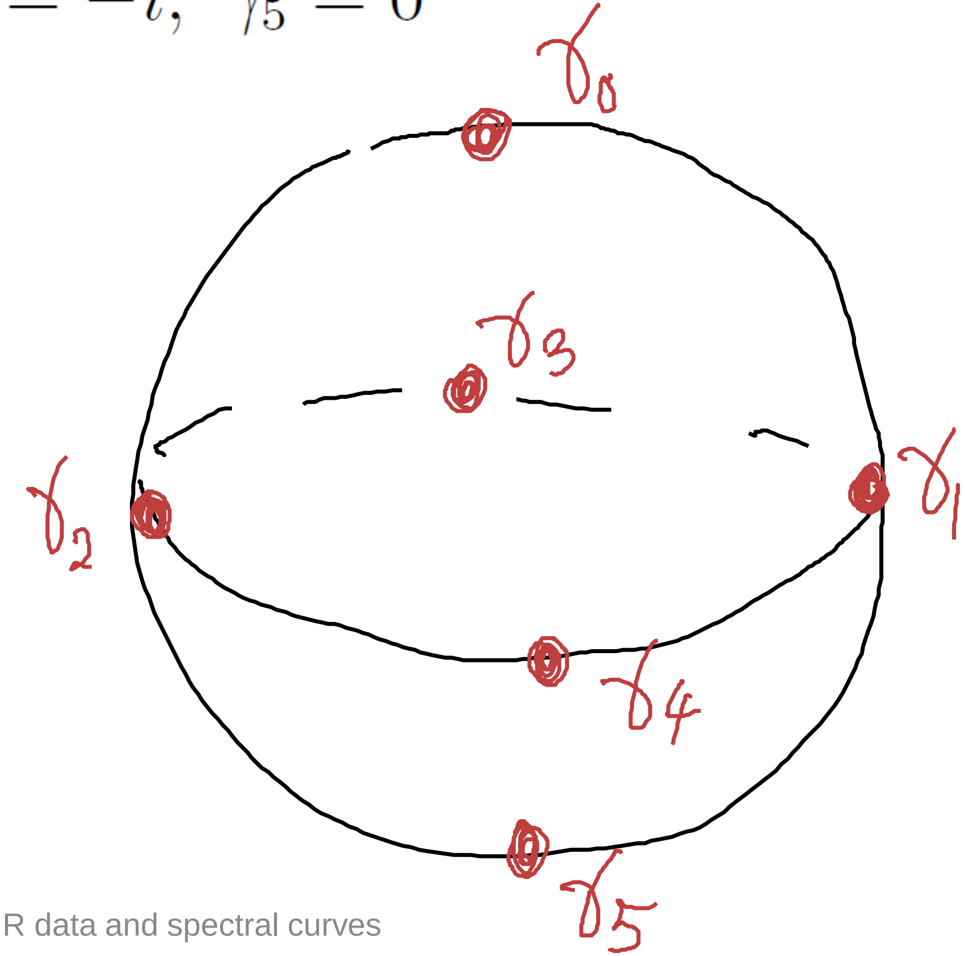


D4 five-monopole family

$$\gamma_0 = \infty,\ \gamma_1 = 1,\ \gamma_2 = -1,\ \gamma_3 = i,\ \gamma_4 = -i,\ \gamma_5 = 0$$

$$\lambda_5^2 = 1, \qquad \lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda_4^2$$



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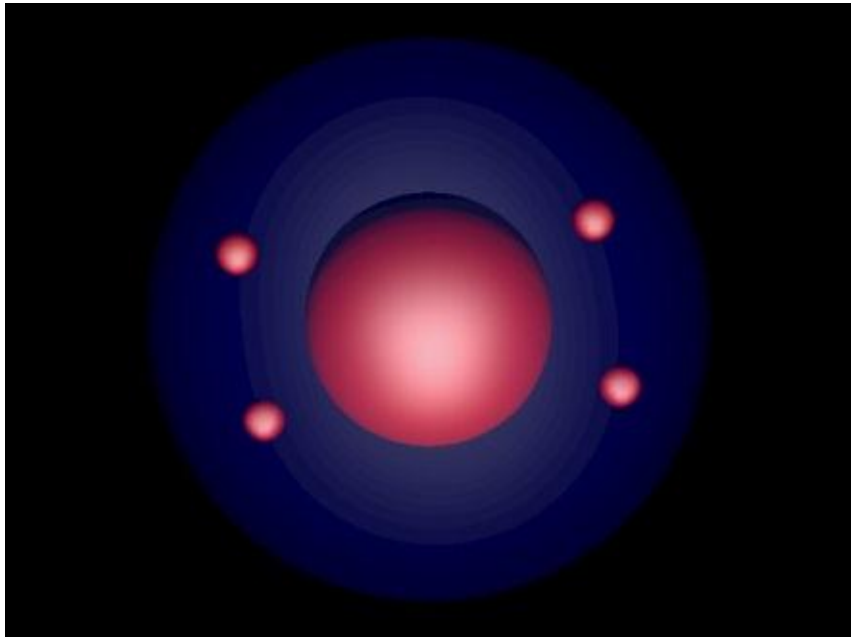

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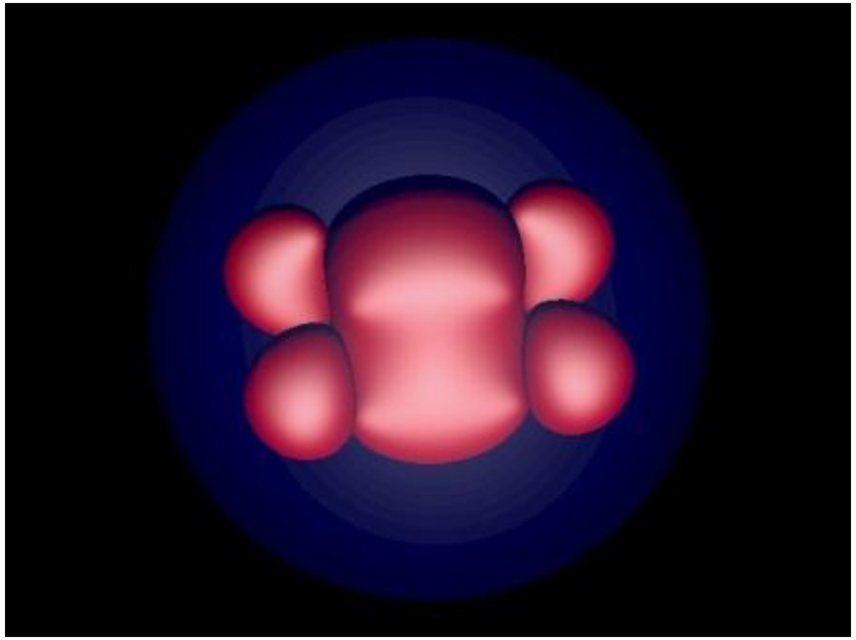

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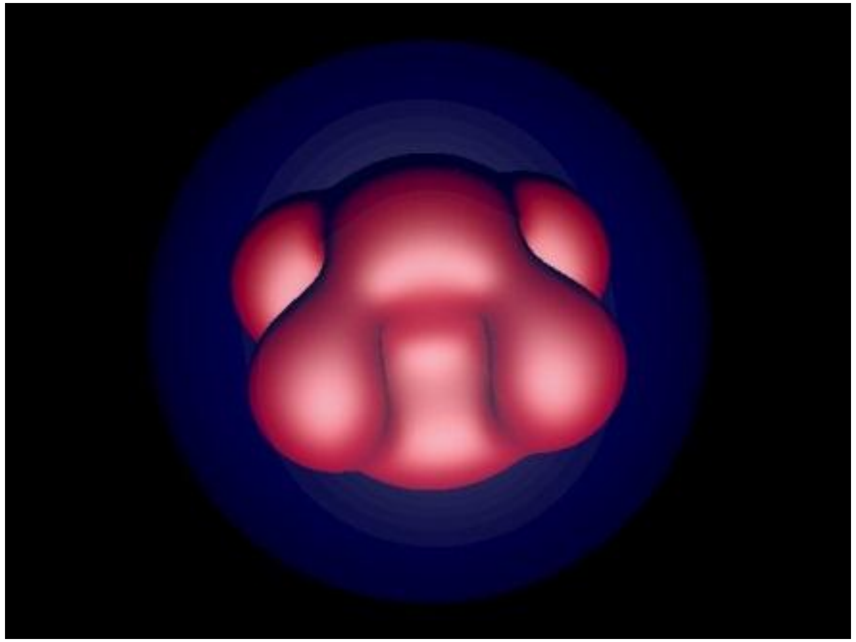

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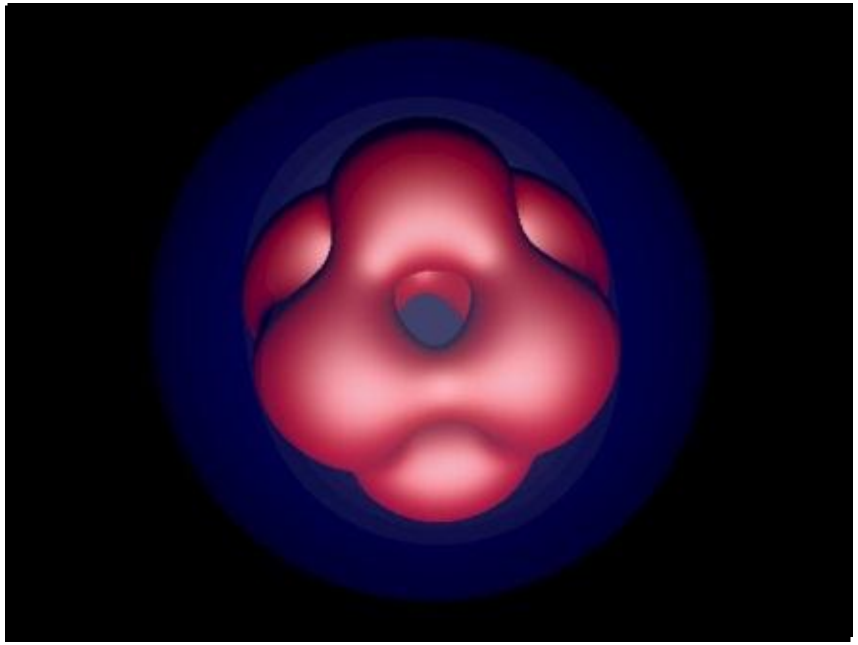

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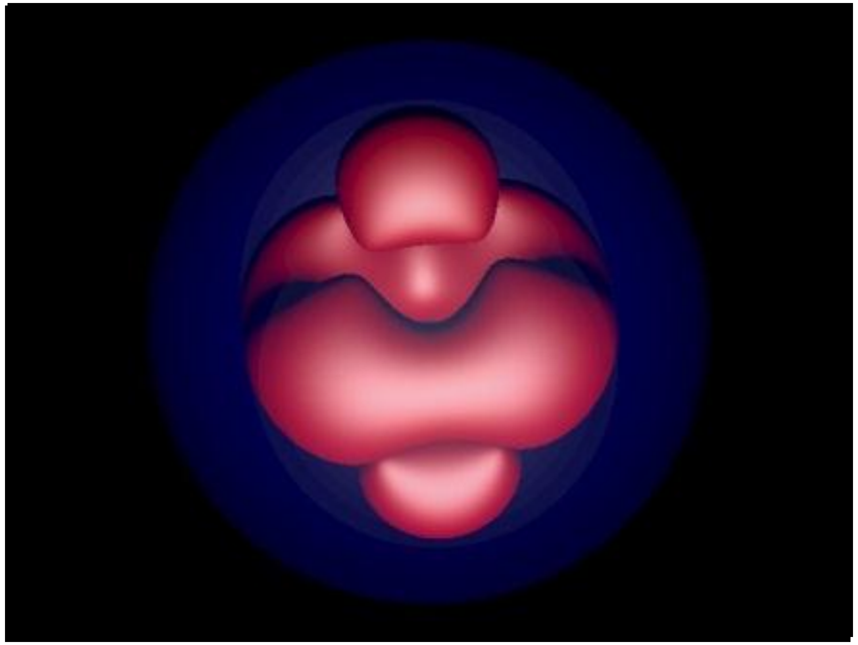

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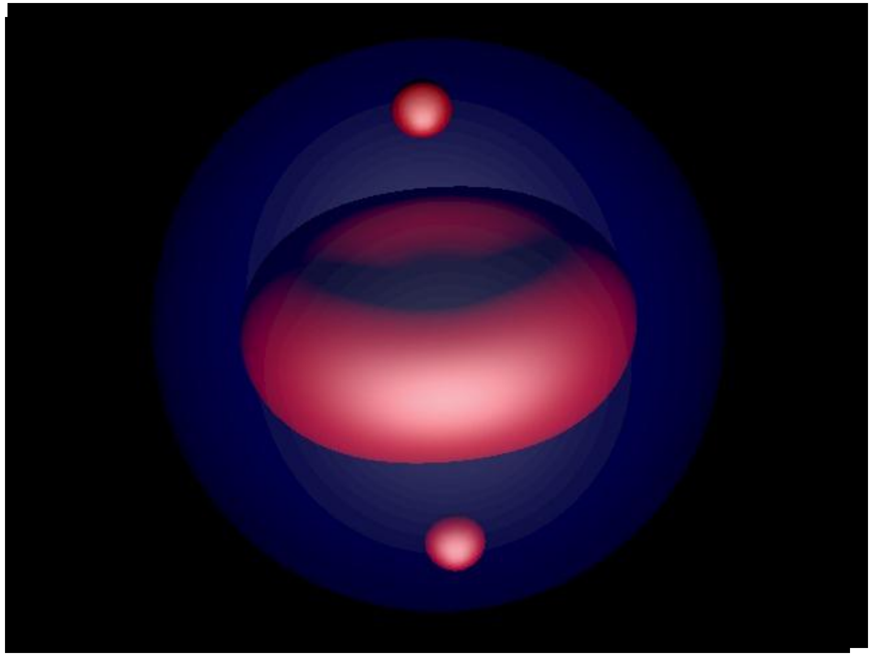
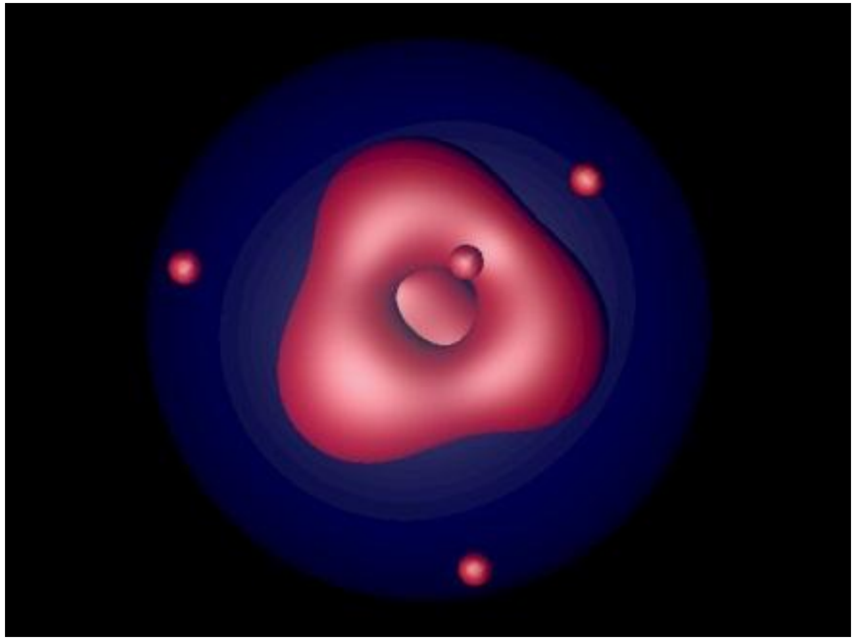


Tetrahedral seven monopole



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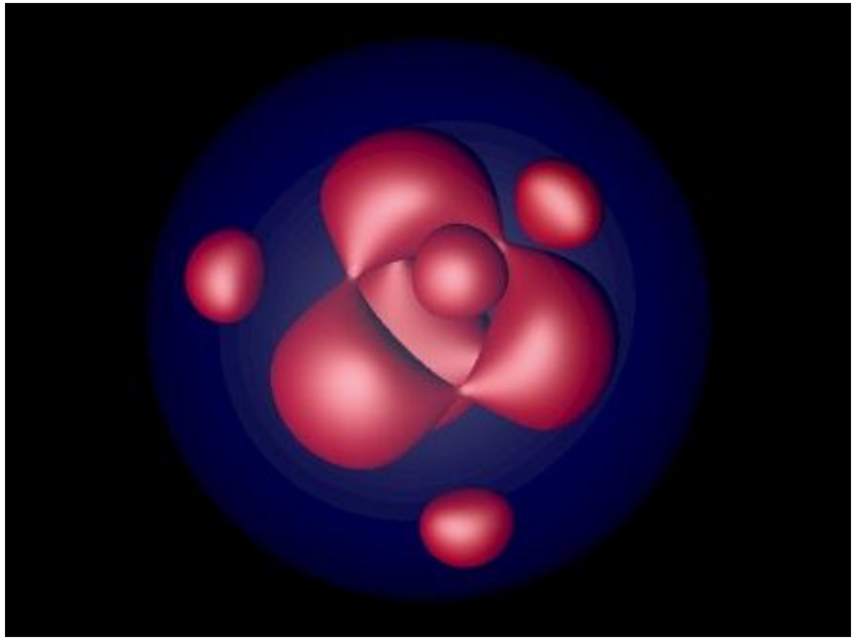

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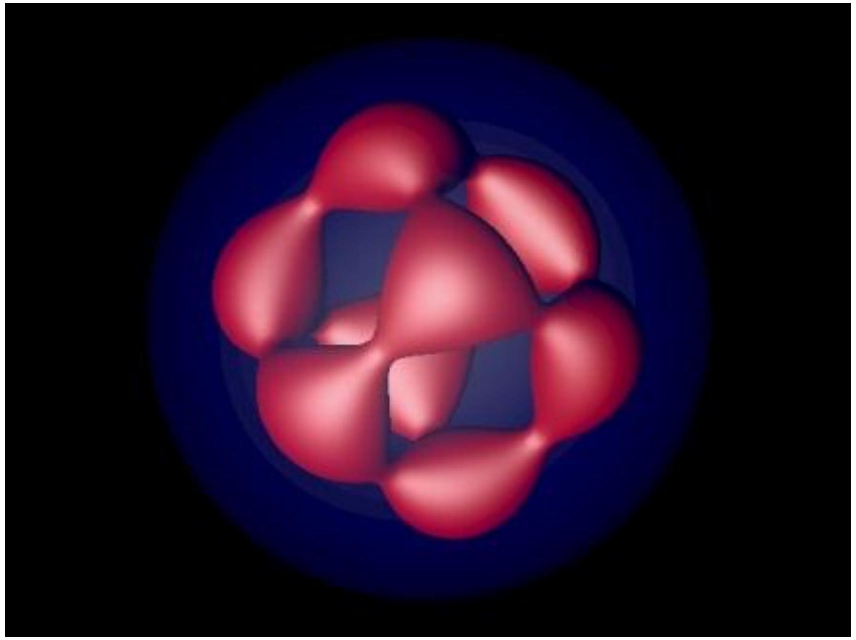

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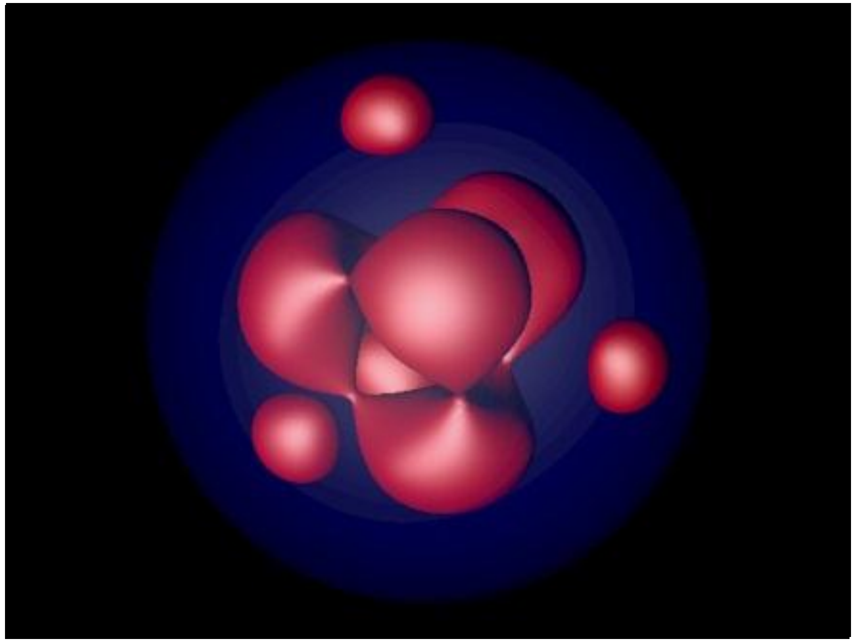

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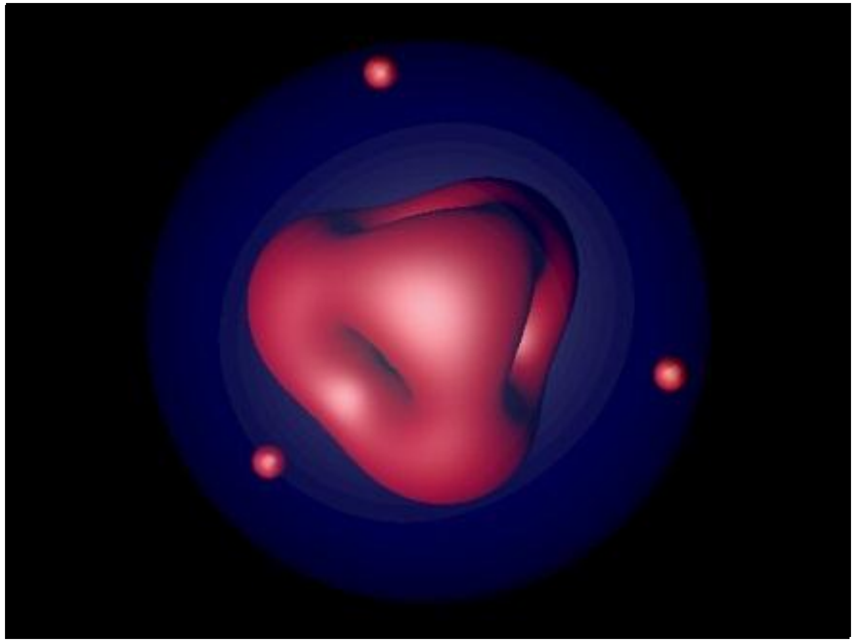
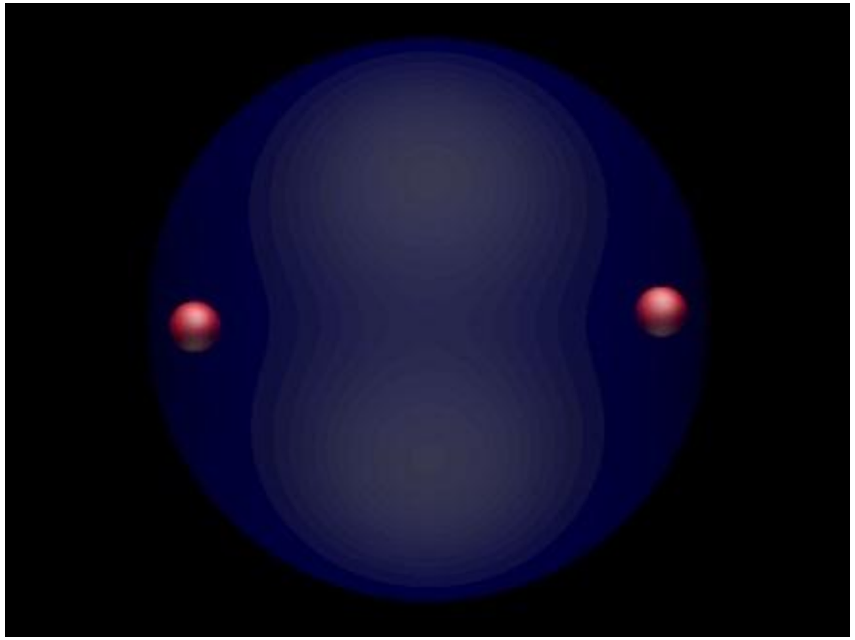


Cyclic and Dihedral



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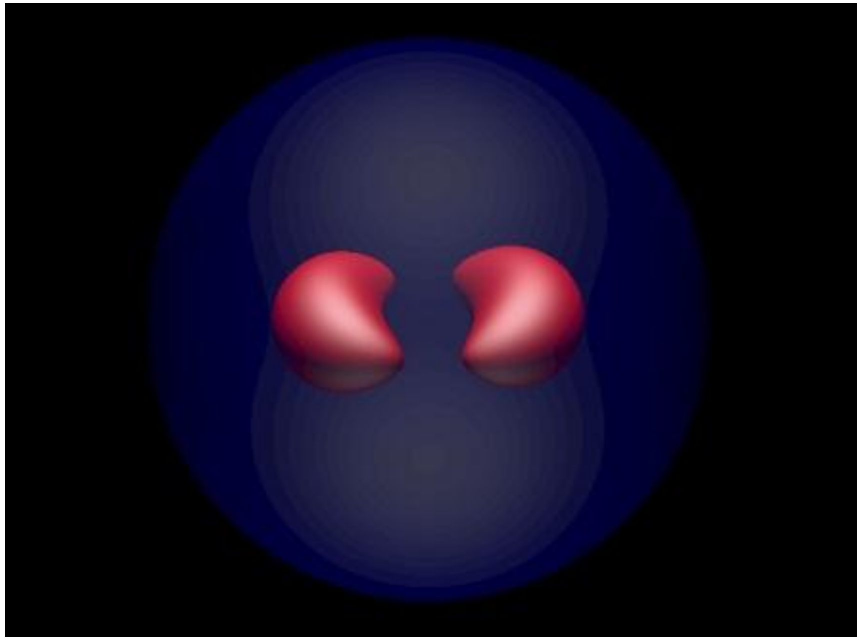

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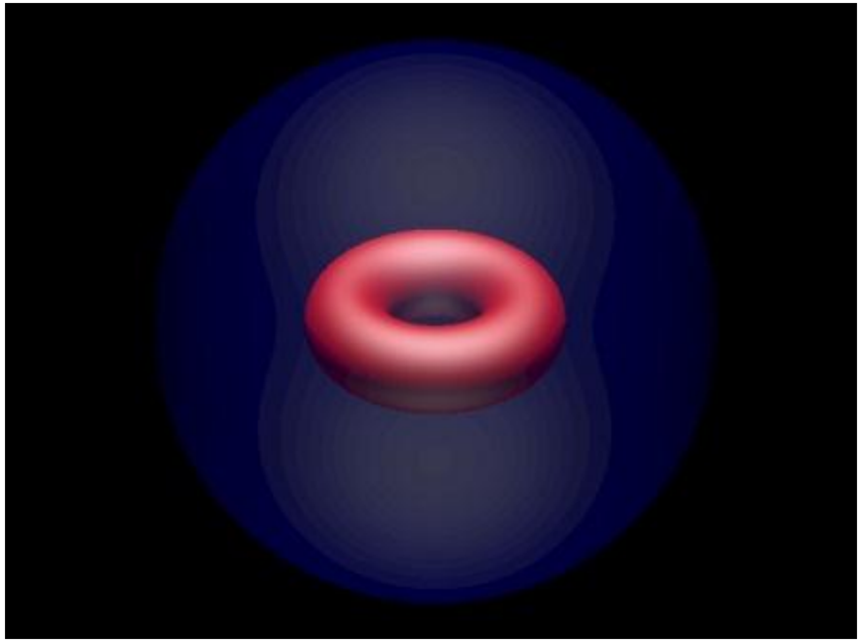

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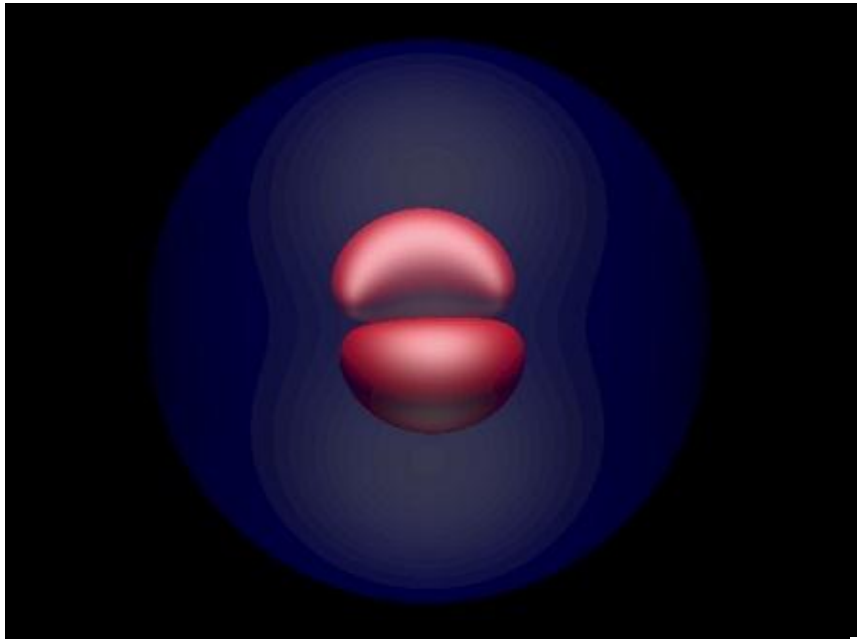

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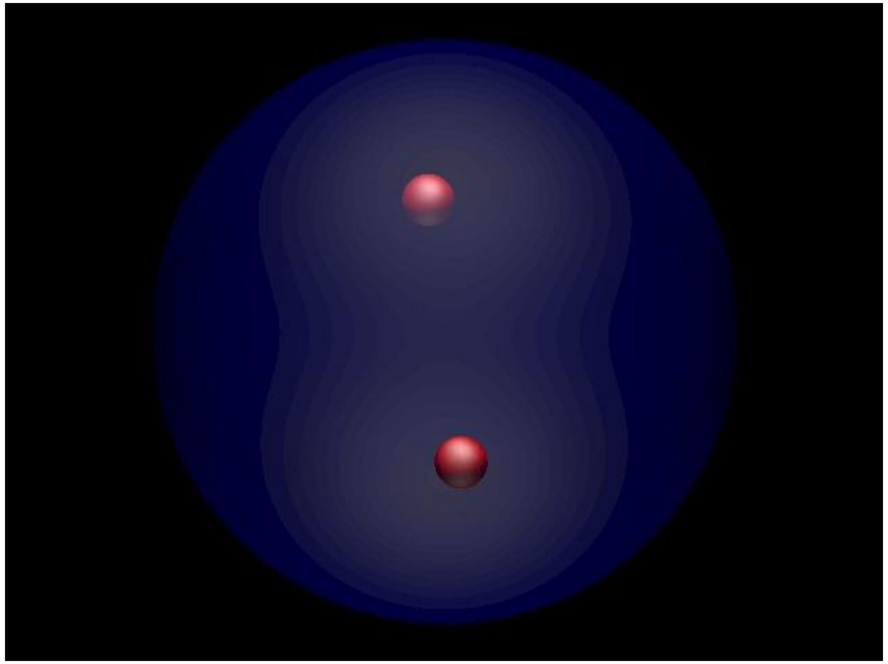
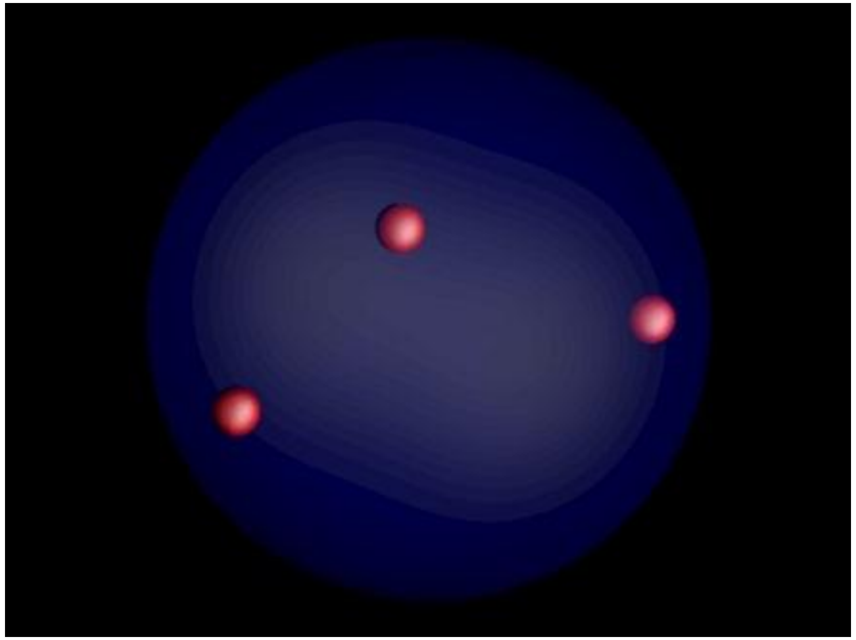


Cyclic three monopole



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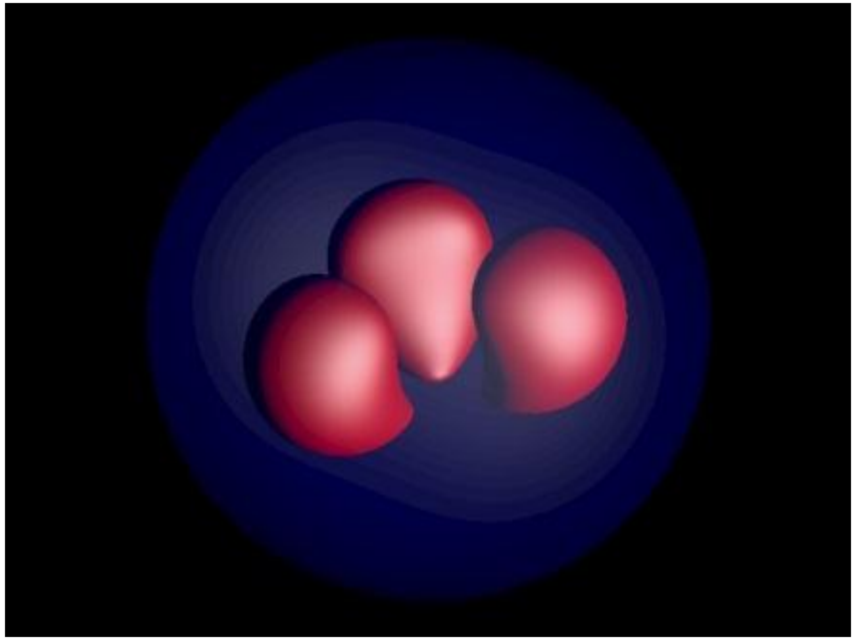

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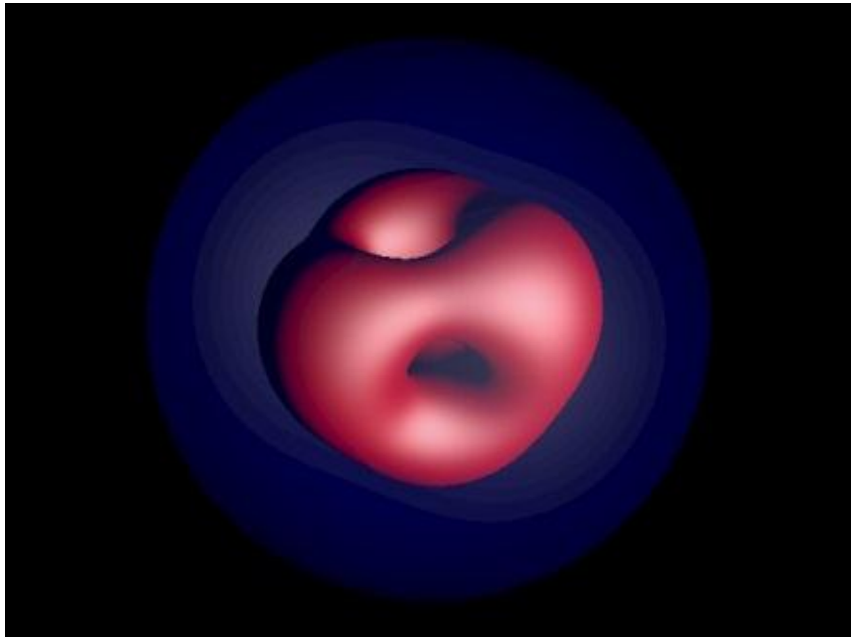

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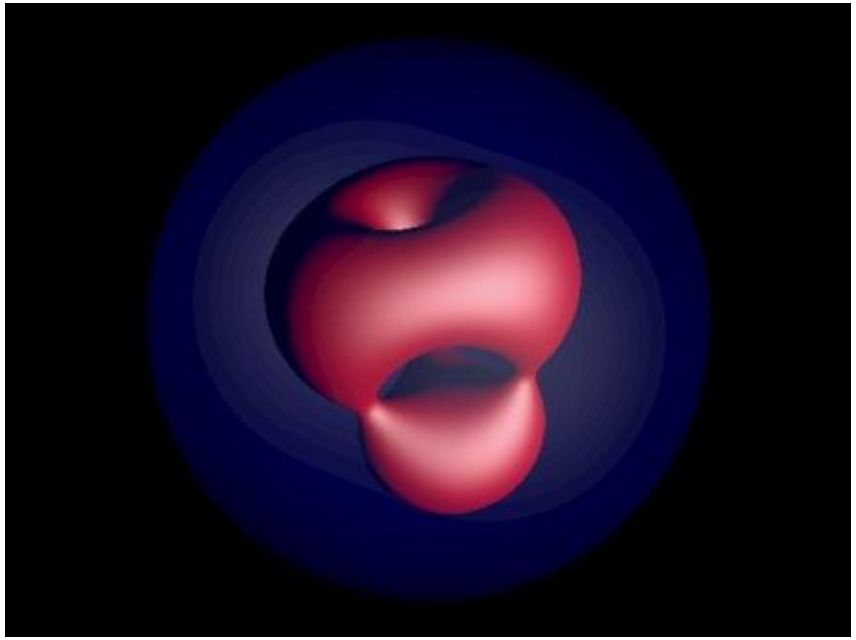

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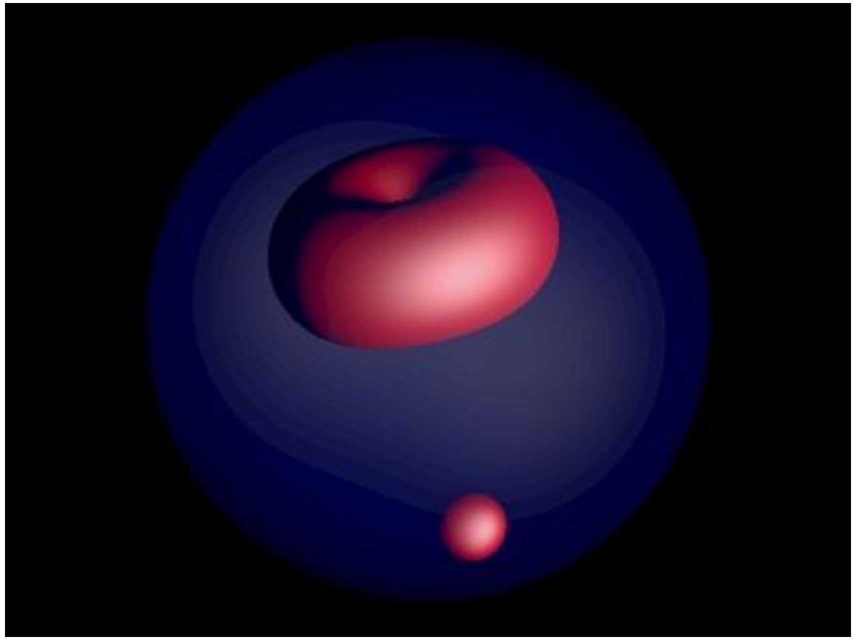

Boundary data

Gauge field at the boundary of the hyperbolic space $\mathcal{A}_z = \frac{1}{2}\partial_z \log h$

Where $h(z,\bar{z})$ is a hermitian metric given by

$$h(z,\bar{z}) = \sum_{j=0}^{N} \lambda_j^2 \prod_{\substack{k=0 \\ k\neq j}}^{N} |z-\gamma_k|^2 = \psi|_{r=0} \prod_{k=0}^{N} |z-\gamma_k|^2$$

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<u>All the information is encoded in the boundary</u>

Moduli space metric

The usual notion of metric diverges in hyperbolic space

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We can define a regularized version, essentially the UV divergent part

We can compute this explicitly for any JNR.
For one monopole for example it reproduces the hyperbolic metric

$$t_1 + it_2 = \gamma_1 \qquad t_3 = \lambda_1 .$$

$$g_{\mu\nu} dt_\mu dt_\nu = \frac{dt_1^2 + dt_2^2 + dt_3^2}{t_3^2}$$

Conclusion

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- We showed many one-parameter families of dihedral and cyclic monopoles, analogue to scattering solutions in flat space