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Distorting General Relativity: Gravity's Rainbow and f(R) gravity at work

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Introduction

- Hamiltonian Formulation of General Relativity and the Wheeler-DeWitt Equation
- The Cosmological Constant as a Zero Point Energy Computation
- Gravity's Rainbow as a tool for computing ZPE
- Gravity's Rainbow and f(R) theory at work

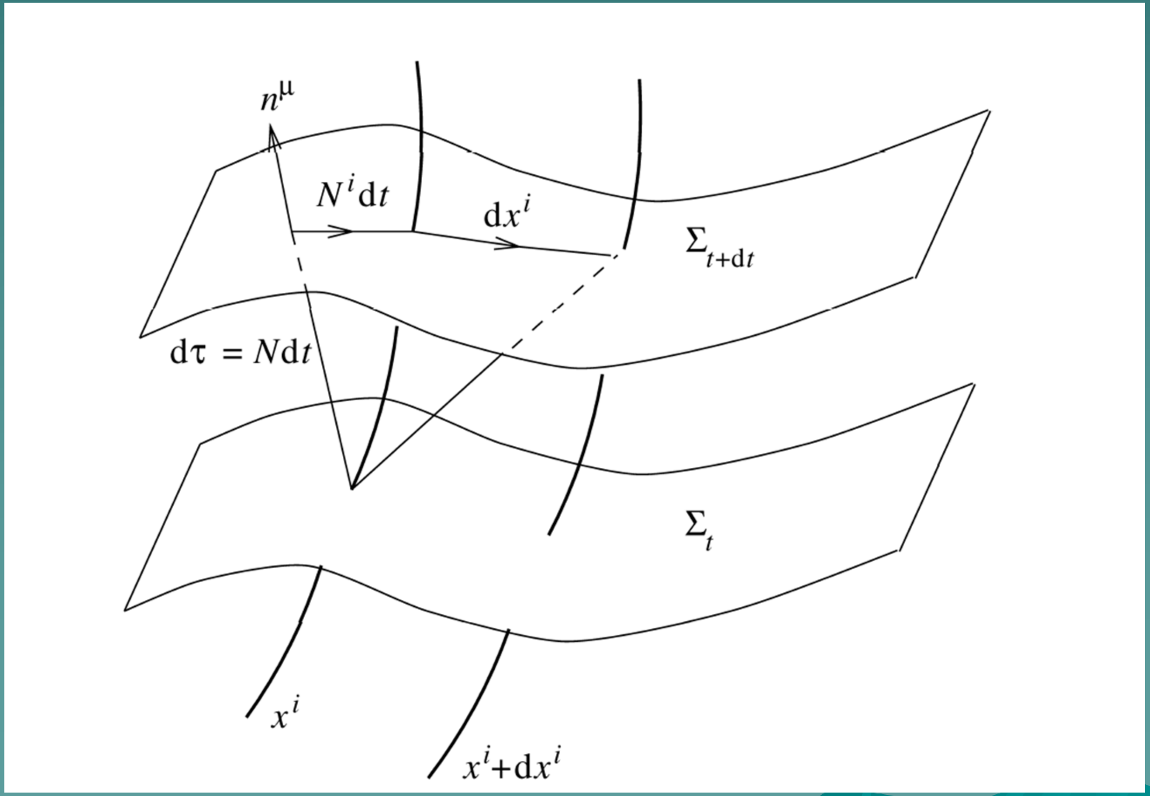
Part I: Hamiltonian Formulation of General Relativity and the Wheeler-DeWitt Equation

Relevant Action for Quantum Cosmology

$$S = \frac{1}{2\kappa}\int_{\mathfrak{M}} d^4x\sqrt{-g}\left({}^4R - 2\Lambda\right) + 2\int_{\partial\mathfrak{M}} d^3x\sqrt{g^{(3)}}K + S_{matter} \qquad \kappa = 8\pi G$$

$G \rightarrow$ Newton's Constant

$\Lambda \rightarrow$ Cosmological Constant



Relevant Action for Quantum Cosmology

$$S=\frac{1}{2\kappa}\int_{\mathfrak{M}} d^4x\sqrt{-g}\left({}^4R-2\Lambda\right)+\frac{1}{\kappa}\int_{\partial\mathfrak{M}} d^3x\sqrt{g^{(3)}}K+S_{matter}$$

ADM Decomposition

$$g_{\mu\nu}=\begin{pmatrix}-N^2+N^kN_k & N_j\\ N_i & g_{ij}^{(3)}\end{pmatrix}\qquad g^{\mu\nu}=\begin{pmatrix}-\frac{1}{N^2} & \frac{N^j}{N^2}\\ \frac{N^i}{N^2} & g^{ij(3)}-\frac{N^iN^j}{N^2}\end{pmatrix}$$

$$\mathrm{ds}^2=g_{\mu\nu}dx^\mu dx^\nu=-N^2dt^2+g_{ij}\left(N^idt+dx^i\right)\left(N^jdt+dx^j\right)$$

N is the lapse function N_i is the shift function

$$K_{ij}=-\frac{1}{2N}\dot{g}_{ij}+\nabla_iN_j+\nabla_jN_i\qquad K=K^{ij}g_{ij}$$

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^k N_k & N_j \\ N_i & g_{ij}^{(3)} \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & g^{ij(3)} - \frac{N^i N^j}{N^2} \end{pmatrix} \qquad \mathrm{ds}^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$S = \frac{1}{2\kappa} \int_{\Sigma\times I} dt d^3x N \sqrt{g^{(3)}} \left(K^{ij} K_{ij} - K^2 + {}^3R - 2\Lambda \right) + S_{\partial(\Sigma\times I)} + S_{matter}$$

Legendre Transformation $\rightarrow$ $H = \int_\Sigma d^3x \left(N_i \mathcal{H}^i + N\mathcal{H} \right) + H_{\partial\Sigma}$

$\mathcal{H} = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} \left({}^3R - 2\Lambda \right) = 0$ Classical Constraint $\rightarrow$ Invariance by time reparametrization

$\mathcal{H}^i = 2\pi^{ij}{}_{|j} = 0$ Classical Constraint $\rightarrow$ Gauss Law

Wheeler-De Witt Equation

B. S. DeWitt, Phys. Rev.**160**, 1113 (1967).

$$\left[\left(2\kappa\right)G_{ijkl}\pi^{ij}\pi^{kl}-\frac{\sqrt{g}}{2\kappa}\left(R-2\Lambda\right)\right]\Psi\left[g_{ij}\right]=0$$

- G_{ijkl} is the super-metric,
- R is the scalar curvature in 3-dim.

Example:WDW for Tunneling

$$ds^2=-N^2dt^2+a^2\left(t\right)d\Omega_3^2 \qquad H\Psi[a]=\left[-\frac{\partial^2}{\partial a^2}-\frac{q}{a}\frac{\partial}{\partial a}+\frac{9\pi^2}{4G^2}\left(a^2-\frac{\Lambda}{3}a^4\right)\right]\Psi[a]=0$$

Formal Schrödinger Equation with zero eigenvalue whose solution is a linear combination of Airy's functions (q=-1 Vilenkin Phys. Rev. D 37, 888 (1988).) containing expanding solutions

Part II: The Cosmological Constant as a Zero Point Energy Computation

Wheeler-De Witt Equation

B. S. DeWitt, Phys. Rev.**160**, 1113 (1967).

Reconsider the WDW Equation as an Eigenvalue Problem

- Λ can be seen as an eigenvalue
- $\Psi[g_{ij}]$ can be considered as an eigenfunction

Define

$$\hat{\Lambda}_\Sigma = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} R \qquad \Lambda(\vec{x}) = -\frac{\sqrt{g}}{\kappa} \Lambda_C$$

$$\int D[g_{ij}] \Psi^*[g_{ij}] \hat{\Lambda}_\Sigma \Psi[g_{ij}] = \int D[g_{ij}] \Psi^*[g_{ij}] \Lambda(\vec{x}) \Psi[g_{ij}]$$

Wheeler-De Witt Equation

B. S. DeWitt, Phys. Rev.**160**, 1113 (1967).

$$\frac{1}{V}\frac{\int D\mu[h]\Psi^*[h]\int_\Sigma d^3x\hat{\Lambda}_\Sigma\Psi[h]}{\int D\mu[h]\Psi^*[h]\Psi[h]}=-\frac{\Lambda}{\kappa}$$

Induced Cosmological "Constant"

$$D\mu[h]=D\left[h_{ij}^{\perp}\right]D\left[\xi_j^T\right]D[h]J$$

Solve this infinite dimensional PDE with a Variational Approach

Ψ is a trial wave functional of the gaussian type

Schrödinger Picture

Spectrum of Λ depending on the metric

Energy (Density) Levels

Canonical Decomposition

M. Berger and D. Ebin, J. Diff. Geom.**3**, 379 (1969). J. W. York Jr., J. Math. Phys., **14**, 4 (1973); Ann. Inst. Henri Poincaré **A 21**, 319 (1974).

$$\begin{cases} g_{ij} \to \bar{g}_{ij} + h_{ij} \\ N \to N \\ N_i \to 0 \end{cases} \quad \Leftrightarrow \quad h_{ij} = \frac{1}{3} h g_{ij} + \left(L\xi\right)_{ij} + h_{ij}^{\perp} \quad \Leftrightarrow \quad \nabla^i \left(h_{ij} - \frac{1}{3} g_{ij} h \right) = 0 = g^{ij} h_{ij}$$

Equivalent in 4D to $h_\mu^0 = h_\mu^\mu = \nabla^\mu h_\mu^\nu = 0$ No Ghosts Contribution

$$\frac{1}{V} \frac{\int D\mu[h] \Psi^*[h] \int_\Sigma d^3x \widehat{\Lambda}_\Sigma \Psi[h]}{\int D\mu[h] \Psi^*[h] \Psi[h]} = -\frac{\Lambda}{\kappa}$$

$$D\mu[h] = D\left[h_{ij}^{\perp}\right] D\left[\xi_j^T\right] D[h] J$$

- h is the trace (spin 0)
- $(L\xi)_{ij}$ is the gauge part [spin 1 (transverse) + spin 0 (long.)]. F.P determinant (ghosts)
- $h^{\perp}_{ij}$ transverse-traceless □ graviton (spin 2)

Gravity’s Rainbow

Doubly Special Relativity

G. Amelino-Camelia, Int.J.Mod.Phys. D 11, 35 (2002); gr-qc/001205.
G. Amelino-Camelia, Phys.Lett. B 510, 255 (2001); hep-th/0012238.

$$E^2 g_1^2\left(E/E_P\right) - p^2 g_2^2\left(E/E_P\right) = m^2$$

$$\lim_{E/E_P\to 0} g_1\left(E/E_P\right) = \lim_{E/E_P\to 0} g_2\left(E/E_P\right) = 1$$

Curved Space Proposal *Gravity’s Rainbow*
[J. Magueijo and L. Smolin, Class. Quant. Grav. 21, 1725 (2004) arXiv:gr-qc/0305055].

$$ds^2 = -\frac{N(r)\,dt^2}{g_1^2\left(E/E_P\right)} + \frac{dr^2}{\left(1-\frac{b(r)}{r}\right) g_2^2\left(E/E_P\right)} + \frac{r^2}{g_2^2\left(E/E_P\right)} d\theta^2 + \frac{r^2}{g_2^2\left(E/E_P\right)} \sin^2\theta\, d\varphi^2$$

$N(r) = \exp\left(-2\Phi(r)\right)$ $\Phi(r)$ is the redshift function

$b(r)$ is the shape function Condition $\rightarrow b(r_0) = r_0 \quad r \in [r_0, +\infty)$

Eliminating Divergences using Gravity's Rainbow

[R.G. and G.Mandanici, Phys. Rev. D 83, 084021 (2011), arXiv:1102.3803 [gr-qc]]

$$\Delta_2 = \underbrace{\left(\Delta h\right)_{ij} - 4R_{ia}h^a_j + Rh_{ij}}_{\text{Modified Lichnerowicz operator}}$$

$$\left(\Delta h\right)_{ij} = \underbrace{\Delta h_{ij} - 2R_{ijkl}h^{jl} + R_{ia}h^a_j + R_{ja}h^a_i}_{\text{Standard Lichnerowicz operator}}$$

$$\left(\Delta_2 \tilde{h}^{\perp}\right)_{ij} = \frac{E^2}{g_2^2\left(E\right)}\tilde{h}^{\perp}_{ij}$$

$$\hat{\Lambda}^{\perp}_{\Sigma} = \frac{g_2^3\left(E\right)}{4V}\int_\Sigma d^3x\sqrt{\tilde{\tilde{g}}}\,\tilde{G}^{ijkl}\left[\left(2\kappa\right)\frac{g_1^2\left(E\right)}{g_2^3\left(E\right)}\tilde{K}^{-1,\perp}\left(x,x\right)_{ijkl} + \frac{1}{2\kappa g_2\left(E\right)}\left(\tilde{\Delta}_2\tilde{K}^{\perp}\left(x,x\right)\right)_{ijkl}\right]$$

$$\tilde{K}\left(\vec{x},\vec{y}\right)_{ijkl} := \sum_\tau \frac{\tilde{h}\left(\vec{x}\right)^{(\tau)\perp}_{ij}\tilde{h}\left(\vec{y}\right)^{(\tau)\perp}_{kl}}{2\lambda\left(\tau\right)g_2^4\left(E\right)} \quad \text{(Propagator)}$$

Eliminating Divergences using Gravity's Rainbow

[R.G. and G.Mandanici, Phys. Rev. D 83, 084021 (2011), arXiv:1102.3803 [gr-qc]]

$$\begin{cases} m_1^2(r) = \dfrac{6}{r^2}\left(1-\dfrac{b(r)}{r}\right)+\dfrac{3b'(r)}{2r^2}-\dfrac{3b(r)}{2r^3} \\ m_2^2(r) = \dfrac{6}{r^2}\left(1-\dfrac{b(r)}{r}\right)+\dfrac{b'(r)}{2r^2}+\dfrac{3b(r)}{2r^3} \end{cases}$$

We can define an r-dependent radial wave number

$$k^2(r,l,E_{nl}) = \frac{E_{nl}^2}{g_2^2(E/E_P)} - \frac{l(l+1)}{r^2} - m_i^2(r) \qquad r \equiv r(x)$$

$$\frac{\Lambda}{8\pi G} = -\frac{1}{3\pi^2}\sum_{i=1}^{2}\int_{E^*}^{+\infty} E_i g_1(E/E_P) g_2(E/E_P) \frac{d}{dE_i}\sqrt{\left(\frac{E_i^2}{g_2^2(E/E_P)} - m_i^2(r)\right)^3}\, dE_i$$

Standard Regularization

$$\frac{\Lambda}{8\pi G} = -\frac{1}{16\pi^2}\int_{\sqrt{m_i^2(r)}}^{+\infty} \frac{\omega_i^2}{\left(\omega_i^2 - m_i^2(r)\right)^{\varepsilon-\frac{1}{2}}}\, d\omega_i$$

Gravity's Rainbow and the Cosmological Constant

R.G. and G.Mandanici, Phys. Rev. D 83, 084021 (2011), arXiv:1102.3803 [gr-qc]

Popular Choice...... $\rightarrow$ Not Promising

$$g_1\left(E/E_P\right)=1-\eta\left(\frac{E}{E_P}\right)^n$$

$$g_2\left(E/E_P\right)=1$$

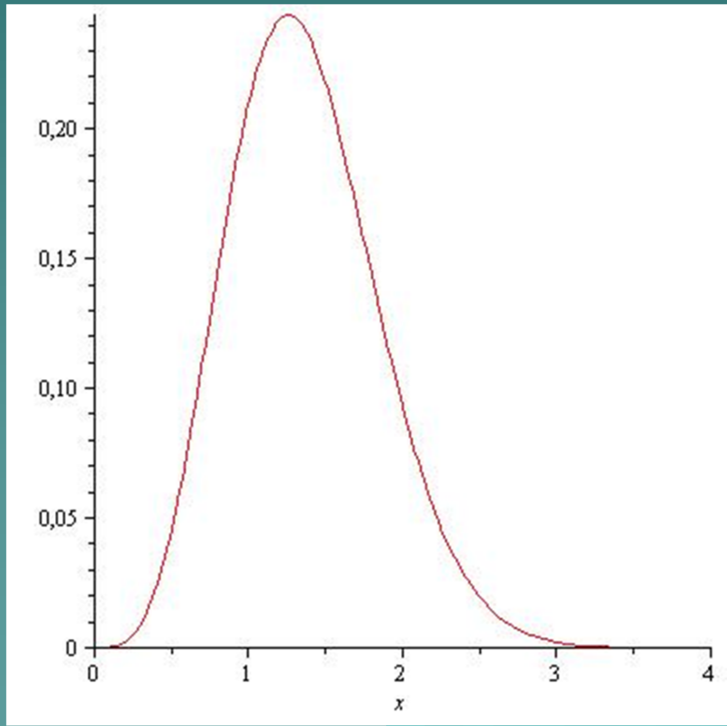
Failure of Convergence

$$g_1\left(E/E_P\right)=\exp\left(-\alpha\frac{E^2}{E_P^2}\right)\left(1+\beta\frac{E}{E_P}\right)$$

$$g_2\left(E/E_P\right)=1$$

Minkowski - de Sitter - Anti-de Sitter

$$m_1^2\left(r\right)=m_2^2\left(r\right)=m_0^2\left(r\right)\rightarrow x=\sqrt{m_0^2\left(r\right)/E_P^2}$$



Comparison with the Noncommutative Approach

[R.G. & P. Nicolini Phys. Rev. D 83, 064021 (2011); 1006.4518 [gr-qc]]

Noncommutative ST $\left[x^{\mu},x^{\nu}\right]=\mathrm{i}\theta^{\mu\nu}$

Number of Nodes $\frac{d^3\vec{x}d^3\vec{k}}{(2\pi)^3}\xrightarrow[\text{Version}]{\text{NonCommutative}}\frac{d^3\vec{x}d^3\vec{k}}{(2\pi)^3}\exp\left(-\frac{\theta}{4}k_i^2\right)$

$$\rho_i(\theta)\propto\int_{\sqrt{m_i^2(r)}}^{+\infty}\left(\omega_i^2-m_i^2(r)\right)^{\frac{3}{2}}\exp\left(-\frac{\theta}{4}k_i^2\right)d\omega_i$$

$$k_i^2=\omega_i^2-m_i^2(r)$$

$$\lim_{M\to 0}\frac{\Lambda}{8\pi G}=0$$

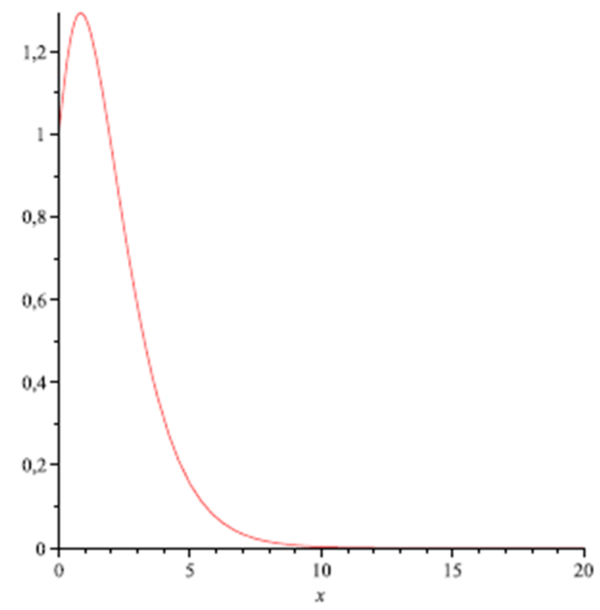


FIG. 1: Plot of $\Lambda/8\pi G$ as a function of the scale invariant y (z) which depends on the backgound choice.

Problem

Prescription: a metric which has the property of being asymptotically flat MUST satisfy the Minkowski limit

$$\lim_{M\to 0}\frac{\Lambda}{8\pi G}=0$$

In analogy with the SU(2) Yang Mills Constant ChromoMagnetic Field

$$\lim_{H\to 0}\frac{E}{V}=\frac{H^2}{2}+\frac{11}{48\pi^2}(eH)^2\ln\left(\frac{eH}{\mu^2}-\frac{1}{2}\right)$$

→ Some background, like the Schwarzschild metric, does not satisfy the Minkowski limit

Gravity’s Rainbow and f(R) theory at work

(R.G., **JCAP 1306 (2013) 017** arXiv:1210.7760)

For a f(R) theory in 4D

$$-\frac{\Lambda}{\kappa}-\frac{1}{2\kappa V}\int_{\Sigma} d^3x\sqrt{g}\,\frac{\mathcal{R}f'\left(\mathcal{R}\right)-f\left(\mathcal{R}\right)}{f'\left(\mathcal{R}\right)}=\sqrt{h\left(\mathcal{R}\right)}\frac{1}{4}\sum_{\tau}\left[\sqrt{E_1^2\left(\tau\right)}+\sqrt{E_2^2\left(\tau\right)}\right].$$

Transformation rule under Gravity’s Rainbow

$$\mathcal{R}\to\mathcal{R}_{g_1\ g_2}=g_2^2\left(E\right)\tilde{R}+g_1^2\left(E\right)\left(\tilde{K}_{ij}\tilde{K}^{ij}-\left(\tilde{K}\right)^2-2\nabla_{\mu}\left(\tilde{K}\tilde{u}^{\mu}+\tilde{a}^{\mu}\right)\right).$$

Gravity's Rainbow and f(R) theory at work

$$\frac{\Lambda}{\kappa}=-\frac{1}{2\kappa V}\int_{\Sigma}d^{3}x\sqrt{g}\frac{\mathcal{R}f'\left(\mathcal{R}_{g_1\ g_2}\right)-f\left(\mathcal{R}_{g_1\ g_2}\right)}{f'\left(\mathcal{R}_{g_1\ g_2}\right)}$$

$$-\frac{1}{\pi^2}\sum_{i=1}^{2}\int_{E^*}^{+\infty}\sqrt{h\left(\mathcal{R}_{g_1\ g_2}\right)}E_i^2 g_1\left(E\right)\sqrt{\frac{E_i^2}{g_2^2\left(E\right)}-m_i^2\left(r\right)}d\left(\frac{E_i}{g_2\left(E\right)}\right).$$

For spherically symmetric backgrounds

$$\mathcal{R}_{g_1\ g_2}=g_2^2\left(E\right)\tilde{R}=2g_2^2\left(E\right)\frac{b'\left(r\right)}{r^2}$$

Gravity's Rainbow and f(R) theory at work

$$f(\mathcal{R}) = \mathcal{R} + f(R).$$

In the $\mathcal{ADM}$ formulation, the Lagrangian density becomes

$$\mathcal{L} = \frac{N}{2\kappa}\sqrt{g} f(\mathcal{R}) = \frac{N}{2\kappa}\sqrt{g}\left[R + K_{ij}K^{ij} - (K)^2 - 2\nabla_\mu\left(Ku^\mu + a^\mu\right) + f(R)\right],$$

$f(R)$ is an arbitrary function of the 3D scalar curvature R

In this case

$$\Lambda^{f(R)} = \Lambda - \frac{1}{2V}\frac{\left\langle \Psi \left| \int_\Sigma d^3x \sqrt{g} f(R) \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} = \Lambda - \frac{1}{2V}\int_\Sigma d^3x \sqrt{g} f(R).$$

$$\frac{\Lambda^{f(R)}}{8\pi G} = -\frac{1}{3\pi^2}\sum_{i=1}^{2}\int_{E^*}^{+\infty} E_i g_1(E) g_2(E) \frac{d}{dE_i}\sqrt{\left(\frac{E_i^2}{g_2^2(E)} - m_i^2(r)\right)^3} dE_i.$$

Gravity’s Rainbow and f(R) theory at work

$$\begin{cases} f\left(\mathcal{R}\right)_{\mathcal{R}=0} = \begin{cases} f'\left(0\right)2E_P^2\left(3\pi-8\right)/\left(\pi c_1^2\right) & x=0 \\ 0 & x>0 \end{cases} \\ f\left(R\right)_{|R=0} = \begin{cases} -(6\pi-16)E_P^2/\left(\pi c_1^2\right) & x=0 \\ 0 & x>0 \end{cases} \end{cases},$$

For this setting the Schwarzschild background can pass the Minkowski test limit

Conclusions and Outlooks

- Application of Gravity's Rainbow can be considered to compute divergent quantum observables.
- Neither Standard Regularization nor Renormalization are required. This also happens in NonCommutative geometries.
- The f(R) function can be related to the potential part of the Horava-Lifshits theory without detailed balanced condition.
- Application to Traversable Wormholes, Topology Change, Particle Propagation, Black Hole Entropy,Tunneling and Inflation.