On Higher Spin Algebras in different dimensions

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Motivation

- Recent advances in High Energy Physics are mostly connected to discoveries of new symmetries.
- String theory contains infinitely many massive particles of arbitrary high spin. They are believed to be Higgsed version of some massless spectrum.
- Interacting field theories of Higher Spin particles face many difficulties. There are strong restrictions on higher spin interactions.
- Extended theories of gravity, that include Higher Spin particles can have better quantum behavior as compared to the familiar Einstein-Hilbert gravity.

Introduction

- HS algebras are defined as Lie algebras of global symmetries of a theory, containing HS spectrum.
- HS algebras are usually infinite-dimensional.
- They contain space-time isometry (A)dS algebra as subalgebra (usually this is the biggest finite-dimensional subalgebra).
- All known HS algebras can be defined as a quotient of the Universal Enveloping Algebra of the (A)dS isometry algebra by an ideal.

HS algebras: A first look

Spin s symmetric field is a gauge field, described by a rank s double traceless symmetric tensor, with a gauge parameter, that is a rank s-1 traceless symmetric tensor.

$$\delta_\varepsilon \varphi_{\mu_1 \cdots \mu_s} = \bar{\nabla}_{(\mu_1} \varepsilon_{\mu_2 \cdots \mu_s)} + t_{\mu_1 \cdots \mu_s}(\varphi, \varepsilon)$$

We restrict ourselves to global symmetries, imposing Killing tensor equations

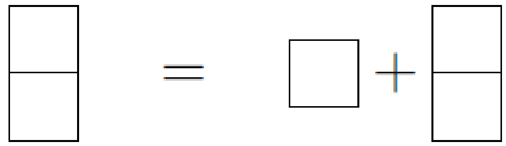
$$0 = \left[\delta_\varepsilon \varphi_{\mu_1 \cdots \mu_s} \right]_{\varphi=0} = \bar{\nabla}_{(\mu_1} \varepsilon_{\mu_2 \cdots \mu_s)}$$

Solution to the Killing equations

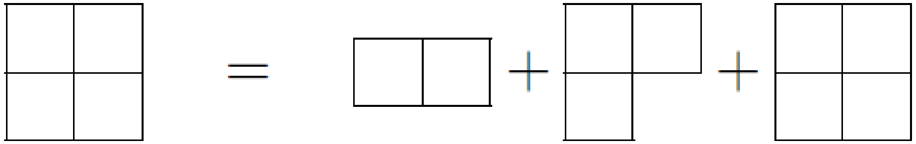
Killing tensors define the vector space of HS algebra.

$SO(D+1)$ $\qquad\qquad$ $SO(D)$

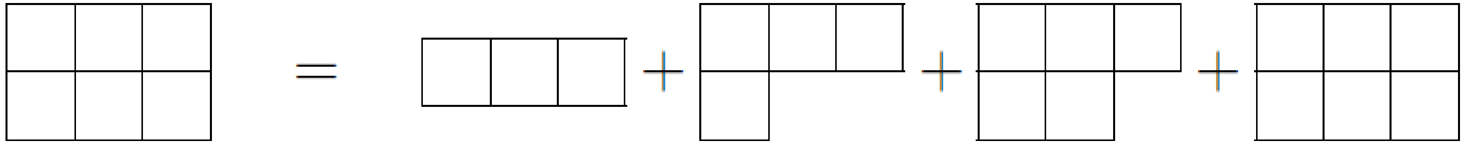
s=2



s=3



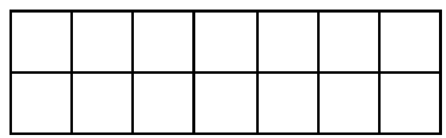
s=4



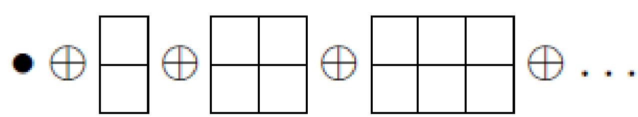
Vector space of the HS algebra

We use SO(D-1,2) or SO(D,1) connection of (A)dS$_D$.

HS algebras are thus span by generators, that are parameterized by two row rectangular Young diagrams of SO(D+1)



Vector space of HS algebra is the space of two row rectangular Young diagrams of SO(D+1)



Requirements on HS algebra

- Spin two part is gravity

$$[\![M_{ab} \, , M_{cd}]\!] = 2 \left(\eta_{a[c} M_{d]b} - \eta_{b[c} M_{d]a} \right)$$

- All the fields couple to gravity minimally (equivalence principle)

$$[\![M_{a_1 b_1, \ldots, a_r b_r} \, , M_{cd}]\!] = 2 \sum_{k=1}^{r} \eta_{a_k[c} M_{\ldots, d] b_k, \ldots} - \eta_{b_k[c} M_{\ldots, d] a_k, \ldots}$$

Universal Enveloping Algebra construction

Higher Spin algebra can be defined as a quotient algebra. Take the Universal Enveloping Algebra of (A)dS isometry algebra, which is defined as a quotient of the tensor algebra of $\mathfrak{so}_{D+1}$ with the ideal generated by

$$I_{abcd} = M_{ab} \otimes M_{cd} - M_{cd} \otimes M_{ab} - [\![M_{ab} \, , \, M_{cd}]\!]$$

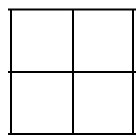
So class representatives are *GL(D+1)* tensors

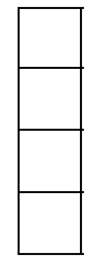
$$M_{a_1 b_1 | \cdots | a_n b_n} := \frac{1}{n!} \sum_{\sigma \in S_n} M_{a_{\sigma(1)} b_{\sigma(1)}} \otimes \cdots \otimes M_{a_{\sigma(n)} b_{\sigma(n)}}$$

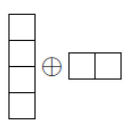
They contain Killing tensors, but not only.

UEA Construction

At the quadratic level we have decomposition to GL(D+1) tensors of the following symmetry type:

$$M^{(a_1}{}_{(b_1|}{}^{a_2)}{}_{b_2)} \sim$$ 

$$M_{[a_1 b_1|a_2 b_2]} \sim$$ 

The first one has traceless $\mathfrak{so}_{D+1}$ tensor part (which is a Killing tensor), and trace part. The algebra, that is a quotient by the two sided ideal, generated by the two elements  contains only (all the) Killing tensors and therefore is a proper Higher Spin algebra (usually referred to as Eastwood – Vasiliev algebra).

UEA Construction

Ideal generating elements are

$$J_{ab} = M_{(a}{}^{c}{}_{|b)c} - \frac{1}{D+1}\,\eta_{ab}\,M^{cd}{}_{|cd}\,, \qquad J_{abcd} := M_{[ab|cd]}$$

Quotienting these elements automatically fixes all the Casimirs

$$C_{2n} = M_{a_1}{}^{a_2}{}_{|\,a_2}{}^{a_3}{}_{|\,\cdots\,|\,a_{2n}}{}^{a_1}$$

In D=5, we have one-parameter family of ideal generating elements, and corresponding HS algebras

$$V^{\lambda}_{abcd} = M_{[ab|cd]} - \lambda\,\epsilon_{abcdef}\,M^{ef}$$

HS algebras in different dimensions

- In any dimensions defined as quotient star product algebra with vector generating elements.
- In 3d, one-parameter family of HS algebras is defined as a star product algebra with spinor generating elements (deformed oscillator).
- In 4d the unique algebra, that contains one copy of generators of each spin, is naturally defined with Weyl spinor generating elements.
- In 5d one-parameter family of algebras can be defined (analogously to 3d) as star product (quotient-) algebra with spinor generating elements.

HS algebras of Classical Lie algebras

In order the obtain HS algebra, we quotient the UEA with an ideal corresponding to the tensors which are not Killing.

The reduction of generators can be carried out by choosing the proper representation of the isometry algebra, that is small enough to project all the generators except for Killing tensors.

It turns out that these representations are in fact the minimal representations.

The kernel of the minimal representation forms an ideal in the UEA, so-called Joseph ideal $\mathcal{J}(\mathfrak{g})$.

Formal definition of HS algebras

The kernel of the minimal representation in the UEA is the Joseph ideal. HS algebra is the algebra of the symmetries of the minimal representation:

$$hs(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})/\mathcal{J}(\mathfrak{g})$$

As a vector space, it is the dual of the space of polynomials in the minimal coadjoint orbit.

We will denote the product in HS algebras by $*$

Which can be written symbolically as $* := \circ/\sim$

HS Algebras of Classical Lie algebras

We start with an algebra $\mathfrak{sp}_{2N}$

$$[\![N_{AB}\,,\; N_{CD}]\!] = \Omega_{A(C}\, N_{D)B} + \Omega_{B(C}\, N_{D)A}$$

With

$$\Omega_{AB} = -\Omega_{BA}$$

We adopt following index notation

$$A = \alpha\, a\,, \qquad \alpha = \pm\,, \quad a = 1, 2, \ldots, N$$

So that

$$\Omega_{\alpha a\, \beta b} = \epsilon_{\alpha\beta}\, \eta_{ab}\,, \qquad \epsilon_{\pm\mp} = \pm 1$$

Special linear algebra $\mathfrak{sl}_N$

$$L^a_b := \eta^{ac} N_{+c\,-b} - \frac{1}{N}\,\delta^a_b\,\eta^{cd}\,N_{+c\,-d}$$

$$[\![\, L^a_b\,,\;\; L^c_d \,]\!] = \delta^a_d\, L^c_b - \delta^c_b\, L^a_d$$

Orthogonal algebra $\mathfrak{so}_N$

$$M_{ab} \;:=\; N_{+a\,-b} - N_{+b\,-a}$$

$$[\![\, M_{ab}\,, M_{cd} \,]\!] = 2i\,(\eta_{a[c}\, M_{d]b} - \eta_{b[c}\, M_{d]a})$$

Dual vector space

We introduce dual vector space elements

$$N(U)=\frac{1}{2}N_{AB}U^{AB} \qquad L(V)=L_b^a V_a^b \qquad M(W)=\frac{1}{2}M_{ab}W^{ab}$$

Minimal coadjoint orbits for classical Lie algebras are defined by following quadratic equations

$$\mathcal{O}_{\min}(\mathfrak{sp}_{2N}) \quad : \quad U^{A[B}U^{D]C}=0\,,$$

$$\mathcal{O}_{\min}(\mathfrak{sl}_N) \quad : \quad V_{[a}^b V_{c]}^d = 0 = V_a^b V_b^c\,,$$

$$\mathcal{O}_{\min}(\mathfrak{so}_N) \quad : \quad W^{a[b}W^{cd]}=0=W^{ab}W_b{}^c$$

With solutions

$$U^{AB}=u^A u^B\,, \quad V_b^a = v_+^a v_{-b} \quad [v_+\cdot v_- = 0]\,, \quad W^{ab}=w_+^{[a}w_-^{b]} \quad [w_\alpha\cdot w_\beta = 0]$$

HS algebras of Classical Lie algebras

The generating functions of the elements of HS algebra, corresponding to classical Lie algebra, is given by

$$hs(\mathfrak{sp}_{2N}) \quad : \qquad N(U) = 1 + \sum_{n=0}^{\infty} \frac{1}{2^n\, n!} N_{A_1 B_1, \dots, A_n B_n}\, U^{A_1 B_1} \dots U^{A_n B_n} ,$$

$$hs_\lambda(\mathfrak{sl}_N) \quad : \qquad L(V) = 1 + \sum_{n=0}^{\infty} \frac{1}{n!} L^{a_1 \cdots a_n}_{b_1 \cdots b_n}\, V^{b_1}_{a_1} \cdots V^{b_n}_{a_n} ,$$

$$hs(\mathfrak{so}_N) \quad : \qquad M(W) = 1 + \sum_{n=0}^{\infty} \frac{1}{2^n\, n!} M_{a_1 b_1, \dots, a_n b_n}\, W^{a_1 b_1} \dots W^{a_n b_n} ,$$

Corresponding HS generators are

$$N_{A_1 B_1, \dots, A_n B_n} , \qquad L^{a_1 \cdots a_n}_{b_1 \cdots b_n} , \qquad M_{a_1 b_1, \dots, a_n b_n}$$

Generating functions of HS algebra generators

Minimal representation of $\mathfrak{sp}_{2N}$ is described by oscillators y_A

$$N_{AB} = y_A\, y_B$$

Endowed with Moyal product $\star$

$$(f \star g)(y) = \exp\left(\Omega_{AB}\, \partial_{y_A}\, \partial_{z_B}\right) f(y)\; g(z)\Big|_{z=y}$$

$hs(\mathfrak{sp}_{2N})$ is generated by polynomials of $y_A\, y_B$

$$N_{A_1 B_1, \ldots, A_n B_n} = y_{A_1}\, y_{B_1} \cdots y_{A_n}\, y_{B_n}$$

With Gaussian generating function

$$N(U) = \exp\left(\frac{1}{2}\, y_A\, U^{AB}\, y_B\right)$$

In order to address the algebraic structure of $hs_\lambda(\mathfrak{sl}_N)$ we consider the dual pair

$$(\,GL_1\,,\,GL_N\,) \subset Sp_{2N}$$

The $\mathfrak{u}_1$ - centralizer in $hs(\mathfrak{sp}_{2N})$ consists of elements, satisfying the following condition

$$\left[\,y_+\cdot y_-\,,\;f(y)\,\right]_\star = \left(y_+\cdot\partial_{y_+} - y_-\cdot\partial_{y_-}\right) f(y) = 0$$

The solution space is generated by

$$\tilde{L}(\tilde{V}) = \exp\left(y_+\cdot\tilde{V}\cdot y_-\right)$$

With

$$\tilde{V}^b_{[a}\,\tilde{V}^d_{c]} = 0 \quad \Leftrightarrow \quad \tilde{V}^b_a = \tilde{v}_{+a}\,\tilde{v}^b_-$$

We call this algebra $hs(\mathfrak{gl}_N)$. It makes use of the same Moyal product.

HS algebra of $\mathfrak{sl}_N$ is a quotient of the $hs(\mathfrak{gl}_N)$ by a relation

$$K_\lambda := y_+ \cdot y_- - N\,\lambda \sim 0$$

corresponding HS algebra is called $hs_\lambda(\mathfrak{sl}_N)$.

Consider following isomorphism

$$\begin{aligned} \rho_\lambda \;:\; hs(\mathfrak{gl}_N) \quad &\to \quad hs_\lambda(\mathfrak{gl}_N)\,, \\ f(y) \quad &\mapsto \quad \rho_\lambda(f)(y) = e^{\lambda\,\partial_{y_+}\cdot\,\partial_{y_-}}\, f(y) \end{aligned}$$

$hs_\lambda(\mathfrak{gl}_N)$ admits a deformed star product

$$(f \star_\lambda g)(y) = \rho_\lambda\big(\,\rho_\lambda^{-1}(f) \star \rho_\lambda^{-1}(g)\,\big)(y)$$

$$= \exp\Big[\,\big(\partial_{y_+}\cdot\partial_{z_-} - \partial_{z_+}\cdot\partial_{y_-}\big) + \lambda\,\big(\partial_{y_+}\cdot\partial_{z_-} + \partial_{z_+}\cdot\partial_{y_-}\big)\,\Big]\, f(y)\;g(z)\,\Big|_{z=y}$$

$hs_\lambda(\mathfrak{sl}_N)$ is a quotient of $hs_\lambda(\mathfrak{gl}_N)$ by a relation

$$\rho_\lambda(K_\lambda) = K_0 \sim 0$$

For the algebra $\mathfrak{so}_N$, we consider the dual pair

$$(Sp_2, O_N) \subset Sp_{2N}$$

The $\mathfrak{sp}_2$ - centralizer of the $hs(\mathfrak{sp}_{2N})$ consists of elements, satisfying

$$\left[y_\alpha \cdot y_\beta \,, f(y) \right]_\star = \left(y_\alpha \cdot \partial_{y^\beta} + y_\beta \cdot \partial_{y^\alpha} \right) f(y) = 0$$

The solution space is generated by

$$\tilde{M}(\tilde{W}) = \exp\left(\frac{1}{2} y_{+a} \tilde{W}^{ab} y_{-b} \right)$$

with

$$\tilde{W}^{a[b} \tilde{W}^{cd]} = 0 \quad \Leftrightarrow \quad \tilde{W}^{ab} = \tilde{w}_+^{[a} \tilde{w}_-^{b]}$$

We call this algebra $\widetilde{hs}(\mathfrak{so}_N)$. It makes use of the same Moyal product.

HS algebra $hs(\mathfrak{so}_N)$ is a quotient of the $\widetilde{hs}(\mathfrak{so}_N)$ by

$$K_{\alpha\beta} := y_\alpha \cdot y_\beta \sim 0$$

Analogously to the $hs_\lambda(\mathfrak{sl}_N)$ construction, one can consider quotient of the algebra $\widetilde{hs}(\mathfrak{so}_N)$ by a more general relation

$$y_\alpha \cdot y_\beta \sim N\,\lambda_{\alpha\beta}$$

The resulting algebra does not correspond to the minimal representation, and therefore describe bigger spectrum of particles, as compared to Eastwood-Vasiliev algebra. Additional generators are those of partially massless fields. These generalizations of HS algebra are under investigation (work in progress).

Special cases of D=3,5

The existence of one-parameter family of HS algebras in D=3 and 5 dimensions is connected to the algebra isomorphisms of $\mathfrak{so}_3$ and $\mathfrak{so}_6$ to $\mathfrak{sl}_2$ and $\mathfrak{sl}_4$ respectively.

For $\mathfrak{sl}_N$ algebras, the corresponding "HS algebras" are the quotients of the UEA by two-sided ideal, generated by the following element

$$I^{\lambda}{}^{ac}_{bd} = L^{[a}_{b} \otimes L^{c]}_{d} + \delta^{[a}_{(b} L^{c]}_{d)} + \lambda\, \delta^{[a}_{[b} L^{c]}_{d]} + \frac{1}{4}(\lambda^2 - 1)\, \delta^{[a}_{[b}\, \delta^{c]}_{d]}$$

Trace

We are interested in structure constants ${C_{ab}}^c$ of HS algebras, defined by

$$T_a * T_b = {C_{ab}}^c \, T_c$$

Where T_a - s are generators of HS algebra $hs(\mathfrak{g})$

We define the trace of the element of HS algebra as the coefficient of the identity

$$\mathrm{Tr}\left[c_0 + c^a \, T_a \right] = c_0$$

Bilinear form and structure constants

The following bilinear form is symmetric and invariant

$$B_{ab} = \mathrm{Tr}\,[\,T_a * T_b\,]$$

Analogously, the trilinear form is simply connected to the structure constant

$$C_{abc} = \mathrm{Tr}\,[\,T_a * T_b * T_c\,] = C_{ab}{}^{d}\; B_{dc}$$

Bilinear form and structure constant

$\mathfrak{sp}_{2N}$ series

$$\mathcal{B}(U) = \frac{1}{\sqrt{1 + \frac{\langle U_1\, U_2 \rangle}{4}}}\,, \qquad \mathcal{C}(U) = \frac{1}{\sqrt{1 + \frac{\langle U_1\, U_2\rangle + \langle U_2\, U_3\rangle + \langle U_3\, U_1\rangle + \langle U_1\, U_2\, U_3\rangle}{4}}}\,.$$

$\mathfrak{sl}_N$ series

$$\mathcal{B}(V) = {}_3F_2\Big(\tfrac{N}{2}\,(1+\lambda)\,,\ \tfrac{N}{2}\,(1-\lambda)\,,\ 1\,;\ \tfrac{N}{2}\,,\ \tfrac{N+1}{2}\,;\ -\tfrac{1}{4}\,\langle V_1\, V_2\rangle\Big)\,,$$

$$\begin{aligned}\mathcal{C}(V) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^k \binom{k}{\ell} \frac{\left(\frac{N(1+\lambda)}{2}\right)_{2\,k-\ell}\left(\frac{N(1-\lambda)}{2}\right)_{k+\ell}}{(N)_{3k}} \times \\ \times \left[\langle V_1\, V_2\rangle + \langle V_2\, V_3\rangle + \langle V_3\, V_1\rangle + \langle V_1\, V_2\, V_3\rangle\right]^{k-\ell} \\ \times \left[\langle V_1\, V_2\rangle + \langle V_2\, V_3\rangle + \langle V_3\, V_1\rangle - \langle V_3\, V_2\, V_1\rangle\right]^{\ell}\,.\end{aligned}$$

Bilinear form and structure constant

$\mathfrak{so}_N$ series

$$\mathcal{B}(W) = {}_2F_1\left(2\,,\ \tfrac{N-4}{2}\,;\ \tfrac{N-1}{2}\,;\ -\tfrac{1}{8}\,\langle W_1\, W_2\rangle\right),$$

$$\begin{aligned}\mathcal{C}(W) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m}{m!}\,\frac{1}{n!}\,\frac{\left(\frac{N-4}{2}\right)_{m+2n}(2)_{m+2n}}{\left(\frac{N-1}{2}\right)_{m+3n}\ 8^{m+3n}} \times \\ \times \left[\langle W_1\, W_2\rangle + \langle W_2\, W_3\rangle + \langle W_3\, W_1\rangle + \langle W_1\, W_2\, W_3\rangle\right]^m \\ \times \left[\langle W_1\, W_2\rangle\langle W_2\, W_3\rangle\langle W_3\, W_1\rangle + 2\,\langle W_1\, W_2\, W_3\rangle^2\right]^n .\end{aligned}$$

Isomorphisms

The simplest example is the isomorphism $\mathfrak{sl}_2 = \mathfrak{sp}_2$

Since

$$ {}_3F_2\left(\tfrac{N}{2}\,(1+\lambda)\,,\ \tfrac{N}{2}\,(1-\lambda)\,,\ 1\,;\ \tfrac{N}{2}\,,\ \tfrac{N+1}{2}\,;\ -z\right) = \frac{1}{\sqrt{1+z}} \qquad [N=2, \lambda=\tfrac{1}{2}] $$

which is pointing to the isomorphism

$$ hs_{\frac{1}{2}}(\mathfrak{sl}_2) \simeq hs(\mathfrak{sp}_2) $$

From

$$ {}_2F_1\left(2\,,\ \tfrac{N-4}{2}\,;\ \tfrac{N-1}{2}\,;\ -z\right) = \frac{1}{\sqrt{1+z}} \qquad [N=5] $$

we conclude that

$$ hs(\mathfrak{so}_5) \simeq hs(\mathfrak{sp}_4) $$

Isomorphisms

From yet another identity we find the connection between $hs(\mathfrak{so}_6)$ and $hs_\lambda(\mathfrak{sl}_4)$

$$
{}_2F_1\Big(2\,,\,\tfrac{N-4}{2}\,;\,\tfrac{N-1}{2}\,;\,-z\Big) = {}_3F_2\Big(\tfrac{N'}{2}\,(1+\lambda)\,,\,\tfrac{N'}{2}\,(1-\lambda)\,,\,1\,;\,\tfrac{N'}{2}\,,\,\tfrac{N'+1}{2}\,;\,-z\Big)
$$

$$
\big[N=6,\ N'=4,\ \lambda=0\big]\,.
$$

From which we deduce

$$
hs(\mathfrak{so}_6) \simeq hs_0(\mathfrak{sl}_4)
$$

More on algebra $hs_\lambda(\mathfrak{sl}_N)$

For the special values of λ , satisfying

$$N(1 \pm \lambda) = -2\,M\,, \qquad M \in \mathbb{N}$$

$hs_\lambda(\mathfrak{sl}_N)$ develops an infinite dimensional ideal, while the factor algebra becomes finite dimensional HS algebra with generators of spin $2, \ldots, M$.

These algebras are special linear algebras

$$\mathfrak{sl}_{(M)_{N-1}}$$

Where

$$(M)_{N-1} := M(M+1)\cdots(M+N-2)$$

Spin 1 field can be added to the spectrum with no cost.

Outlook

- Generalization of HS algebras to include mixed symmetry and/or partially massless spectrum.
- Most general physical theories that make use of these HS algebras (*is there any HS theory within Lagrangian formulation?*).
- HS algebras corresponding to theories with more than one graviton.

Thank you for your attention!