

Classical Electrodynamics of charged massless particles

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based on F. Azzurli and K. Lechner, *Electromagnetic fields and potentials generated by massless charged particles*, [arXiv:1401.5721[hep-th]].

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Some issues in classical Electrodynamics for point-like particles I

- ▶ Self-interaction not included in Lorentz force $\frac{dp^\mu}{ds} = eF^{\mu\nu}u_\nu$
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It does not conserve the energy!
- ▶ $F^{\mu\nu}(x)$ and $j^\mu(x) = e \int u^\mu \delta^4(x - y(s)) ds$ are distributions but the energy-momentum tensor of the EM field

$$T_{em}^{\mu\nu} = F^{\mu\rho}F_{\rho}^{\nu} + \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}$$

is not: $T_{em} \sim \frac{1}{L^4}$, not integrable for $L \sim 0$.

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 \implies postulate $\mathcal{T}_{em}^{\mu\nu}$



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Conservation of $\mathcal{T}_{em}^{\mu\nu} + T_{particle}^{\mu\nu} \implies$ Lorentz-Dirac force with radiation reaction (and some other minor issue)

$$\frac{dp^\mu}{ds} = eF^{\mu\nu} u_\nu + \frac{e^2}{6\pi} \left(\frac{d^2 u^\mu}{ds^2} + \left(\frac{du}{ds} \right)^2 u^\mu \right)$$

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With Maxwell's equations we obtain a consistent theory (at the classical level) in agreement with $\hbar \rightarrow 0$ limit of QED!

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Can we do the same for massless charges? (and learn some lesson?)

Some notation

For massive particles moving at speed $V < 1$ capital letters

- ▶ $Y^\mu(\lambda)$ world-line parametrized by λ
- ▶ $U^\mu(\lambda) = \frac{dY^\mu}{d\lambda}(\lambda)$ four-velocity
- ▶ $W^\mu(\lambda) = \frac{dU^\mu}{d\lambda}(\lambda)$ four-acceleration
- ▶ $J^\mu = e \int u^\mu \delta^4(x - y(s)) d\lambda$ four-current

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For massless particles lower case letters for

$y^\mu(\lambda)$, $u^\mu(\lambda)$, $w^\mu(\lambda)$, j^μ and calligraphic font for the fields $\mathcal{F}^{\mu\nu}$...

Liénard-Wichert fields

Define retarded time $\lambda_r(x)$ and

$$L^\mu = x^\mu - Y^\mu(\lambda_r(x)) \quad , \quad L^2 = 0 \quad L^0 \geq 0$$

light-like *causal*

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$$C^{\mu\nu} = \frac{e}{4\pi} \frac{(U^\mu L^\nu - U^\nu L^\mu) U^2}{(UL)^3}$$

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Naively set $V = 1$. On

$$\Gamma^\mu(\lambda, b) = y^\mu(\lambda) + bu^\mu(\lambda), \quad b > 0$$

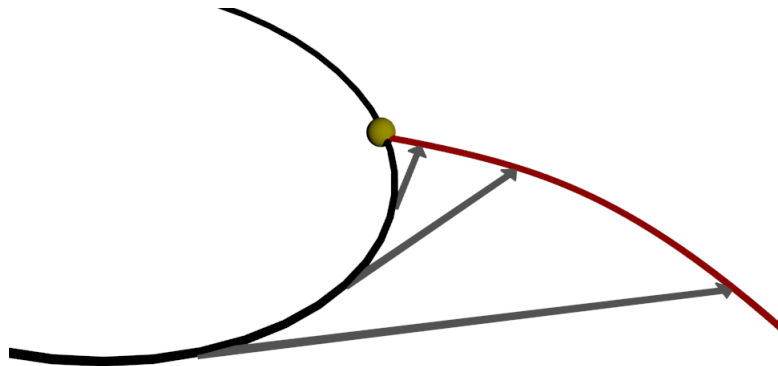
$$l^\mu = \Gamma^\mu - y^\mu \propto u^\mu \implies ul \propto u^2 = 0 \implies C^{\mu\nu}, \mathcal{R}^{\mu\nu}|_\Gamma \rightarrow +\infty$$

Notice: Γ has a border: $l^0 > 0 \implies b > 0$. 

String of singularity

Corresponding string at time t (using t as parameter)

$$\vec{\gamma}(b) = \vec{y}(t - b) + b \frac{d\vec{y}}{dt}(t - b)$$



Green function method fails

The method ($V = 1$):

$$\partial_\mu \mathcal{F}^{\mu\nu} = j^\nu, \partial_{[\alpha} \mathcal{F}_{\beta\gamma]} = 0, \quad \Longrightarrow \quad \square \mathcal{A}^\mu = j^\mu, \quad \partial_\mu \mathcal{A}^\mu = 0$$

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$$\square G(x) = \delta^4(x) \implies G(x) = \frac{1}{2\pi} H(x^0) \delta(x^2) \longrightarrow \mathcal{A}^\mu = G * j^\mu$$

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For rectilinear uniform motion at $V = 1$ one gets

$$\mathcal{A}_{RU}^\mu = \frac{e}{4\pi} \frac{u^\mu}{ux} H(ux) \notin \mathcal{S}'!$$

Rectilinear uniform motion

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Absorb divergences with gauge transformations:

$$\tilde{A}_{RU}^\mu = A_{RU}^\mu + \partial^\mu \Lambda, \quad \mathcal{A}_{RU}^\mu = \lim_{V \rightarrow 1} \tilde{A}_{RU}^\mu \in \mathcal{S}'$$

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$$\mathcal{F}_{RU}^{\mu\nu} = \partial^\mu \mathcal{A}_{RU}^\nu - \partial^\nu \mathcal{A}_{RU}^\mu = \text{Lim}_{V \rightarrow 1} F_{RU}^{\mu\nu}$$

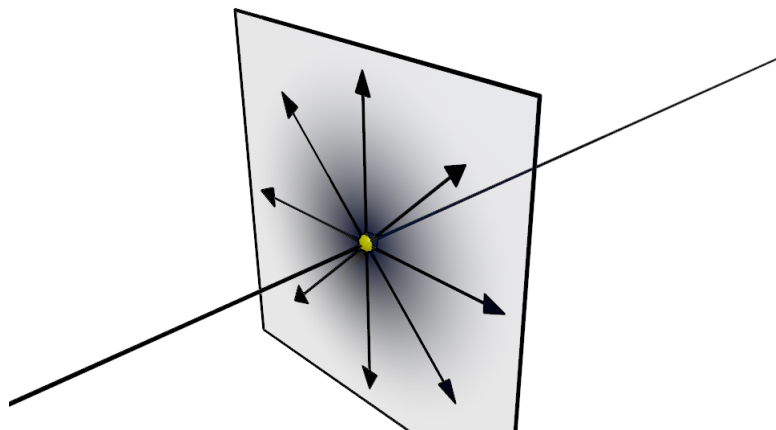
and it solves the Maxwell equation for j_{RU}^μ

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The shockwave

Resulting field: the shockwave ($v^\mu = \frac{dy^\mu}{dy^0}$)

$$\mathcal{F}_{RU}^{\mu\nu} = C_{RU}^{\mu\nu} = \frac{e}{2\pi} \frac{v^\mu x^\nu - v^\nu x^\mu}{x^2} \delta(vx)$$



The four potential

Green function method provides

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$$\mathcal{A}^\mu(\varphi) = \frac{e}{4\pi} \int \frac{u^\mu}{r} \varphi(t + r, \vec{x} + \vec{y}(t)) d^4x$$

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Not integrable in region $t \sim -r \rightarrow -\infty$ and $\vec{x} \sim -\vec{y}(t) \rightarrow \infty$.

Bounded motion

$\vec{y}(t) \xrightarrow{t \rightarrow -\infty} \infty \implies \mathcal{A}^\mu \in \mathcal{S}' \implies \mathcal{F}^{\mu\nu}$ given by derivatives
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Easier to use a regularisation!

$$Y^0(\lambda) = \frac{y^0(\lambda)}{V}, \quad \vec{Y}(\lambda) = \vec{y}(\lambda) \implies \lim_{V \rightarrow 1} J^\mu = j^\mu$$

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$$Q^{\mu\nu} = e \int_0^{+\infty} b (u^\mu w^\nu - u^\nu w^\mu) db \int \delta^4(x - \Gamma(\lambda, b)) d\lambda$$

Poincaré dual of $\Gamma^\mu \implies \partial_\mu Q^{\mu\nu} = j^\nu \implies \partial_\mu \mathcal{P}(R^{\mu\nu}) = \frac{1}{2} j^\mu$

Currents

$$\left\{ \begin{array}{l} \partial_{[\mu} C_{\nu\rho]} = 0 \\ \partial_{\mu} C^{\mu\nu} = J^{\nu} + K^{\nu} \end{array} \right\}, \quad \left\{ \begin{array}{l} \partial_{[\mu} R_{\nu\rho]} = 0 \\ \partial_{\mu} R^{\mu\nu} = -K^{\nu} \end{array} \right.$$

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\Downarrow

$$\begin{aligned} \partial_{\mu} C^{\mu\nu} &= 0 \\ \partial_{\mu} \mathcal{R}^{\mu\nu} &= j^{\nu} \end{aligned}$$

Unbounded motion I

$$\begin{array}{ccc} \vec{y} & \xrightarrow{t \rightarrow -\infty} & t\vec{v}_\infty \\ & \Downarrow & \\ \vec{\gamma}(b) & = & \vec{y}(t-b) + b \frac{d\vec{y}}{dt}(t-b) \\ b \rightarrow +\infty & \hookrightarrow & b\vec{v}_\infty \end{array}$$

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Same gauge transformation used for A_{RU}^μ in the direction of \vec{v}_∞ :

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Same strategy as before

$$\mathcal{R}^{\mu\nu} = \lim_{V \rightarrow 1} R^{\mu\nu} = \mathcal{P}(R^{\mu\nu}) + \frac{1}{2} Q^{\mu\nu}$$

Unbounded motion II

This time

$$\partial_\mu \mathcal{R}^{\mu\nu} = -\lim_{V \rightarrow 1} K^\mu = e \int [u^\mu(\lambda) \varphi(\Gamma(\lambda, b))]_{b=+\infty}^{b=0} d\lambda = j^\nu - j_{RU}^\nu$$

where j_{RU} current of rectilinear motion along \vec{v}_∞

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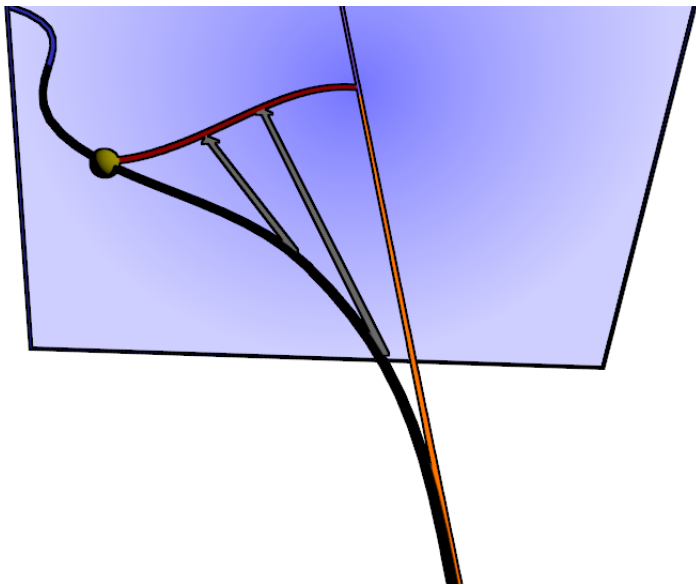
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Unbounded motion generate:

1. A radiation field
2. A shockwave
3. A string of singularities that ends on the shockwave



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Thanks for your kind attention.