Black Holes in Supergravity

Antonio Gallerati

Politecnico di Torino - Dip. DISAT

XXIV Convegno Nazionale di Fisica Teorica
28-31 Maggio 2014, Cortona

Based on:

JHEP **1401** (2014) 053 by L. Andrianopoli, A. Gallerati, M. Trigiante

JHEP **05** (2013) 071 by L. Andrianopoli, R. D'Auria, A. Gallerati, M. Trigiante

JHEP **12** (2012) 078 by L. Andrianopoli, R. D'Auria, P. Giaccone, M. Trigiante;

Outline

Stationary, asymptotically Flat Black Holes in D=4 SUGRAS

Stationary, asymptotically Flat Black Holes in D=4 SUGRAS

Supergravity bosonic field content

n_S scalar fields ϕ^r; n_V vector fields A_μ^Λ; Graviton $g_{\mu\nu}$

Stationary, asymptotically Flat Black Holes in D=4 SUGRAS

Supergravity bosonic field content

n_S scalar fields ϕ^r; n_V vector fields A^Λ_μ; Graviton $g_{\mu\nu}$

$D=3$ description of $D=4$ stationary solutions

- Metric: $ds^2 = -e^{2U}\,(dt + \omega_i\,dx^i)^2 + e^{-2U}\,g_{ij}\,dx^i dx^j$

Stationary, asymptotically Flat Black Holes in D=4 SUGRAS

Supergravity bosonic field content

n_S scalar fields ϕ^r; n_V vector fields A^Λ_μ; Graviton $g_{\mu\nu}$

$D=3$ description of $D=4$ stationary solutions

- Metric: $ds^2 = -e^{2U}\,(dt + \omega_i\,dx^i)^2 + e^{-2U}\,g_{ij}\,dx^i dx^j$
- Solution to a $D=3$ Euclidean theory obtained from time-reduction from the $D=4$ one (Breitenlohner, Gibbons, Maison)

Stationary, asymptotically Flat Black Holes in D=4 SUGRAS

Supergravity bosonic field content

n_S scalar fields ϕ^r; n_V vector fields A^Λ_μ;   Graviton $g_{\mu\nu}$

$D = 3$ description of $D = 4$ stationary solutions

- Metric: $ds^2 = -e^{2U}\,(dt + \omega_i\,dx^i)^2 + e^{-2U}\,g_{ij}\,dx^i dx^j$
- Solution to a $D = 3$ Euclidean theory obtained from time-reduction from the $D = 4$ one (Breitenlohner, Gibbons, Maison)
- Dualizing vectors into scalars in $D = 3$ we end up with a sigma model describing $n = 2 + n_S + 2n_V$ scalars ϕ^I coupled to gravity

Stationary, asymptotically Flat Black Holes in D=4 SUGRAS

Supergravity bosonic field content

n_S scalar fields ϕ^r; n_V vector fields A^Λ_μ; Graviton $g_{\mu\nu}$

$D = 3$ description of $D = 4$ stationary solutions

- Metric: $ds^2 = -e^{2U}(dt + \omega_i\, dx^i)^2 + e^{-2U}\, g_{ij}\, dx^i dx^j$
- Solution to a $D = 3$ Euclidean theory obtained from time-reduction from the $D = 4$ one (Breitenlohner, Gibbons, Maison)
- Dualizing vectors into scalars in $D = 3$ we end up with a sigma model describing $n = 2 + n_S + 2n_V$ scalars ϕ^I coupled to gravity

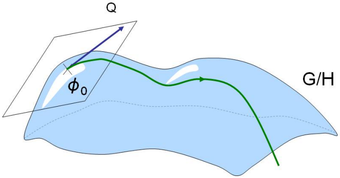
$$e^{-1}\mathcal{L} = \frac{R_3}{2} - \frac{1}{2} G_{IJ}(\phi)\, \partial_i \phi^I \partial^i \phi^J \quad \Rightarrow \quad (\phi^I) \in \mathscr{M}^{(3)}_{scal} = \frac{G}{H}$$

$$\underline{\textit{Field dualization}}: \quad \omega \to a, \quad A^\Lambda_\mu \to \mathcal{Z}^M = (\mathcal{Z}^\Lambda, \mathcal{Z}_\Lambda)$$

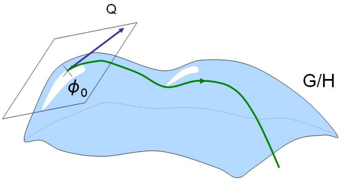
- Spherical symmetry:
  $\phi^I = \phi^I(\tau)$, solution is a geodesic on $\mathscr{M}_{scal}^{(3)}$

- Spherical symmetry: $\phi^I = \phi^I(\tau)$, solution is a geodesic on $\mathscr{M}^{(3)}_{scal}$
- Geodesic uniquely defined by initial point $\phi^I_0 = \phi^I(\tau = 0)$ and initial velocity $Q \in T_{\phi_0}(G/H)$

- Spherical symmetry: $\phi^I = \phi^I(\tau)$, solution is a geodesic on $\mathscr{M}^{(3)}_{scal}$
- Geodesic uniquely defined by initial point $\phi^I_0 = \phi^I(\tau = 0)$ and initial velocity $Q \in T_{\phi_0}(G/H)$

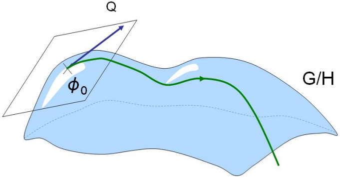


- Spherical symmetry: $\phi^I = \phi^I(\tau)$, solution is a geodesic on $\mathscr{M}^{(3)}_{scal}$
- Geodesic uniquely defined by initial point $\phi^I_0 = \phi^I(\tau = 0)$ and initial velocity $Q \in T_{\phi_0}(G/H)$

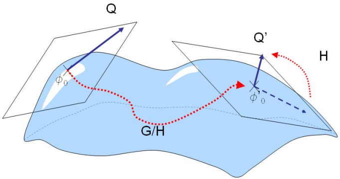


- Isometry group G is the global symmetry of the $D = 3$ theory
- Action of G on a geodesic (ϕ_0, Q)
- Fix $\phi_0 \equiv 0$, G-orbit of geodesic is H-orbit of $Q \in T_0$

- Spherical symmetry: $\phi^I = \phi^I(\tau)$, solution is a geodesic on $\mathscr{M}^{(3)}_{scal}$
- Geodesic uniquely defined by initial point $\phi^I_0 = \phi^I(\tau = 0)$ and initial velocity $Q \in T_{\phi_0}(G/H)$



- Isometry group G is the global symmetry of the $D=3$ theory
- Action of G on a geodesic (ϕ_0, Q)
- Fix $\phi_0 \equiv 0$, G-orbit of geodesic is H-orbit of $Q \in T_0$



- Axisymm. sol. $\phi = \phi(\tau, \theta)$, still defined by unique point $\phi_0 = \lim_{\tau \to 0} \phi(\tau, \theta)$ and vector $Q \in T_{\phi_0}$. Fix G/H by setting $\phi_0 = 0$.

- Axisymm. sol. $\phi=\phi(\tau,\theta)$, still defined by unique point $\phi_0=\lim_{\tau\to 0}\phi(\tau,\theta)$ and vector $Q\in T_{\phi_0}$. Fix G/H by setting $\phi_0=0$.

  Tangent space at the origin $T_0\sim\mathfrak{K}$ subspace of the Lie algebra $\mathfrak{g}$ of G.

  Q is a matrix in $\mathfrak{K}$

- Axisymm. sol. $\phi = \phi(\tau, \theta)$, still defined by unique point $\phi_0 = \lim_{\tau \to 0} \phi(\tau, \theta)$ and vector $Q \in T_{\phi_0}$. Fix G/H by setting $\phi_0 = 0$.

  Tangent space at the origin $T_0 \sim \mathfrak{K}$ subspace of the Lie algebra $\mathfrak{g}$ of G.

  Q is a matrix in $\mathfrak{K}$

- Velocity vector Q is the *Noether-charge matrix*:

$$Q = \frac{1}{4\pi} \int_{S_2} {}^*J = M_{ADM}\, K_0 + \Sigma^r\, K_r + n_{NUT}\, K_\bullet + p^\Lambda K_\Lambda + q_\Lambda K^\Lambda \quad \in \ \mathfrak{K}$$

$J = J_i\, dx^i$ being the Noether current. Q does not contain angular momentum $\mathscr{J}$!

- Axisymm. sol. $\phi=\phi(\tau,\theta)$, still defined by unique point $\phi_0=\lim_{\tau\to 0}\phi(\tau,\theta)$ and vector $Q\in T_{\phi_0}$. Fix G/H by setting $\phi_0=0$.

  Tangent space at the origin $T_0\sim\mathfrak{K}$ subspace of the Lie algebra $\mathfrak{g}$ of G.

  Q is a matrix in $\mathfrak{K}$

- Velocity vector Q is the *Noether-charge matrix*:

$$Q=\frac{1}{4\pi}\int_{S_2}{}^*J=M_{ADM}\,K_0+\Sigma^r\,K_r+n_{NUT}\,K_\bullet+p^\Lambda K_\Lambda+q_\Lambda K^\Lambda\quad\in\ \mathfrak{K}$$

  $J=J_i\,dx^i$ being the Noether current. Q does not contain angular momentum $\mathscr{J}$!

- Define new $\mathfrak{g}$-matrix Q_ψ capturing rotation:

$$Q_\psi=-\frac{3}{4\pi}\int_{S_2^\infty}\psi_{[i}J_{j]}\,dx^i\wedge dx^j=\mathscr{J}\,K_\bullet+\ldots\quad\in\ \mathfrak{K}\qquad(\psi=\partial_\varphi)$$

  Q and Q_ψ represent independent vectors in T_0. Static solution $\to$ $Q_\psi=0$

Global symmetry and regularity

- Action of G on the solution $\Rightarrow$ action of H on Q, Q_ψ:

$$Q \;\rightarrow\; Q' = h^{-1}\,Q\,h\,, \qquad Q_\psi \;\rightarrow\; Q'_\psi = h^{-1}\,Q_\psi\,h \qquad (h \in H)$$

Global symmetry and regularity

- Action of G on the solution $\Rightarrow$ action of H on Q, Q_ψ:

$$Q \to Q' = h^{-1}\,Q\,h\,, \qquad Q_\psi \to Q'_\psi = h^{-1}\,Q_\psi\,h \qquad (h \in H)$$

- Kerr-Newman solution $(m,\,p,\,q,\,\mathscr{J})$. *Regularity* : $\boxed{m^2 - \frac{p^2+q^2}{2} \geq \frac{\mathscr{J}^2}{m^2}}$

Global symmetry and regularity

- Action of G on the solution $\Rightarrow$ action of H on Q, Q_ψ:

$$Q \to Q' = h^{-1}\, Q\, h\,, \qquad Q_\psi \to Q'_\psi = h^{-1}\, Q_\psi\, h \qquad (h \in H)$$

- Kerr-Newman solution $(m,\, p,\, q,\, \mathscr{J})$. *Regularity* : $\boxed{m^2 - \frac{p^2+q^2}{2} \geq \frac{\mathscr{J}^2}{m^2}}$

- Q, Q_ψ diagonalizable matrices:

$$\frac{k}{2}\,\mathrm{Tr}(Q^2) = m^2 - \frac{p^2+q^2}{2}\,, \qquad \mathrm{Tr}(Q_\psi^2) = \frac{\mathscr{J}^2}{m^2}\,\mathrm{Tr}(Q^2)$$

Global symmetry and regularity

- Action of G on the solution $\Rightarrow$ action of H on Q, Q_ψ:

$$Q \rightarrow Q' = h^{-1}\, Q\, h\,, \qquad Q_\psi \rightarrow Q'_\psi = h^{-1}\, Q_\psi\, h \qquad (h \in H)$$

- Kerr-Newman solution $(m,\, p,\, q,\, \mathscr{J})$. *Regularity* : $m^2 - \frac{p^2+q^2}{2} \geq \frac{\mathscr{J}^2}{m^2}$

- Q, Q_ψ diagonalizable matrices:

$$\frac{k}{2}\,\mathrm{Tr}(Q^2) = m^2 - \frac{p^2+q^2}{2}\,, \qquad \mathrm{Tr}(Q_\psi^2) = \frac{\mathscr{J}^2}{m^2}\,\mathrm{Tr}(Q^2)$$

- Regularity condition can be written in a G-invariant form:

$$\frac{k}{2}\,\mathrm{Tr}(Q^2) \geq \frac{\mathrm{Tr}(Q_\psi^2)}{\mathrm{Tr}(Q^2)}$$

"$=$" holds for extremal $(T=0)$ solutions

General non-extremal solution

- Act on the well known Kerr solution with *Ehlers+Harrison transformations*

  $\Rightarrow$ obtain general STU-model solution

General non-extremal solution

- Act on the well known Kerr solution with *Ehlers+Harrison transformations*

  $\Rightarrow$ obtain general STU-model solution

- Use coset-space geometry to find the new form of the scalar fields in term of the Harrison β-parameters

$$\Rightarrow \quad \phi, \quad U, \quad a, \quad \mathcal{Z}$$

General non-extremal solution

- Act on the well known Kerr solution with *Ehlers+Harrison transformations*

  $\Rightarrow$ obtain general STU-model solution

- Use coset-space geometry to find the new form of the scalar fields in term of the Harrison β-parameters

$$\Rightarrow \quad \phi, \quad U, \quad a, \quad \mathcal{Z}$$

- Solve the dualization integral equation and get the form of the metric relevant quantities and of the 4-dimensional vectors

$$\mathbb{F}^M = \begin{pmatrix} F^{\Lambda}_{\mu\nu} \\ G_{\Lambda\,\mu\nu} \end{pmatrix} = d\mathcal{Z}^M \wedge (dt+\omega) + e^{-2U}\mathbb{C}^{MN}\mathcal{M}_{(4)NP} \ *_3 d\mathcal{Z}^P$$

$$d\omega = -e^{-4U} \ *_3 (da + \mathcal{Z}^T\mathbb{C}d\mathcal{Z}) \quad \Rightarrow \quad \omega$$

$$(local\ integration) \quad \mathbb{F}^M = dA^M \quad \Rightarrow \quad A^M$$

Above results were not present in literature

Singular limits...

- Limits of non-extremal axisymmetric solution studied in specific contexts (Heterotic Sugra, Kaluza Klein theories...)

Singular limits...

- Limits of non-extremal axisymmetric solution studied in specific contexts (Heterotic Sugra, Kaluza Klein theories...)

- General, frame-independent, geometric prescription though singular Harrison transformations.

  Harrison generators $(\mathbb{J}_M) = (\mathbb{J}_\Lambda, \mathbb{J}^\Lambda)$ in $\mathfrak{H}$ are in one-to-one correspondence with $(\mathcal{P}^M) = (p^\Lambda, q_\Lambda)$.

Singular limits...

- Limits of non-extremal axisymmetric solution studied in specific contexts (Heterotic Sugra, Kaluza Klein theories...)

- General, frame-independent, geometric prescription though singular Harrison transformations.

  Harrison generators $(\mathbb{J}_M) = (\mathbb{J}_\Lambda, \mathbb{J}^\Lambda)$ in $\mathfrak{H}$ are in one-to-one correspondence with $(\mathcal{P}^M) = (p^\Lambda, q_\Lambda)$.

- Act on the Kerr solution $(m_K, \mathscr{J}_K)$ by means of the Harrison transformation:

$$\mathcal{O} = \exp\left(\sum_l \log(\beta_l)\, \mathbb{J}_l\right)$$

The resulting solution is a non extremal rotating one, coupled to scalar fields, with charges in the normal form

- Rescale the Harrison parameters and the original angular momentum as

$$\beta_\ell \;\to\; {m_K}^{\pm 1}\,\beta'_\ell\,; \quad \mathscr{J}_K \;\to\; {m_K}^2\,\Omega$$

and then send $m_K \to 0$ while keeping β'_ℓ and Ω fixed.

- Rescale the Harrison parameters and the original angular momentum as

$$\beta_\ell \to m_K{}^{\pm 1}\, \beta'_\ell\,; \quad \mathscr{J}_K \to m_K{}^2\, \Omega$$

and then send $m_K \to 0$ while keeping β'_ℓ and Ω fixed.

1. Only for choices of "$\pm$" yielding

   $$I_4(p,q) < 0 \quad \text{(nonBPS solution)}$$

   there is a residual $\mathscr{J} \neq 0$ $\Longrightarrow$ under-rotating single-center solution

- Rescale the Harrison parameters and the original angular momentum as

$$\beta_\ell \rightarrow m_K{}^{\pm 1}\, \beta'_\ell\,; \quad \mathscr{J}_K \rightarrow m_K{}^2\, \Omega$$

and then send $m_K \rightarrow 0$ while keeping β'_ℓ and Ω fixed.

1. Only for choices of "$\pm$" yielding

   $$I_4(p,q) < 0 \quad \text{(nonBPS solution)}$$

   there is a residual $\mathscr{J} \neq 0$ $\Longrightarrow$ under-rotating single-center solution

2. For choices of "$\pm$" yielding

   $$I_4(p,q) > 0 \quad \text{(BPS and nonBPS solutions)}$$

   no residual rotation $(Q_\psi = 0)$ $\Longrightarrow$ extremal static solutions

Attractor mechanism

Static, sph.-symm. extremal solutions exhibit the attractor mechanism:

Attractor mechanism

Static, sph.-symm. extremal solutions exhibit the attractor mechanism:

The scalars (effectively coupled to the b.h.) evolve towards values at the horizon which are fixed in terms of p^Λ, q_Λ. The near horizon geometry (and thus the entropy S) only depends on the quantized charges.

Attractor mechanism

Static, sph.-symm. extremal solutions exhibit the attractor mechanism:

The scalars (effectively coupled to the b.h.) evolve towards values at the horizon which are fixed in terms of p^Λ, q_Λ. The near horizon geometry (and thus the entropy S) only depends on the quantized charges.

$$S_{non-extr.}(\phi_0, p, q) = \frac{A_{non-extr.}(\phi_0, p, q)}{4} \quad \xrightarrow{c \to 0} \quad S_{extr.}(p, q) = \pi\sqrt{|I_4(p, q)|}$$

Attractor mechanism

Static, sph.-symm. extremal solutions exhibit the attractor mechanism:

The scalars (effectively coupled to the b.h.) evolve towards values at the horizon which are fixed in terms of p^Λ, q_Λ. The near horizon geometry (and thus the entropy S) only depends on the quantized charges.

$$S_{non-extr.}(\phi_0, p, q) = \frac{A_{non-extr.}(\phi_0, p, q)}{4} \quad \stackrel{c\to 0}{\longrightarrow} \quad S_{extr.}(p,q) = \pi\sqrt{|I_4(p,q)|}$$

- In the extremal under-rotating limit ($I_4 < 0$) similar behavior for $\mathscr{J}$ [arXiv:1310.7886]:

Attractor mechanism

Static, sph.-symm. extremal solutions exhibit the attractor mechanism:

The scalars (effectively coupled to the b.h.) evolve towards values at the horizon which are fixed in terms of p^Λ, q_Λ. The near horizon geometry (and thus the entropy S) only depends on the quantized charges.

$$S_{non-extr.}(\phi_0, p, q) = \frac{A_{non-extr.}(\phi_0, p, q)}{4} \quad \xrightarrow{c\to 0} \quad S_{extr.}(p, q) = \pi \sqrt{|I_4(p, q)|}$$

- In the extremal under-rotating limit ($I_4 < 0$) similar behavior for $\mathscr{J}$ [arXiv:1310.7886]:

$$\mathscr{J}_{non-extr.}(\phi_0, p, q) \longrightarrow \mathscr{J}_{extr.}(p, q) = \frac{\mathscr{J}_K}{2\, m_K^2} \sqrt{|I_4(p, q)|}$$

manif. G_4-invariance, independent of ϕ_0: *"Attractor mechanism" at work for* $\mathscr{J}$.

Attractor mechanism

Static, sph.-symm. extremal solutions exhibit the attractor mechanism:

The scalars (effectively coupled to the b.h.) evolve towards values at the horizon which are fixed in terms of p^Λ, q_Λ. The near horizon geometry (and thus the entropy S) only depends on the quantized charges.

$$S_{non-extr.}(\phi_0, p, q) = \frac{A_{non-extr.}(\phi_0, p, q)}{4} \quad \xrightarrow{c\to 0} \quad S_{extr.}(p, q) = \pi\sqrt{|I_4(p, q)|}$$

- In the extremal under-rotating limit ($I_4 < 0$) similar behavior for $\mathscr{J}$ [arXiv:1310.7886]:

$$\mathscr{J}_{non-extr.}(\phi_0, p, q) \longrightarrow \mathscr{J}_{extr.}(p, q) = \frac{\mathscr{J}_K}{2\, m_K^2}\sqrt{|I_4(p, q)|}$$

  manif. G_4-invariance, independent of ϕ_0: *"Attractor mechanism" at work for* $\mathscr{J}$.

- The entropy:

$$S_{non-extr.}(\phi_0, p, q) \longrightarrow S_{extr.}(p, q) = \pi\sqrt{|I_4(p, q)|}\sqrt{1 - \mathscr{J}_K^2/m_K^4}$$

Conclusions

- Introduced a new tool for studying axisymmetric solution: $\mathfrak{g}$-valued Q_ψ

Conclusions

- Introduced a new tool for studying axisymmetric solution: $\mathfrak{g}$-valued Q_ψ
- Defined general geometric prescription for passing from the Kerr-orbit to (nilpotent) orbits describing extremal under-rotating and static orbits;

Conclusions

- Introduced a new tool for studying axisymmetric solution: $\mathfrak{g}$-valued Q_ψ
- Defined general geometric prescription for passing from the Kerr-orbit to (nilpotent) orbits describing extremal under-rotating and static orbits;
- Obtained general explicit solution for the STU model

Conclusions

- Introduced a new tool for studying axisymmetric solution: $\mathfrak{g}$-valued Q_ψ
- Defined general geometric prescription for passing from the Kerr-orbit to (nilpotent) orbits describing extremal under-rotating and static orbits;
- Obtained general explicit solution for the STU model

Work in progress:

- Attractor mechanism

Conclusions

- Introduced a new tool for studying axisymmetric solution: $\mathfrak{g}$-valued Q_ψ
- Defined general geometric prescription for passing from the Kerr-orbit to (nilpotent) orbits describing extremal under-rotating and static orbits;
- Obtained general explicit solution for the STU model

Work in progress:

- Attractor mechanism
- Black Holes in AdS geometry

Conclusions

- Introduced a new tool for studying axisymmetric solution: $\mathfrak{g}$-valued Q_ψ
- Defined general geometric prescription for passing from the Kerr-orbit to (nilpotent) orbits describing extremal under-rotating and static orbits;
- Obtained general explicit solution for the STU model

Work in progress:

- Attractor mechanism
- Black Holes in AdS geometry

Explicit form of the solution $p_0 q^i$

4-D scalars

$$z_i = \epsilon_i - i\, e^{\varphi_i} \qquad (i = 1, 2, 3)$$

Explicit form of the solution $p_0 q^i$

4-D scalars

$$\boxed{z_i = \epsilon_i - i\, e^{\varphi_i}} \qquad (i = 1, 2, 3)$$

in terms of the 3-D scalar fields

$$\epsilon_1 = \frac{2\, m\, \alpha \cos\theta\, (c_2\, s_3\, s_4\, c_5 - s_2\, c_3\, c_4\, s_5)}{\alpha^2 \cos^2\theta\, (r + 2\, m\, {s_2}^2)(r + 2\, m\, {s_5}^2)}, \qquad e^{\varphi_1} = \frac{\rho^4}{\alpha^2 \cos^2\theta\, (r + 2\, m\, {s_2}^2)(r + 2\, m\, {s_5}^2)},$$

$$\epsilon_2 = \epsilon_1\, (2 \leftrightarrow 3)\,, \qquad e^{\varphi_2} = e^{\varphi_1}\, (2 \leftrightarrow 3)\,,$$

$$\epsilon_3 = \epsilon_1\, (2 \leftrightarrow 4)\,, \qquad e^{\varphi_3} = e^{\varphi_1}\, (2 \leftrightarrow 4)$$

Explicit form of the solution $p_0 q^i$

4-D scalars

$$z_i = \epsilon_i - i\, e^{\varphi_i} \qquad (i = 1, 2, 3)$$

in terms of the 3-D scalar fields

$$\epsilon_1 = \frac{2\, m\, \alpha \cos\theta\, (c_2\, s_3\, s_4\, c_5 - s_2\, c_3\, c_4\, s_5)}{\alpha^2 \cos^2\theta\, (r + 2\, m\, s_2{}^2)(r + 2\, m\, s_5{}^2)}, \qquad e^{\varphi_1} = \frac{\rho^4}{\alpha^2 \cos^2\theta\, (r + 2\, m\, s_2{}^2)(r + 2\, m\, s_5{}^2)},$$

$$\epsilon_2 = \epsilon_1\, (2 \leftrightarrow 3)\,, \qquad e^{\varphi_2} = e^{\varphi_1}\, (2 \leftrightarrow 3)\,,$$

$$\epsilon_3 = \epsilon_1\, (2 \leftrightarrow 4)\,, \qquad e^{\varphi_3} = e^{\varphi_1}\, (2 \leftrightarrow 4)$$

with

$$c_\ell = \cosh\left(\log\sqrt{\beta_\ell}\right) = \frac{1 + \beta_\ell}{2\sqrt{\beta_\ell}}\,,$$

$$s_\ell = \sinh\left(\log\sqrt{\beta_\ell}\right) = \frac{-1 + \beta_\ell}{2\sqrt{\beta_\ell}}\,, \qquad (\ell = 2, 3, 4, 5)\,,$$

$$\rho^4 = (\alpha^2 \cos^2\theta\, (r + 2m\, s_2{}^2)(r + 2m\, s_3{}^2))\, (\alpha^2 \cos^2\theta\, (r + 2m\, s_4{}^2)(r + 2m\, s_5{}^2)) - 4\alpha^2 m^2 (c_2 c_3 s_4 s_5 - s_2 s_3 c_4 c_5)^2 \cos^2\theta\,.$$

4-D vectors

A_{φ}^{M}

$M = 1, \ldots, 8$ (symplectic index)

4-D vectors

$$A_\varphi^M$$

$M = 1, \ldots, 8$ (symplectic index)

with explicit form

$$A_\varphi^1 = -\frac{\sqrt{2}\, m\, \Delta\, \cos\theta\, c_5\, s_5}{\tilde{\Delta}}\,, \quad A_\varphi^2 = -\frac{\sqrt{2}\, m\, \alpha\, \sin^2\theta\, (c_2\, s_3\, s_4\, s_5 (2\,m - r) + r\, s_2\, c_3\, c_4\, c_5)}{\tilde{\Delta}}\,,$$

$$A_\varphi^6 = -A_\varphi^1\, (5 \leftrightarrow 2)\,, \qquad A_\varphi^3 = A_\varphi^2\, (2 \leftrightarrow 3)\,,$$

$$A_\varphi^7 = -A_\varphi^1\, (5 \leftrightarrow 3)\,, \qquad A_\varphi^4 = A_\varphi^2\, (2 \leftrightarrow 4)\,,$$

$$A_\varphi^8 = -A_\varphi^1\, (5 \leftrightarrow 4)\,, \qquad A_\varphi^5 = A_\varphi^2\, (2 \leftrightarrow 5)\,.$$

4-D vectors

$$A_{\varphi}^{M}$$

$M = 1, \ldots, 8$ (symplectic index)

with explicit form

$$A_{\varphi}^{1} = -\frac{\sqrt{2}\, m\, \Delta\, \cos\theta\, c_5\, s_5}{\tilde{\Delta}}\,, \quad A_{\varphi}^{2} = -\frac{\sqrt{2}\, m\, \alpha\, \sin^2\theta\, (c_2\, s_3\, s_4\, s_5 (2\,m - r) + r\, s_2\, c_3\, c_4\, c_5)}{\tilde{\Delta}}\,,$$

$$A_{\varphi}^{6} = -A_{\varphi}^{1}\,(5 \leftrightarrow 2)\,, \qquad A_{\varphi}^{3} = A_{\varphi}^{2}\,(2 \leftrightarrow 3)\,,$$

$$A_{\varphi}^{7} = -A_{\varphi}^{1}\,(5 \leftrightarrow 3)\,, \qquad A_{\varphi}^{4} = A_{\varphi}^{2}\,(2 \leftrightarrow 4)\,,$$

$$A_{\varphi}^{8} = -A_{\varphi}^{1}\,(5 \leftrightarrow 4)\,, \qquad A_{\varphi}^{5} = A_{\varphi}^{2}\,(2 \leftrightarrow 5)\,.$$

with

$$\Delta = (r-m)^2 - (m^2 - \alpha^2)\,,$$

$$\tilde{\Delta} = \Delta - \alpha^2 \sin^2\theta\,.$$

- ***Relevant physical quantities*** $M_{ADM}, \; \Gamma^M, \; \mathscr{J}, \; S$

- ***Relevant physical quantities*** $M_{ADM}, \ \Gamma^M, \ \mathscr{J}, \ S$

$$\boldsymbol{M_{ADM}} = \frac{1}{4}\, m \, \left(c_2^2 + c_3^2 + c_4^2 + c_5^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2\right) \ ,$$

$$\boldsymbol{\Gamma^M} = \left(\sqrt{2}\, m\, c_5\, s_5, 0, 0, 0, 0, -\sqrt{2}\, m\, c_2\, s_2, -\sqrt{2}\, m\, c_3\, s_3, -\sqrt{2}\, m\, c_4\, s_4\right) \ ,$$

$$\mathscr{J} = m\, \alpha\, (P_c - P_s) \qquad \left(\xrightarrow{extr.\ case} \quad \mathscr{J}^{\,\boldsymbol{(extr)}} \propto \sqrt{|I_4^{(extr)}|} \right)$$

$$\boldsymbol{S} = 2\pi\, m \left(m\, (P_c - P_s) + c_{ex}\, (P_c - P_s)\right) ,$$

$$\boldsymbol{I_4} = -\, 4\, p_0\, q^1\, q^2\, q^3 = \ 16\, m^4\, P_c\, P_s \, .$$

- ***Relevant physical quantities*** M_{ADM}, Γ^M, $\mathscr{J}$, S

$$\boldsymbol{M_{ADM}} = \frac{1}{4}\, m\, \left(c_2^2 + c_3^2 + c_4^2 + c_5^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2\right)\,,$$

$$\boldsymbol{\Gamma^M} = \left(\sqrt{2}\, m\, c_5\, s_5, 0, 0, 0, 0, -\sqrt{2}\, m\, c_2\, s_2, -\sqrt{2}\, m\, c_3\, s_3, -\sqrt{2}\, m\, c_4\, s_4\right)\,,$$

$$\mathscr{J} = m\, \alpha\, (P_c - P_s) \qquad \left(\xrightarrow{extr.\ case} \quad \mathscr{J}^{\,(\mathbf{extr})} \propto \sqrt{|I_4^{(extr)}|}\right)$$

$$\boldsymbol{S} = 2\pi\, m\, \Big(m\, (P_c - P_s) + c_{ex}\, (P_c - P_s)\Big)\,,$$

$$\boldsymbol{I_4} = -\,4\, p_0\, q^1\, q^2\, q^3 = \; 16\, m^4\, P_c\, P_s\,.$$

$$\left(\text{with} \quad P_c = c_2\, c_3\, c_4\, c_5\,, \quad P_s = s_2\, s_3\, s_4\, s_5\,, \quad c_{ex} = \sqrt{m^2 - \alpha^2}\right).$$