Numerical study of the quark antiquark static potential in the temporal gauge

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Outline

1. Yang-Mills theory in the continuum and on the lattice

2. Feynman propagation kernel

3. Numerical results

4. Conclusions

Scope

- Non perturbative study of potential: in QCD perturbative calculations are allowed just for short distances.
  - In the STATIC approximation the quark masses are infinite.
  - We do not observe free gluons and quarks, a good PARAMETRIZATION for the POTENTIAL is

$$V(r) = A + \frac{B}{r} + \sigma r,$$

    we can study the linear term just in a non perturbative way. The coefficient is called the string tension.

Yang-Mills theory in the continuum

- Invariance of the lagrangian under transformations of the symmetry group $SU(3)$
- Gauge fields or connections are the fundamental variables

$$A_\mu^a \lambda_a = A_\mu(x),$$

they transform like

$$A_\mu^\Omega = \Omega(x) A_\mu(x) \Omega^\dagger(x) + i\Omega(x)\partial_\mu \Omega^\dagger(x);$$

- Invariants: starting from

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + i[A_\mu, A_\nu]$$

$\Rightarrow$ trace of a local power

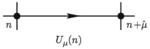
$$tr(F_{\mu\nu}F^{\rho\sigma}) = tr(\Omega F_{\mu\nu}\Omega^\dagger \; \Omega F^{\rho\sigma}\Omega^\dagger)$$

- Action Yang-Mills (pure)

$$S_G[A] = \frac{1}{2g^2} \int tr(F^{\mu\nu} F_{\mu\nu})$$

Yang-Mills theory on the lattice

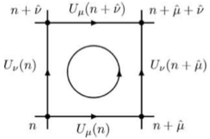
- Fundamental variables are the links $U_\mu(n)$ defined on a discrete space Λ ($\mathbf{R}^4 \to \mathbf{Z}^4$) and finite space $\Lambda \subset \mathbf{Z}^4$



  connect two points of the lattice
- The invariants are built multiplying four link variables. In this way we have an holonomy, a closed loop which define the plaquette

$$U_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_{-\mu}(n+\hat{\mu}+\hat{\nu})U_{-\nu}(n+\hat{\nu}) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu(n+\hat{\nu})^+U_\nu(n)^+$$

  taking the trace.



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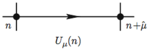

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  taking the trace.

- The Wilson Action

$$S_G[U] = \frac{2}{g^2}\sum_{n\in\Lambda}\sum_{\mu<\nu}\mathcal{R}e\left(tr\left[I - U_{\mu\nu}(n)\right]\right).$$

Constraints on the discrete Action

The Wilson action in terms of $\beta = \frac{6}{g^2}$ is

$$S_G[U] = \frac{\beta}{3} \sum_{n \in \Lambda} \sum_{\mu < \nu} \mathcal{R}e \left(tr \left[I - U_{\mu\nu}(n) \right] \right).$$

The action must respects the constraints:

1. the gauge inveriance, translated for a discrete spacetime;
2. in the limit $a \to 0$, the continuum limit, has to reproduce the Yang-Mills action.

Feynman propagation kernel

- In Quantum Mechanics we can write the probability amplitude $< x_b|e^{-iHT}|x_a >$ as

$$U(x_a, x_b; T) = \int \mathcal{D}x(t)\; e^{i\;S}$$

Feynman propagation kernel.

- In a Yang-Mills theory, in the temporal gauge $(A_0 = 0)$:

$$K(\mathbf{A}_2, \mathbf{A}_1; T_2 - T_1) = \int \mathcal{D}\Omega \int_{\mathbf{A}(\mathbf{x},T_1)=\mathbf{A}_1(\mathbf{x})}^{\mathbf{A}(\mathbf{x},T_2)=\mathbf{A}_2^{\Omega}(\mathbf{x})} \mathcal{D}\mathbf{A}(\mathbf{x}, t) e^{-S_{YM}(\mathbf{A}, A_0=0)}$$

where $\mathcal{D}\Omega$ the invariant measure on the group. The propagation kernel is the Euclidean version for the probability amplitude $< \mathbf{A}_2|e^{-iHT}|\mathbf{A}_1 >$ to go to the configuration $\mathbf{A}_2$, at the time T_2, starting from $\mathbf{A}_1$ at the time T_1.

Spectral decomposition

The propagation kernel, in the temporal gauge $A_0 = 0$, without external sources, in terms of eigenstates of the Hamiltonian is:

$$K(\mathbf{A}_1, \mathbf{A}_2; T_2 - T_1) = \sum_n e^{-E_n(T_2 - T_1)} \psi_n(\mathbf{A}_2) \psi_n^*(\mathbf{A}_1)$$

where the functional $\psi_n(\mathbf{A}) =< \mathbf{A}|n >$ is the representation of the eigenstate of the Hamiltonian in terms of the eigenstates of the field operators $\hat{\mathbf{A}}(x)$: $\hat{\mathbf{A}}(x)|\mathbf{A} >= \mathbf{A}(x)|\mathbf{A} >$.

Kernel in presence of external sources $q\ \bar{q}$

(G.C. Rossi and M. Testa, Phys. Rev. D 87,085014 (2013))

- The propagation kernel for a Yang-Mills theory in presence of external sources $q\ \bar{q}$ in x and y is

$$K(\mathbf{A}_2, s_2, r_2, \mathbf{A}_1, s_1, r_1; T) = \int_{\mathcal{G}_0} \mathcal{D}\Omega\ \Omega_{s_2 s_1}(\mathbf{x}) \Omega^\dagger_{r_2 r_1}(\mathbf{y}) \tilde{K}(\mathbf{A}_2^\Omega, \mathbf{A}_1; T)$$

  where
  - $\mathcal{G}_0$ group of time-independent gauge transformations that tend to the identity at spatial infinity;
  - $\mathcal{D}\Omega$ is the invariant Haar measure over the group.

- The states, which are the basis of the spectral decomposition, are eigenstates of the Hamiltonian $H\psi_k(\mathbf{A}, s, r) = E_k \psi_k(\mathbf{A}, s, r)$ with eigenvalue E_k:

$$K(\mathbf{A}_2, s_2, r_2, \mathbf{A}_1, s_1, r_1; T) = \sum_k e^{-E_k T} \psi_k(\mathbf{A}_2, s_2, r_2) \psi_k^*(\mathbf{A}_1, s_1, r_1).$$

Theory with $q\ \bar{q}$ sources

(G.C. Rossi and M. Testa, Phys. Rev. D 87,085014 (2013))

- The kernel is symmetric under global color rotation, $[\hat{U}(V), \hat{H}] = 0$ and the eigenstates of the Hamiltonian in the sector $q\bar{q}$ are of the form:

$$\psi(\mathbf{A}, s_1, s_2) = [\phi(\mathbf{A})\mathbb{1} + \phi_a(\mathbf{A})\lambda^a]_{s_1 s_2} \equiv \phi(\mathbf{A})\mathbb{1} + \phi_a(\mathbf{A})\lambda^a.$$

- Under a global rotation

$$U(V)\psi(\mathbf{A}) \equiv \psi^V(\mathbf{A}) = V\psi(\mathbf{A}^V)V^+ = \phi(\mathbf{A}^V)\mathbb{1} + \phi_a(\mathbf{A})V\lambda^a V^\dagger,$$

we have:

1. the *orbital color*, which comes from $\mathbf{A} \to \mathbf{A}^V$;
2. the *color spin* which comes from the action of V on the source indexes.

Eigenstates of H

(G.C. Rossi and M. Testa, Phys. Rev. D 87,085014 (2013))
For $\mathbf{A} = \mathbf{0}$

$$\psi^V(\mathbf{0}) = V\psi(\mathbf{0})V^+ = \phi(\mathbf{0})\mathbb{1} + V\lambda^a V^+\phi_a(\mathbf{0});$$

we have three possibilities (otherwise $\psi^V(\mathbf{0})$ could be in a reducible representation $\mathbb{1} \oplus \mathbb{8}$):

1. $\phi(\mathbf{0}) \neq 0$ con $\phi_a(\mathbf{0}) = 0$;
2. $\phi(\mathbf{0}) = 0$ con $\phi_a(0) \neq 0$;
3. $\phi(\mathbf{0}) = \phi_a(\mathbf{0}) = 0$.

In the firsts two cases:
if $\phi(\mathbf{0}) \neq 0$ then $\psi(\mathbf{A})$ is in the singlet of spin color;
if $\phi_a(\mathbf{0}) \neq 0$ then $\psi(\mathbf{A})$ is in the octet of spin color.

Kernel with homogeneous boundary conditions

(G.C. Rossi and M. Testa, Phys. Rev. D 87,085014 (2013))

- It can be shown that there are four different types of eigenstates of the energy;
- from the previous statement it follows that the structure of the kernel is

$$K(\mathbf{0}, r_1, r_2; \mathbf{0}, s_1, s_2) =$$

$$= |\phi(\mathbf{0})|^2 \frac{\delta_{s_1 s_2} \delta_{r_1 r_2}}{N_c} e^{-E^{[S]}T} + \sum_a |\phi_a(\mathbf{0})|^2 \sum_b \lambda^b_{r_1 r_2} \lambda^b_{s_1 s_2} e^{-E^{[Ad]}T} + \ldots$$

We study these boundary conditions with a lattice simulation.
The condition $\mathbf{A} = \mathbf{0}$ corresponds to the links at the boundary $U_1 = U_2 = U_3 = \mathbb{1}$.

Extraction of the potential

(G.C. Rossi and M. Testa, Phys. Rev. D 87,085014 (2013))
Singlet correlator:

$$P_1(K(R,T)) = \frac{1}{3}\int \mathcal{D}\Omega\, tr(\Omega(\underline{x})\Omega^\dagger(\underline{y}))\tilde{K}(i\Omega^\dagger\underline{\nabla}\Omega, \underline{0}).$$

Octet correlator:

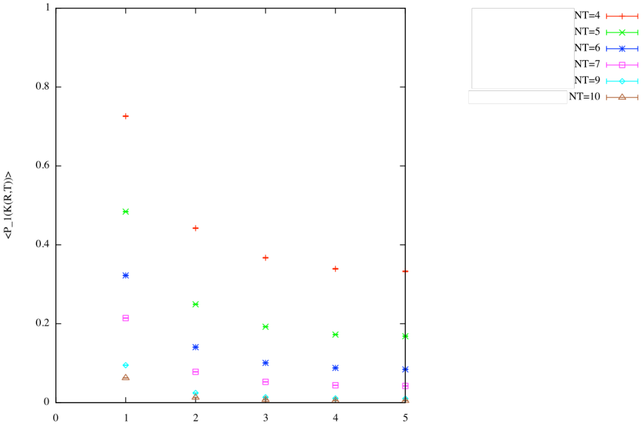
$$P_8(K(R,T)) = 2\int \mathcal{D}\Omega\, tr(\Omega(\underline{x})\lambda^a\Omega^\dagger(\underline{y})\lambda^a)\tilde{K}((\Omega^\dagger\underline{\nabla}\Omega, \underline{0}).$$

Note that no gauge fixing is needed.
Under the hypothesis, verified in perturbation theory, there is one state and the singlet potential is given by

$$\hat{V}_1 = -ln < tr\, P_1(K(R,T)) > .$$

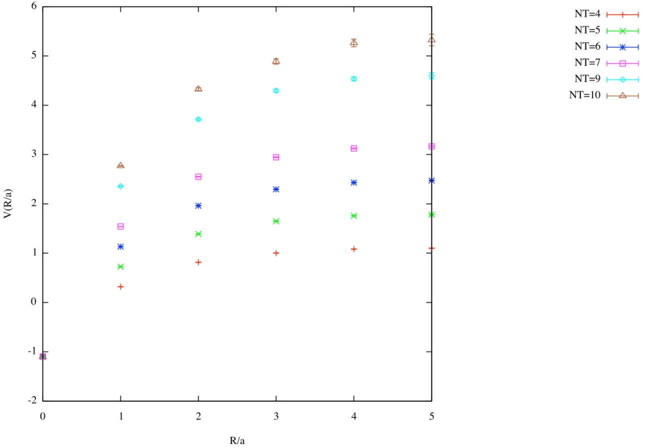
Numerical results

Singlet correlators in adimensional units with parameters: $\beta = 6$, 4000 configs, lattice extension $N^3 = 10^3$, $N_T = 4, 5, 6, 7, 9, 10$; a is the lattice spacing ($a \simeq (0.1 - 0.05)$fm). The statistical error was estimated by a jackknife method.



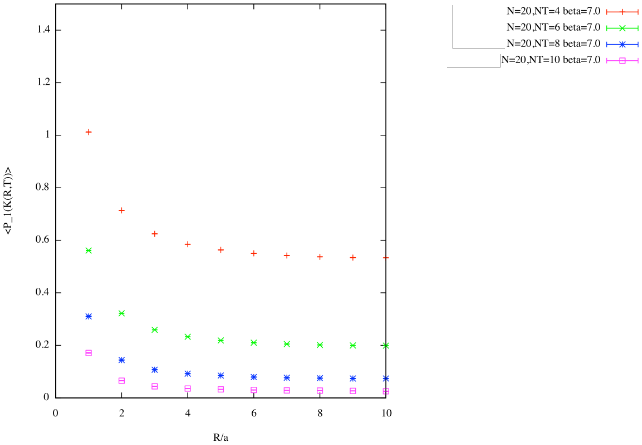
Numerical results

Singlet potential in adimensional units varying N_T, $\beta = 6$ lattice extension $N^3 = 10^3, N_T = 4, 5, 6, 7, 9, 10$, (4000 configs).



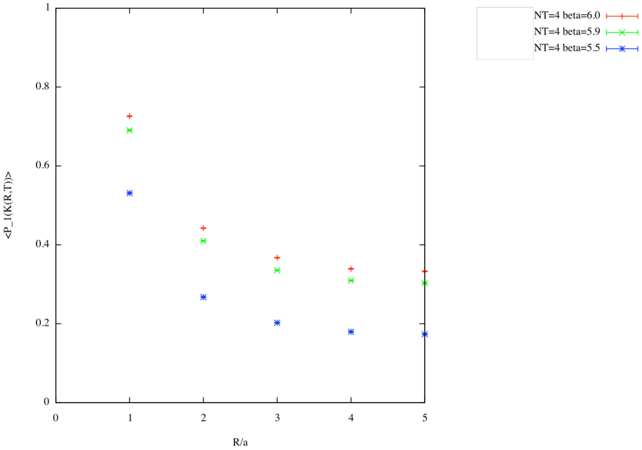
Numerical results

Singlet correlators from which we extract potentials in adimensional units $\beta = 7$ lattice extension $N^3 = 20^3$, $N_T = 4, 6, 8, 10$, (4000 configs).



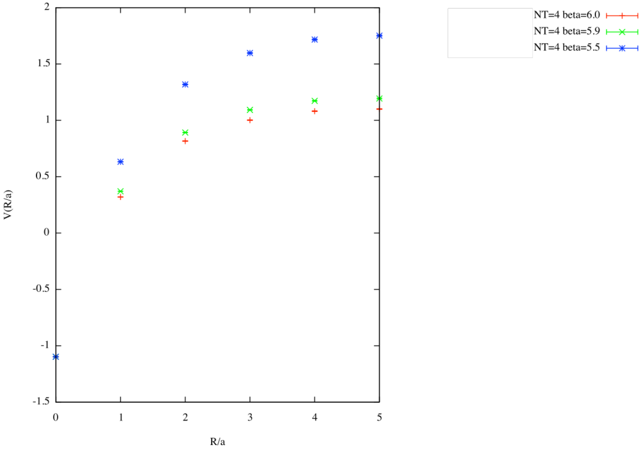
Numerical results

Singlet correlators which give the singlet potentials in adimensional units varying β lattice extension N^3 x $N_T = 10^3$ x 4



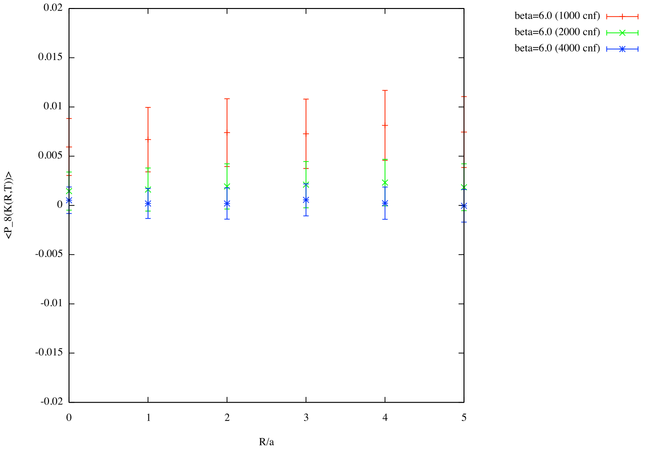
Numerical results

Singlet potentials in adimensional units varying β lattice extension N^3 x $N_T = 10^3$ x 4



Numerical results

Correlators in the octet channel in adimensional units varying the configuration number, $\beta = 6$, lattice extension $N^3 \times N_T = 10^3 \times 4$



Conclusions

We have performed numerical calculations of the correlators in the singlet and the octet channels for a pure Yang-Mills theory in presence of two static sources with homogeneous boundary conditions.

- For the singlet corraltor we find a discrepancy between homogeneous boundary conditions and the periodic ones consistent with the multilevel calculation (M. Lüsher and Weisz, hep-lat/0108014, JHEP 09 (2001) 010, 2001):
  $< P^* P >_{hom} = 1,4(7) \times 10^{-4}$ vs $< P^* P >_{per} = 2.48(2) \times 10^{-4}$,
  using the parameters $\frac{T}{a} = 6$ $\frac{r}{a} = 6$ $\beta = 5.7$.

  At present we are exploring higher statistic and checking other possible causes of the discrepancy (L. Giusti, A.L. Guerrieri, S. Petrarca, A. Rubeo, M. Testa, to be published).

- In the octet channel the signal is zero. We suspect that this is due to the fact that the integration over $\mathcal{G}$ is in fact extended to the group of *all gauge transformations*, not vanishing to the infinity, thus averaged to zero (O. Philipsen and M. Wagner, Phys.Rev. D89 014509, 2014).

Thank you for your attention!

BackUp

Number of points on the lattice $V = N^3 \cdot N_T$

$V_{min} = \ \ 10^3 \cdot 4 = 4000$, $\qquad V_{max} = \ \ 20^3 \cdot 10 = 80000$

where
$\beta = aN_T = \frac{1}{T}$
$L = a\,N$ spatial length
$T = aN_T$ euclidean time.

Volume
$V_{min} = (0.1\ fm^3)\ V_{max} = (0.8\ fm^3)$

Loop di Wilson:

$$W_{\mathcal{L}}[U] = tr[\prod_{(k,\mu)\in\mathcal{L}} U_\mu(k)].$$

Correlator of the Wilson loop in the static approximation:

$$< W_L > \propto e^{-V(r)t}(1 + O(e^{-\Delta E\, t})) = e^{-V(r)an_t}(1 + O(e^{-\Delta E\, a\, n_t})).$$

The lowest value of the energy, E_1, represents the static quark antiquark potential

$$E_1 = V(r) \qquad r = a|\mathbf{m} - \mathbf{n}|.$$

$$< W_L > \propto e^{-V(r)t}(1 + O(e^{-\Delta E\, t})) = e^{-V(r)an_t}(1 + O(e^{-\Delta E\, a\, n_t})).$$

The Polyakov loop is defined setting $n_t = N_T$, where N_T is the number of points on the lattice in the temporal direction ($aN_T = t$ is the euclidean time)

$$P(\mathbf{m}) = tr[\prod_{j=0}^{N_T-1} U_4(\mathbf{m}, j)]$$

where 4 is the Lorentz index, $\mathbf{m}$ and j are the points on the lattice, respectively spatial and temporal.

Link variable on the lattice

$$U_\mu(n) = e^{iaA_\mu(n)}.$$

Limit $a \to 0$:

$$U_\mu(n) = e^{iaA_\mu(n)} \sim 1 + iaA_\mu(n),$$

- Plaquette $\sim a$

$$U_{\mu\nu}(n) = e^{iaA_\mu(n) + ia^2 \partial_\mu A_\nu(n)};$$

- Wilson action $\sim a^2$

$$S_G[U] = \frac{2}{g^2} \sum_{n \in \Lambda} \sum_{\mu < \nu} \mathcal{R}e\, tr[\mathbb{1} - U_{\mu\nu}(n)].$$

In the continuum: the *path-ordered* of the exponential of the integral which contains the gauge field along the path $\mathcal{C}_{xy}$ which links the points x y (gauge transporter)

$$G(xy) = \mathcal{P}e^{i\int_{\mathcal{C}_{xy}} A\cdot ds}.$$

Comparing $G(xy)$ with $U_\mu(n)$ we can see that we approximate the path length with the value of the field in the starting point, this is true at the first order, $O(a)$.

Studying the potential just considering the self interaction is equivalent to take the lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} D_0 \psi + \bar{\psi} m \psi$$

in the limit of infinite mass m. In this limit the solution is the Wilson loop.

Kernel (with all the possible sources)

$$K(\mathbf{A}_2, \mathbf{A}_1; T_2 - T_1) = \int \mathcal{D}\Omega \int_{\mathbf{A}(\mathbf{x},T_1)=\mathbf{A}_1(\mathbf{x})}^{\mathbf{A}(\mathbf{x},T_2)=\mathbf{A}_2^{\Omega}(\mathbf{x})} \mathcal{D}\mathbf{A}(\mathbf{x}, t) e^{-S_{YM}(\mathbf{A}, A_0=0)}$$

Kernel with sources quark antiquark

$$K(\mathbf{A}_2, s_2, r_2, \mathbf{A}_1, s_1, r_1) = \sum_k e^{-E_k T} \psi_k(\mathbf{A}_2, s_2, r_2) \psi_k^*(\mathbf{A}_1, s_1, r_1)$$

Symmetries of the kernel

- Invariance of the kernel without Gauss constraint under gauge transformations

$$\tilde{K}(\mathbf{A}_2^{\Omega}, T_2; \mathbf{A}_1^{\Omega} T_1) = \tilde{K}(\mathbf{A}_2, T_2; \mathbf{A}_1 T_1),$$
$$\Rightarrow [\hat{H}, \hat{U}(\Omega)] = 0$$

  $\hat{U}(\Omega)$ unitary operators
  $\Rightarrow \hat{H}$ has to be diagonal in the subspaces which correspond to the irreducible representations of the gauge group.

Gauss constraint in QCD:

$$\partial_i E_i^a = g\rho^a + g f^{abc} A_i^b E_i^c \tag{1}$$

Static potential parametrization

$$V(r) = A + \frac{B}{r} + \sigma r$$

where $B = -\frac{4}{3}\alpha_S$.
Coulombian behavior dominates at short distances where perturbation theory works;
linear confining behavior dominates at long distances where non perturbative calculations work.