Local thermodynamical equilibrium and the hydrodynamic β frame

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Outline

- Local equilibrium in quantum relativistic theory
 The β frame
- rightarrow Landau vs β frame

Motivations:Relativistic hydrodynamics of strongly interacting fluids (QGP, cold atoms...)

Full quantum-relativistic treatment

$$T^{\mu\nu}(x) \equiv tr\left(\hat{\rho} \widehat{T}^{\mu\nu}(x)\right)_{ten}$$

Covariance maintained throughout

Kinetic-theory free

Local thermodynamical equilibrium

Maximization of entropy with fixed *densities* of conserved currents

$$S = -tr(\hat{g}_{le} \log \hat{g}_{le})$$
 max. w.r.t. \hat{g}_{le}

$$\begin{split} n_{\mu} \mathrm{tr}(\widehat{\rho}_{\mathrm{LE}} \, \widehat{T}^{\mu\nu}(x))_{\mathrm{ren}} &= n_{\mu} \langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{LE}} \equiv n_{\mu} T_{\mathrm{LE}}^{\mu\nu}(x) = n_{\mu} T^{\mu\nu}(x) \\ n_{\mu} \mathrm{tr}(\widehat{\rho}_{\mathrm{LE}} \widehat{j}^{\mu}(x))_{\mathrm{ren}} &= n_{\mu} \langle \widehat{j}^{\mu}(x) \rangle_{\mathrm{LE}} \equiv n_{\mu} j_{\mathrm{LE}}^{\mu}(x) = n_{\mu} j^{\mu}(x) \end{split}$$

As entropy is a non-conserved global quantity in LTE, it requires the specification of one frame, namely a 3d spacelike hypersurface with its normal *n* at any time (a spacetime foliation)



The vector field *n*(*x*) must be vorticity-free:

$$\epsilon_{\mu\nu\rho\sigma}n^{\nu}(\partial^{\rho}n^{\sigma}-\partial^{\sigma}n^{\rho})=0$$

Maximize:

$$-\mathrm{tr}(\widehat{\rho}\log\widehat{\rho}) + \int_{\Sigma(\tau)} \mathrm{d}\Sigma \ n_{\mu} \left[\left(\langle \widehat{T}^{\mu\nu}(x) \rangle - T^{\mu\nu}(x) \right) \beta_{\nu}(x) - \left(\langle \widehat{j}^{\mu}(x) \rangle - j^{\mu}(x) \right) \xi(x) \right]$$

The 5 Lagrange multiplier functions $\beta(x)$ and $\xi(x)$ become the primordial thermodynamic fields

$$\beta^{\mu}(x) = \frac{1}{T} u^{\mu}(x)$$
$$\tilde{\xi}(x) = \frac{\mu(x)}{T(x)}$$

The solution

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left[-\int_{\Sigma(\tau)} \mathrm{d}\Sigma \ n_{\mu} \left(\widehat{T}^{\mu\nu}(x)\beta_{\nu}(x) - \xi(x)\widehat{j}^{\mu}(x)\right)\right]$$

$$n_{\mu}T_{\rm LE}^{\mu\nu}[\beta,\xi,n] = n_{\mu}T^{\mu\nu} \qquad n_{\mu}j_{\rm LE}^{\mu}[\beta,\xi,n] = n_{\mu}j^{\mu}$$

This operator, as expected, depends on the hypersurface Σ It is independent if

$$\partial_{\mu}\beta_{\nu} + \partial_{\nu}\beta_{\mu} = 0 \qquad \qquad \partial_{\mu}\xi = 0.$$

 β Killing vector field

Global equilibrium

$$\beta_{\mu} = b_{\mu} + \omega_{\mu\nu} x^{\nu}$$

Reproduces all known forms of global equilibrium

$$\begin{cases} b = (\frac{1}{T}, \underline{0}) \longrightarrow e^{-\hat{H}/T} \\ \omega = 0 \end{cases}$$
$$\begin{cases} b = cost. \\ \omega = 0 \end{pmatrix} e^{-\hat{D}} \hat{P} \\ \begin{pmatrix} b = cost. \\ \omega = 0 \end{pmatrix} e^{-\hat{D}} \hat{P} \\ \begin{pmatrix} b = (\frac{1}{T}, \underline{0}) \\ \omega = (\frac{0}{01} \underline{\omega}) \end{pmatrix} \longrightarrow e^{-\hat{H}/T} + \underline{\omega} \hat{J}/T \end{cases}$$

$$|f \beta^{\mu} = b^{\mu} \cos t.$$

 $\xi = const.$

$$\langle \hat{T}^{\mu\nu} \rangle_{e_q} = (P + \dot{P}) \frac{\beta^{\mu}\beta^{\nu}}{\beta^2} - \dot{P}g^{\mu\nu}$$

LOCAL equilibrium depends on Σ

A choice has to be made. The simplest is to define LTE on the hypersurface orthogonal to β itself

$$n = vers \beta$$

The β frame

Equations

defining β

and ξ

 $\beta_{\mu}T_{\rm LE}^{\mu\nu}[\beta,\xi] = \beta_{\mu}T^{\mu\nu} \qquad \beta_{\mu}j_{\rm LE}^{\mu}[\beta,\xi] = \beta_{\mu}j^{\mu},$

This is possible if β is vorticity-free. If it is not, the definition has to be modified, but it can still be done



 vers β is the four-velocity of an ideal relativistic thermometer, that is an object able to reach equilibrium with respect to energy and momentum echange with the fluid in x

Relativistic hydro in the β frame

Equation of state Ideal s.e.t.

$$p(\beta^2, \xi) \qquad T_{id}^{\mu\nu} = -2 \frac{\partial p}{\partial \beta^2} \beta^{\mu} \beta^{\nu} - p g^{\mu\nu}, \qquad -2 \frac{\partial p}{\partial \beta^2} = \frac{\rho + p}{\beta^2} = \frac{h}{\beta^2}.$$
All gradients in *u*.*T* can be recasted as gradients of β

$$\sqrt{\beta^2} \Delta_{\mu\nu} D\beta^{\nu} + \frac{1}{2} \nabla_{\mu} \beta^2 = \beta_\lambda \Delta_{\mu\nu} (\partial^{\lambda} \beta^{\nu} + \partial^{\nu} \beta^{\lambda})$$

$$\sqrt{\beta^2} \Delta_{\mu\nu} D\beta^{\nu} + \frac{1}{2} \nabla_{\mu} \beta^2 = \beta^2 A^{\mu} + \frac{1}{2} \nabla_{\mu} \frac{1}{T^2} = \frac{1}{T^2} \left(A_{\mu} - \frac{1}{T} \nabla_{\mu} T \right)$$

$$\begin{aligned} \nabla_{\mu}u^{\nu} &= \nabla_{\mu}\frac{\beta^{\nu}}{\sqrt{\beta^{2}}} = \beta^{\nu}\left(-\frac{1}{2}\right)(\beta^{2})^{-3/2}\nabla_{\mu}\beta^{2} + \frac{1}{\sqrt{\beta^{2}}}\nabla_{\mu}\beta^{\nu} \\ &= \frac{1}{\sqrt{\beta^{2}}}\left(-\frac{\beta^{\nu}\beta^{\rho}}{\beta^{2}}\nabla_{\mu}\beta_{\rho} + \nabla_{\mu}\beta^{\nu}\right) = \frac{1}{\sqrt{\beta^{2}}}\Delta^{\rho\nu}\nabla_{\mu}\beta_{\rho}, \end{aligned}$$

$$\nabla_{\mu}u^{\mu} = \frac{1}{\sqrt{\beta^2}} \nabla_{\mu}\beta^{\mu}.$$

The transport coefficients

$$\partial_{\mu}T^{\mu\nu} = \partial_{\mu}\mathrm{tr}(\widehat{\rho}\,\widehat{T}^{\mu\nu})_{\mathrm{ren}} = \mathrm{tr}(\widehat{\rho}\,\partial_{\mu}\widehat{T}^{\mu\nu})_{\mathrm{ren}} = 0,$$

where ρ is the fixed density operator in the Heisenberg picture

The hydrodynamical problem is to determine the evolution of the mean values starting from LTE condition at some time TO

$$\hat{\boldsymbol{\rho}} \equiv \hat{\boldsymbol{\rho}}_{L\epsilon}(\tau_0) = \frac{1}{Z_{L\epsilon}(\tau_0)} \quad \exp\left[-\int_{\tau_0} d\Sigma n_{\mu}(\hat{\boldsymbol{\tau}}^{\mu\nu}\beta_{\nu} - \tilde{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\gamma}}^{\mu\nu})\right]$$

Gauss theorem

$$-\int_{\Sigma(\tau_0)} d\Sigma n_{\mu} \left(\widehat{T}^{\mu\nu} \beta_{\nu} - \widehat{j}^{\mu} \xi \right) = -\int_{\Sigma(\tau')} d\Sigma n_{\mu} \left(\widehat{T}^{\mu\nu} \beta_{\nu} - \widehat{j}^{\mu} \xi \right) + \int_{\Omega} d\Omega \left(\widehat{T}^{\mu\nu} d_{\mu} \beta_{\nu} - \widehat{j}^{\mu} d_{\mu} \xi \right),$$

$$(ovariant derivative)$$

$$(vanishing flux)$$

$$\Gamma = \sum_{\Sigma(\tau_0)} \sum_{\Sigma(\tau_0)} \sum_{\Gamma} \sum_{\Sigma(\tau_0)} \sum_{\Gamma} \sum_$$

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\int_{\Sigma(\tau')} \mathrm{d}\Sigma \ n_{\mu} \left(\widehat{T}^{\mu\nu}\beta_{\nu} - \widehat{j}^{\mu}\xi\right) + \int_{\Omega} \mathrm{d}\Omega \ \left(\widehat{T}^{\mu\nu}d_{\mu}\beta_{\nu} - \widehat{j}^{\mu}d_{\mu}\xi\right)\right]$$

If the system stays close to LTE at any time, the second term is a correction and an expansion can be made from present LTE

Linear response theory

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\int_{\Sigma(\tau')} d\Sigma \, n_{\mu} \left(\widehat{T}^{\mu\nu}\beta_{\nu} - \widehat{j}^{\mu}\xi\right) + \int_{\Omega} d\Omega \, \left(\widehat{T}^{\mu\nu}d_{\mu}\beta_{\nu} - \widehat{j}^{\mu}d_{\mu}\xi\right)\right]$$

$$\widehat{A} \qquad \widehat{B}$$

For small B, using the linear response theory:

$$\langle \widehat{O}(x') \rangle \simeq \langle \widehat{O}(x') \rangle_{\rm LE} - \langle \widehat{O}(x') \rangle_{\rm LE} \langle \widehat{B} \rangle_{\rm LE} + \int_0^1 \mathrm{d}z \ \langle \widehat{O}(x') \mathrm{e}^{z\widehat{A}} \widehat{B} \mathrm{e}^{-z\widehat{A}} \rangle_{\rm LE}$$





Eckart

 $T^{\mu\nu}u_{\nu} = \lambda u_{\perp}^{\mu}$ Landau

Beta

 $\beta_{\mu}T_{\rm LE}^{\mu\nu}[\beta,\xi] = \beta_{\mu}T^{\mu\nu}$

 $\beta_{\mu} j^{\mu}_{\rm LE}[\beta,\xi] = \beta_{\mu} j^{\mu},$

Landau vs β frame (contd)

If the β field is not vorticous then $n = vers \beta$ and one should enforce:

$$\beta_{\mu}\delta T^{\mu\nu} = 0 \qquad \qquad \beta_{\mu}\delta j^{\mu} = 0$$



$$\beta_{\mu}T_{\rm LE}^{\mu\nu}[\beta,\xi] = \beta_{\mu}T^{\mu\nu} \qquad \beta_{\mu}j_{\rm LE}^{\mu}[\beta,\xi] = \beta_{\mu}j^{\mu},$$

These are 5 equations. Very similar to Landau matching conditions, though not exactly the same

If the β field is vorticous, see later

Landau vs β frame (contd)

If
$$\langle \hat{T}_{LE}^{\mu\nu} \rangle = \langle \hat{T}_{id}^{\mu\nu} \rangle$$
 they coincide

Yet, in general

$$T_{\rm LE}^{\mu\nu}(x) = \operatorname{tr}(\widehat{\rho}_{\rm LE}\widehat{T}^{\mu\nu}(x))_{\rm ren} = \frac{1}{Z_{\rm LE}}\operatorname{tr}\left(\exp\left[-\int \mathrm{d}\Sigma_{\mu} \left(\widehat{T}^{\mu\nu}\beta_{\nu} - \xi\widehat{j}^{\mu}\right)\right]\widehat{T}^{\mu\nu}(x)\right)_{\rm ren}$$

Taylor expand about x

$$\exp\left[-\int \mathrm{d}\Sigma_{\mu} \left(\widehat{T}^{\mu\nu}\beta_{\nu} - \xi\widehat{j}^{\mu}\right)\right]$$
$$\simeq \exp\left[-\beta_{\nu}(x)\widehat{P}^{\nu} + \xi(x)\widehat{Q} - \nabla_{\lambda}\beta_{\nu}(x)\int_{T\Sigma}\mathrm{d}\Sigma_{\mu}(y)\,\widehat{T}^{\mu\nu}(y)(y^{\lambda} - x^{\lambda}) + \nabla_{\lambda}\xi(x)\int_{T\Sigma}\mathrm{d}\Sigma_{\mu}(y)\,\widehat{j}^{\mu}(y)(y^{\lambda} - x^{\lambda}) + \dots\right]$$

15

Then

$$T_{\rm LE}^{\mu\nu}(x) \simeq \frac{1}{Z_{\rm eq}(\beta(x),\xi(x))} \operatorname{tr}\left(\exp\left[-\beta_{\nu}(x)\widehat{P}^{\nu} + \xi(x)\widehat{Q}\right]\widehat{T}^{\mu\nu}(x)\right)_{\rm ren} + \mathcal{O}(\partial\beta,\partial\xi).$$

$$\frac{1}{Z} \operatorname{tr} \left(e^{-\beta \cdot \hat{P}} + \frac{1}{2} \widehat{\alpha} \widehat{T}^{m}(x) \right) = T^{m}(x)_{ia} = (P + \beta) \frac{\beta^{m} \beta^{v}}{\beta^{2}} - \beta g^{nv}$$

 $T_{\rm LE}^{\mu\nu}(x) \simeq T_{\rm id}^{\mu\nu}(x) + \mathcal{O}(\nabla\beta, \nabla\xi) = (\rho + p)_{\rm eq} \frac{1}{\beta^2} \beta^{\mu}(x) \beta^{\nu}(x) - p_{\rm eq} g^{\mu\nu} + \mathcal{O}(\partial\beta, \partial\xi),$

Is there some leading correction? If yes, Landau and β differ

β # Landau. An equilibrium calculation: the free scalar field



$$\widehat{\rho} = \frac{1}{Z} \exp[-\widehat{H}/T + \omega \cdot \widehat{J}_z/T] \mathsf{P}_V,$$

$$\widehat{T}^{\mu\nu} = \partial^{(\mu}\widehat{\psi}\partial^{\nu)}\widehat{\psi} - g^{\mu\nu}\widehat{\mathcal{L}} \qquad \widehat{\mathcal{L}} = \frac{1}{2} \left(\partial_{\mu}\widehat{\psi}\partial^{\mu}\widehat{\psi} - m^2\widehat{\psi}^2\right)$$
Keeping in mind
$$\beta_{\mu} = b_{\mu} + \omega_{\mu\nu}x^{\nu} \qquad \omega_{\mu\nu} = -\partial_{\mu}\beta_{\nu} + \partial_{\nu}\beta_{\mu}.$$

in this case

$$\omega_{\mu\nu} = \begin{pmatrix} \circ & \circ \\ & \underline{\omega}_{\top} \\ \circ & \underline{\omega}_{\top} \end{pmatrix}$$

It can be shown by explicit calculation

$$T = \begin{pmatrix} u \cdot T \cdot u & u \cdot T \cdot \hat{\tau} & 0 & 0 \\ u \cdot T \cdot \hat{\tau} & \hat{\tau} \cdot T \cdot \hat{\tau} & 0 & 0 \\ 0 & 0 & \hat{r} \cdot T \cdot \hat{r} & 0 \\ 0 & 0 & 0 & k \cdot T \cdot k \end{pmatrix}$$

 $\hat{r} \cdot T \cdot \hat{r} \neq \hat{k} \cdot T \cdot \hat{k}$

$$\hat{r} \cdot T \cdot \hat{r} - k \cdot T \cdot k = \sum_{M=-\infty}^{+\infty} \sum_{p_T} \int dp_L \; \frac{2}{(2\pi)^2 \,\varepsilon \, R^2 \, J_M^{\prime 2}(p_T R)} \frac{1}{\mathrm{e}^{\beta(\varepsilon - M\omega)} - 1} \left[p_T^2 J_M^{\prime}(p_T r)^2 - p_L^2 J_M(p_T r)^2 \right],$$

< Î^m> ≠ (g+p) uⁿuⁿ - pgⁿ

 $\langle \hat{T}^{\mu\nu} \rangle_{eq} = -\frac{\partial p}{\partial \beta^2} \beta^{\mu} \beta^{\nu} - \frac{1}{\beta} (\beta^2, \xi) g^{\mu\nu} + O((\omega/r)^2)$

Vorticous case

Define the field b(x) inspired by the equilibrium solution

$$\beta_{\nu}(x) \equiv b_{\nu}(x) + \varpi_{\nu\lambda}(x)x^{\lambda},$$

$$\varpi_{\nu\lambda}(x) = -\frac{1}{2}(\partial_{\nu}\beta_{\lambda} - \partial_{\lambda}\beta_{\nu}) - \frac{1}{6}\left(x^{\rho}\partial_{\rho}\partial_{\nu}\beta_{\lambda} - x^{\rho}\partial_{\rho}\partial_{\lambda}\beta_{\nu}\right) + \dots$$

This is not vorticous and can be taken as frame field

$$(\beta_{\mu} - \varpi_{\mu\lambda} x^{\lambda}) \delta T^{\mu\nu} = 0 \qquad \qquad (\beta_{\mu} - \varpi_{\mu\lambda} x^{\lambda}) \delta j^{\mu} = 0$$

Take ξ = 0 and expand the exponent of the LTE operator

$$-\int d\Sigma \ n_{\mu}\widehat{T}^{\mu\nu}\beta_{\nu} = -\int d\Sigma(y) \ n_{\mu}\widehat{T}^{\mu\nu}(b_{\nu} + \varpi_{\nu\lambda}y^{\lambda})$$
$$= -\int d\Sigma(y) \ n_{\mu}\widehat{T}^{\mu\nu}b_{\nu} - \frac{1}{2}\varpi_{\lambda\nu}(y^{\lambda}\widehat{T}^{\mu\nu} - y^{\nu}\widehat{T}^{\mu\lambda})$$

$$\begin{split} &-\int \mathrm{d}\Sigma(y) \ n_{\mu} \left[\widehat{T}^{\mu\nu} b_{\nu} - \frac{1}{2} \varpi_{\lambda\nu} (y^{\lambda} \widehat{T}^{\mu\nu} - y^{\nu} \widehat{T}^{\mu\lambda}) \right] \\ &\simeq -b_{\nu}(x) \int \mathrm{d}\Sigma(y) \ n_{\mu} \widehat{T}^{\mu\nu} - \frac{\partial b_{\nu}}{\partial x^{\rho}} \int \mathrm{d}\Sigma(y) \ n_{\mu} (y^{\rho} - x^{\rho})_{T} \widehat{T}^{\mu\nu} + \frac{1}{2} \varpi_{\lambda\nu}(x) \int \mathrm{d}\Sigma(y) \ n_{\mu} (y^{\lambda} \widehat{T}^{\mu\nu} - y^{\nu} \widehat{T}^{\mu\lambda}) \\ &= -b_{\nu}(x) \widehat{P}^{\nu} - \frac{\partial b_{\nu}}{\partial x^{\rho}} \int \mathrm{d}\Sigma(y) \ n_{\mu} (y^{\rho} - x^{\rho})_{T} \widehat{T}^{\mu\nu} + \frac{1}{2} \varpi_{\lambda\nu}(x) \widehat{J}^{\lambda\nu} \\ &= -b_{\nu}(x) \widehat{P}^{\nu} - \frac{1}{4} (\partial_{\rho} b_{\nu} + \partial_{\nu} b_{\rho}) \int \mathrm{d}\Sigma(y) \ n_{\mu} \left[(y^{\rho} - x^{\rho})_{T} \widehat{T}^{\mu\nu} + (y^{\nu} - x^{\nu})_{T} \widehat{T}^{\mu\rho} \right] + \frac{1}{2} \varpi_{\lambda\nu}(x) \widehat{J}^{\lambda\nu} \end{split}$$

Result

$$-b_{\nu}(x)\widehat{P}^{\nu} + \frac{1}{2}\varpi_{\lambda\nu}(x)\widehat{J}^{\lambda\nu} - \frac{1}{2}(\nabla_{\lambda}\beta_{\nu} + \nabla_{\nu}\beta_{\lambda})\widehat{L}_{x}^{\lambda\nu}$$

$$\widehat{L}_x^{\lambda\nu} \equiv \frac{1}{2} \int_{T\Sigma} \mathrm{d}\Sigma(y) \ n_\mu(y^\lambda - x^\lambda) \ \widehat{T}^{\mu\nu}(y) + (\lambda \leftrightarrow \nu)$$

Taking into account that $\widehat{J}^{\lambda\nu} = \widehat{J}^{\lambda\nu}_x + x^{\lambda}\widehat{P}^{\nu} - x^{\nu}\widehat{P}^{\lambda}$

the LTE exponent can be finally written

$$-\beta_{\nu}(x)\widehat{P}^{\nu} + \frac{1}{2}\varpi_{\lambda\nu}(x)\widehat{J}_{x}^{\lambda\nu} - \frac{1}{2}(\nabla_{\lambda}\beta_{\nu} + \nabla_{\nu}\beta_{\lambda})\widehat{L}_{x}^{\lambda\nu}$$

Local velocity Local vorticity Local shear

Outlook and conclusions

- * The β frame is a well suited frame for strongly interacting fluids, when a kinetic description is impossible
- It allows to treat hydrodynamics in a quantum relativistic framework and to properly express the condition of local thermodynamical equilibrium and to expand from it (Zubarev approach to transport coefficients)
- ^{*} β does not coincide with Landau whenever the stress-energy tensor at LTE differs from the ideal form
- Work in progress (E. Grossi) to derive more consequences and to compare with different approaches