The FDR way to renormalizable and non–renormalizable QFTs

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The Four Dimensional Renormalization Philosophy

1. The **FDR** approach to QFT **defines** a **four-dimensional and UV-free loop-integration** in a way compatible with shift and gauge invariance
2. Having done this, the correct results automatically emerge once the theory is fixed in terms of physical observables by means of a **finite** renormalization relating the parameters of the Lagrangian $\mathcal{L}$ to measured quantities
3. Subtraction of UV infinities
   encoded in the definition of loop integral

R. P., **arXiv:1208.5457** (first paper)

A. M. Donati and R. P., **arXiv:1302.5668** (1-loop EW)

R. P., **arXiv:1305.0419** (effective theories)

R. P., **arXiv:1307.0705** (massless QCD)

A. M. Donati and R. P., **arXiv:1311.5500** (2-loop)

FDR is more convenient than DR because

1. four-dimensional
2. order-by-order renormalization avoided (No counterterms and $\mathcal{L}$ untouched)
3. ℓ-loop integrals are directly re-usable in $(\ell+1)$-loop calculations, with no need of further expanding in ϵ
4. infrared and collinear divergences can be dealt with within the same four-dimensional framework used to cope with the ultraviolet infinities
5. it allows a novel interpretation of non-renormalizable theories in which predictivity is restored

Outline

1. The FDR idea
2. Physical interpretation
3. Bottom-up: Tests (renormalizable QFTs)
4. Top-down: Non-renormalizable QFTs

- Take the *integrand* of a ℓ-loop function

$$J(q_1,\ldots,q_\ell) = J_{\rm INF}(q_1,\ldots,q_\ell) + J_{{\rm F},\ell}(q_1,\ldots,q_\ell)$$

- To avoid the occurrence of infrared divergences due to this separation

$$+i0 = -\mu^2$$

  and $\mu \to 0$ outside integration

- The loop *integrands* in $J_{\rm INF}(q_1,\ldots,q_\ell)$ allowed to depend on μ, **but not on physical scales** $\Rightarrow$ physics in $J_{{\rm F},\ell}(q_1,\ldots,q_\ell)$

- The FDR integral over $J(q_1,\ldots,q_\ell)$ is defined as

$$\int [d^4q_1]\ldots[d^4q_\ell]\, J(q_1,\ldots,q_\ell) \equiv \lim_{\mu\to 0}\int d^4q_1\ldots d^4q_\ell\, J_{{\rm F},\ell}(q_1,\ldots q_\ell)$$

2-loop example

$$J^{\alpha\beta}(q_1,q_2) = \frac{q_1^\alpha q_1^\beta}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}}$$

$$\bar{D}_1 = \bar{q}_1^2 - m_1^2 \qquad \bar{D}_2 = \bar{q}_2^2 - m_2^2 \qquad \bar{D}_{12} = \bar{q}_{12}^2 - m_{12}^2$$
$$q_{12} = q_1 + q_2 \qquad \bar{q}_j^2 = q_j^2 - \mu^2$$

Needed denominator expansion (*FDR defining expansion*) with

$$\frac{1}{\bar{D}_j} = \frac{1}{\bar{q}_j^2} + \frac{m_j^2}{\bar{q}_j^2 \bar{D}_j} \qquad \frac{1}{\bar{q}_{12}^2} = \frac{1}{\bar{q}_2^2} - \frac{q_1^2 + 2(q_1\cdot q_2)}{\bar{q}_2^2 \bar{q}_{12}^2}$$

$$\begin{aligned} J^{\alpha\beta}(q_1,q_2) &= q_1^\alpha q_1^\beta \left\{ \left[\frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{q}_{12}^2}\right] + \left(\frac{1}{\bar{D}_1^3} - \frac{1}{\bar{q}_1^6}\right)\left(\left[\frac{1}{\bar{q}_2^4}\right] - \frac{q_1^2 + 2(q_1\cdot q_2)}{\bar{q}_2^4 \bar{q}_{12}^2}\right) \right. \\ &\quad \left. + \frac{1}{\bar{D}_1^3 \bar{q}_2^2 \bar{D}_{12}} \left(\frac{m_2^2}{\bar{D}_2} + \frac{m_{12}^2}{\bar{q}_{12}^2}\right) \right\} \end{aligned}$$

Then

$$
\begin{aligned}
J^{\alpha\beta}_{\rm INF}(q_1,q_2) &= q_1^\alpha q_1^\beta \left\{ \left[\frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{q}_{12}^2}\right] + \left(\frac{1}{\bar{D}_1^3} - \frac{1}{\bar{q}_1^6}\right)\left[\frac{1}{\bar{q}_2^4}\right] \right\} \\
J^{\alpha\beta}_{{\rm F},2}(q_1,q_2) &= q_1^\alpha q_1^\beta \left\{ \frac{1}{\bar{D}_1^3 \bar{q}_2^2 \bar{D}_{12}} \left(\frac{m_2^2}{\bar{D}_2} + \frac{m_{12}^2}{\bar{q}_{12}^2}\right) \right. \\
&\quad \left. - \left(\frac{1}{\bar{D}_1^3} - \frac{1}{\bar{q}_1^6}\right) \frac{q_1^2 + 2(q_1\cdot q_2)}{\bar{q}_2^4 \bar{q}_{12}^2} \right\}
\end{aligned}
$$

And

$$
\int [d^4q_1][d^4q_2] \frac{q_1^\alpha q_1^\beta}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} = \lim_{\mu\to 0} \int d^4q_1 d^4q_2 \underbrace{J^{\alpha\beta}_{{\rm F},2}(q_1,q_2)}_{q_1^\alpha q_1^\beta \left\{\cdots\right\}}
$$

Formal properties of the FDR integration

i) invariance under shift of any integration variable

ii) simplifications among numerators and denominators

i) + ii) guarantee Gauge Invariance: usual manipulations hold at the integrand level (any graphical proof of Ward Identities holds)

i)

FDR integrals as finite differences of shift invariant UV divergent integrals

$$\int [d^4q_1]\dots[d^4q_\ell]\, J(\{q_i\}) = \lim_{\mu\to 0}\int d^nq_1\dots d^nq_\ell \Big(J(\{q_i\}) - J_{\rm INF}(\{q_i\})\Big)$$

r.h.s. regulated in dimensional regularization ii)

By construction, provided any q_i^2 appearing in the numerator from Feynman rules is also considered as $\bar{q}_i^2 = q_i^2 - \mu^2|_i$. For example, $(\bar{D}_1 = \bar{q}_i^2 - m_1^2)$

$$\int [d^4q_1][d^4q_2] \frac{\bar{q}_1^2}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} = \int [d^4q_1][d^4q_2] \frac{1}{\bar{D}_1^2 \bar{D}_2 \bar{D}_{12}} + m_1^2 \int [d^4q_1][d^4q_2] \frac{1}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}}$$

It works only if in front of the μ^2 term the same denominator expansion used for q_1^2 is performed

$$\int [d^4q_1][d^4q_2] \frac{\mu^2|_1}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} = \lim_{\mu \to 0} \int d^4q_1 d^4q_2 \, \mu^2 \left\{ \cdots \right\}$$

Only one μ^2 exists: $|_1$ only denotes the expansion to be performed

For consistency, irreducible tensor decomposition works as in the following example

$$\int [d^4q_1][d^4q_2] \frac{q_1^\alpha q_1^\beta}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} = \frac{g^{\alpha\beta}}{4} \int [d^4q_1][d^4q_2] \frac{q_1^2}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}}$$

$$= \left(\frac{g^{\alpha\beta}}{4} \int [d^4q_1][d^4q_2] \frac{\bar{q}_1^2 + \mu^2|_1}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} \right)$$

Here the q_1^2 is not deformed together with the denominators because it appears after tensor reduction

Dependence on μ

$$\int [d^4q_1]\dots[d^4q_\ell]\, J(\{q_i\}) = \lim_{\mu\to 0} \int d^nq_1 \dots d^nq_\ell \left(J(\{q_i\}) - J_{\rm INF}(\{q_i\}) \right)$$

1. First term in r.h.s. does not depend on μ, because $\lim_{\mu\to 0}$ can be moved inside integration
2. Any polynomially divergent integral in $J_{\rm INF}$ cannot contribute either, being proportional to positive powers of μ
3. μ dependence of the l.h.s. entirely due to powers of $\ln(\mu/\mu_R)$ generated by the logarithmically divergent subtracted integrals

a) FDR integrals depend on μ *logarithmically*

b) if all powers of $\ln(\mu/\mu_R)$ are moved to the l.h.s. $\lim_{\mu\to 0}$ formally taken by trading $\ln(\mu)$ for $\ln(\mu_r)$

FDR integrals *do not depend on any cut off* but only on the renormalization scale μ_R

- 1-loop example (with cutoff regulator, DR gives the same $\ln$)

$$J(q) = \frac{1}{(\bar{q}^2 - m_0^2)((q+p)^2 - m_1^2 - \mu^2)} = \left[\frac{1}{\bar{q}^4}\right] + J_{\mathrm{F},1}(q)$$

$$\lim_{\mu\to 0} \int_\Lambda d^4q \left[\frac{1}{\bar{q}^4}\right] = \lim_{\mu\to 0} 2i\pi^2 \left(\int_0^{\mu_R} dq + \int_{\mu_R}^{\Lambda} dq\right) \frac{q^3}{(q^2+\mu^2)^2}$$

$$\Uparrow$$

$$-i\pi^2\left(1 + \ln\frac{\mu^2}{\mu_R^2}\right)$$

- μ_R can also be thought as an arbitrary separation scale from the UV regime

$$\int [d^4q]\, J(q) = -i\pi^2 \int_0^1 dx\, \ln\left(\frac{m_0^2 x + m_1^2(1-x) - p^2 x(1-x)}{\mu_R^2}\right)$$

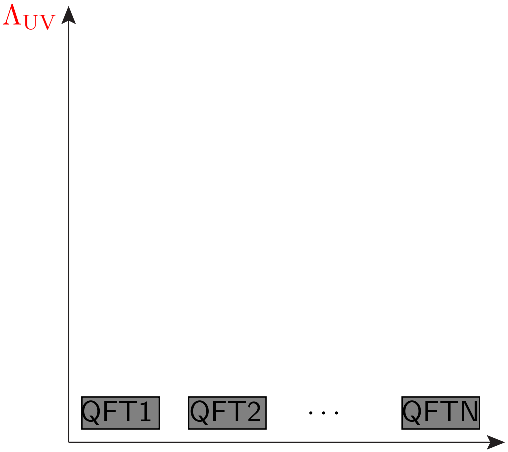
is cutoff independent

In summary, the symbol $\int [d^4q]$ means

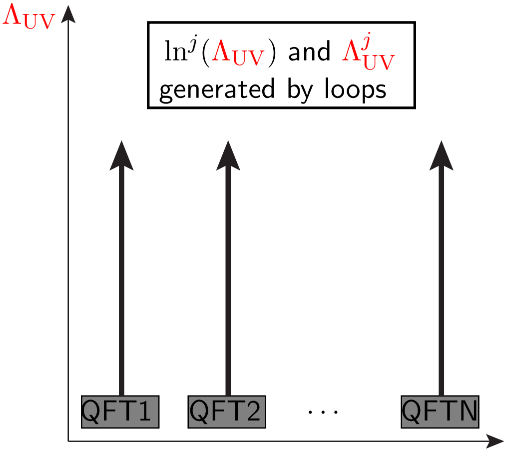
1. Use partial fraction to move all divergences in vacuum integrands treating $\bar{q}^2$ globally
2. Drop all divergent vacuum terms from the integrand
3. Integrate over d^4q
4. Take $\mu \to 0$ until a logarithmic dependence on μ is reached
5. Compute the result in $\mu = \mu_R$ ($\mu \to \mu_R$ in $[d^4q]$ definition)

Physical Interpretation

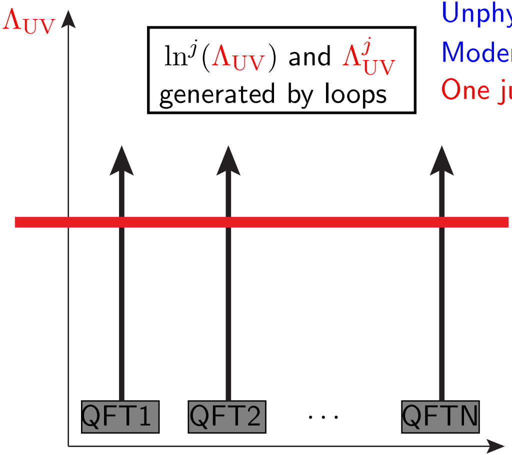
QFTs vs UV cutoff (I)



QFTs vs UV cutoff (II)



QFTs vs UV cutoff (III)



Unphysical if $\Lambda_{\rm UV} \to \infty$

Modern version of *aether*?

One just **ignore** them

What is the cost of throwing away infinities?

- No cost for polynomially divergent infinities (decoupling)
- Only logarithmic infinities influence the physical spectrum (μ_R pops up in physical observables when separating them)
- Physics at $\Lambda_{\rm UV}$ scale manifests itself only logarithmically at lower energies

$$\ln(M_{\rm Higgs}/{\rm GeV}) \sim 5$$
$$\ln(M_{\rm Plank}/{\rm GeV}) \sim 44$$

Hierarchy problem with no BSM particles?

Classification

independent of the number of external legs!

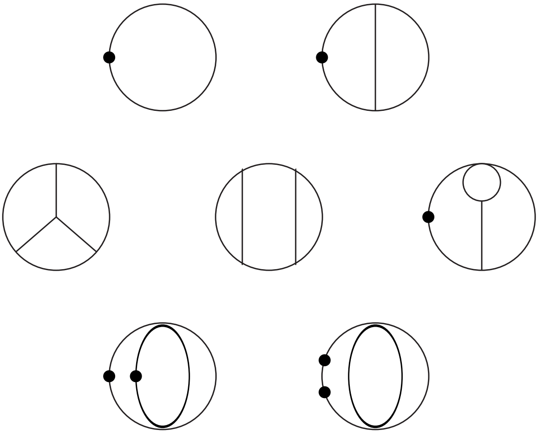
1. $\left[\frac{1}{\bar{q}^4}\right]$ is the only possible **subtracted** 1-loop log divergent **vacuum integrand**

2. At 2 loops $\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_{12}^2}\right]$

3. Five additional log divergent vacuum integrands at 3 loops

$$\left[\frac{1}{\bar{q}_1^2\bar{q}_2^2\bar{q}_3^2\bar{q}_{12}^2\bar{q}_{13}^2((q_2-q_3)^2-\mu^2)}\right] \quad \left[\frac{1}{\bar{q}_1^2\bar{q}_3^2\bar{q}_2^4\bar{q}_{12}^2\bar{q}_{23}^2}\right]$$
$$\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_3^2\bar{q}_{12}^2\bar{q}_{123}^2}\right] \quad \left[\frac{1}{\bar{q}_1^4\bar{q}_2^4\bar{q}_3^2\bar{q}_{123}^2}\right] \quad \left[\frac{1}{\bar{q}_1^6\bar{q}_2^2\bar{q}_3^2\bar{q}_{123}^2}\right]$$

Corresponding 1-, 2- and 3-loop log topologies



Divergent tensor integrands are reducible to combinations of those topologies plus finite constants

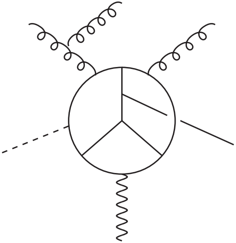
- Infinities are directly put into the vacuum, rather than in the parameter of the Lagrangian

Order by order **vacuum redefinition** dubbed
Topological Renormalization

- The vacuum back-reacts by trading the cutoff μ for μ_R, which, however, drops after a **Finite Renormalization**

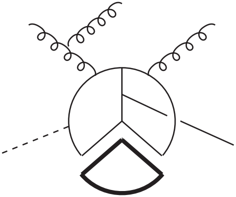
The vacuum is by far more efficient in accommodating infinities than the Lagrangian

Vacuum inside loops (pictorially)



(a)

(b)



(c)

(b) *and* (c) *are* **Vacuum Bubbles** *generated by the generic diagram* (a) *contributing to the interaction*

Finite Renormalization

- Only *finite* $\ln^j(\mu_R)$ remain

  (generated when subtracting log divergent vacuum integs)

- Reabsorbed in physical parameters when fixing the theory:

  NO order-by-order renormalization necessary!

  NO counterterms!

- At 1-loop equivalent to *Dimensional Reduction* in the $\overline{\mathrm{MS}}$ scheme

Consider the Lagrangian of a renormalizable QFT dependent on m parameters p_i $(i = 1 : m)$

$$\mathcal{L}(p_1, \ldots, p_m)$$

Before an observable $\mathcal{O}^{\rm TH}_{m+1}$ can be calculated, p_i must be fixed by means of m measurements

$$\mathcal{O}^{\rm TH}_i(p_1, \ldots, p_m) = \mathcal{O}^{\rm EXP}_i$$

which determine p_i in terms of observables $\mathcal{O}^{\rm EXP}_i$ and corrections computed at the loop level ℓ one is working:

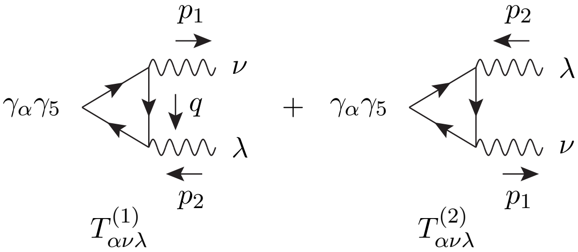
$$p_i = p_i^{\ell-loop}(\mathcal{O}^{\rm EXP}_1, \ldots, \mathcal{O}^{\rm EXP}_m) \equiv \bar{p}_i$$

Then

$$\mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m) \quad \text{with} \quad \frac{\partial\, \mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m)}{\partial \mu_R} = 0$$

is a finite prediction of the QFT

TEST0: The ABJ anomaly

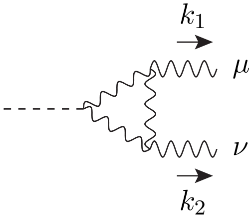


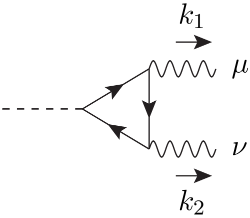
$$p^\alpha T_{\alpha\nu\lambda} = -i\frac{e^2}{4\pi^4}\mathrm{Tr}[\gamma_5 \not{p}_2\gamma_\lambda\gamma_\nu\not{p}_1]\int[d^4q]\,\mu^2\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}$$

$$p^\alpha T_{\alpha\nu\lambda} = \frac{e^2}{8\pi^2}\mathrm{Tr}[\gamma_5\not{p}_2\gamma_\lambda\gamma_\nu\not{p}_1]$$

TEST1: $H \to \gamma(k_1^\mu)\,\gamma(k_2^\nu)$ (generic R_ξ gauge)

Alice M. Donati and R.P., arXiv:1302.5668 [hep-ph]





$\widetilde{\mathcal{M}}_W(\beta)$ | $\widetilde{\mathcal{M}}_f(\eta)$

26 diagrams | 2 diagrams

$$\beta = \frac{4\,M_W^2}{M_H^2} \qquad \eta = \frac{4\,m_f^2}{M_H^2}$$

$$\begin{aligned}
\mathcal{M}^{\mu\nu}(\beta,\eta) &= \left(\widetilde{\mathcal{M}}_W(\beta) + \sum_f N_c Q_f^2\, \widetilde{\mathcal{M}}_f(\eta)\right) T^{\mu\nu} \\
T^{\mu\nu} &= k_1^\nu k_2^\mu - (k_1\cdot k_2)\, g^{\mu\nu} \\
\widetilde{\mathcal{M}}_W(\beta) &= \frac{i\, e^3}{(4\pi)^2 s_W M_W}\left[2 + 3\beta + 3\beta(2-\beta) f(\beta)\right] \\
\widetilde{\mathcal{M}}_f(\eta) &= \frac{-i\, e^3}{(4\pi)^2 s_W M_W}\, 2\eta \left[1 + (1-\eta) f(\eta)\right]
\end{aligned}$$

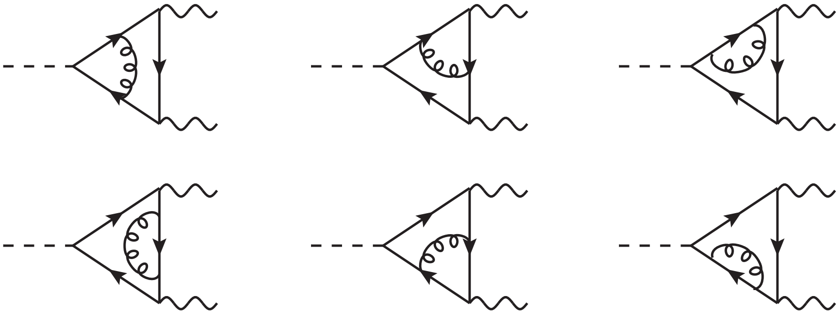
$$f(x) = -\frac{1}{4}\ln^2\left(\frac{1+\sqrt{1-x+i\varepsilon}}{-1+\sqrt{1-x+i\varepsilon}}\right)$$

NOTE:

$$\int [d^4q]\frac{\bar{q}^2 g_{\mu\nu} - 4 q_\mu q_\nu}{(\bar{q}^2-M^2)^3} = \int [d^4q]\frac{-\mu^2}{(\bar{q}^2-M^2)^3} g_{\mu\nu} = -\frac{i\pi^2}{2} g_{\mu\nu}$$

TEST2: gluonic corrections to $\Gamma(\mathbf{H} \to \gamma\gamma)$

Alice M. Donati and R.P., arXiv:1311.5500



12 diagrams

Important facts

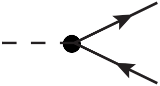
- $$\mathcal{M}^{(2-loop)} = \underbrace{\mathcal{M}^{(1-loop)}}_{\frac{i\alpha}{3\pi v}} \left(1 - \frac{\alpha_S}{\pi}\right) \quad (\text{when } m_{\text{top}} \to \infty)$$

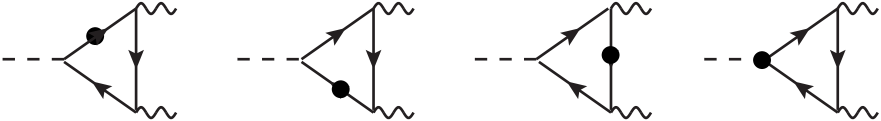
- No integral by integral correspondence between DR and FDR and results coincide only at the very end

- If $m_{\text{top}} \to \infty$ no renormalization (of sub-divergences) needed in FDR

- This understood because FDR avoids spurious $\frac{\epsilon}{\epsilon}$ terms from the beginning

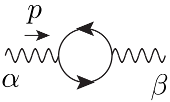
- In DR no renormalization would give a wrong result

 $= -i\,\delta m$  $= -i\,\dfrac{\delta m}{v}$



$$
= \begin{cases} 0 \times \delta m & \text{in FDR} \quad \text{with } \delta m \propto \ln \mu_r \\ \mathcal{O}(\epsilon) \times \delta m & \text{in DR} \quad \text{with } \delta m \propto 1/\epsilon \end{cases}
$$

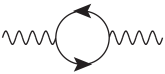
Simple QED example



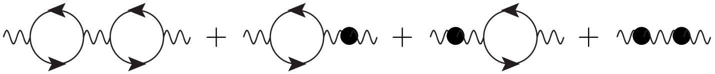
$$= i\,T_{\alpha\beta}\,\Pi(p^2) \qquad T_{\alpha\beta} = g_{\alpha\beta}p^2 - p_\alpha p_\beta$$

$$\Pi(p^2) = \frac{1}{\epsilon}\,\Pi_{-1} + \Pi_0 + \epsilon\,\Pi_1$$

In DR, the corresponding two-loop computation requires the addition of one-loop counterterms such that $\Pi_{-1}\Pi_1$ is avoided

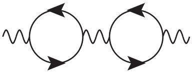
 $+$  $= i\,T_{\alpha\beta}\,\Pi_0 + \mathcal{O}(\epsilon)$

Therefore, at two loops, up to terms $\mathcal{O}(\epsilon)$



$$= iT_{\alpha\beta}\,\Pi_0^2$$

In FDR, the product of two one-loop diagrams **is simply the product of the two finite parts**, so that one directly obtains



$$= i\,T_{\alpha\beta}\,\Pi^2_{\rm FDR}(p^2)$$

with $\Pi_{\rm FDR}(p^2) = \Pi_0$

- The previous example also shows that ℓ-**loop integrals are directly re-usable in (ℓ+1)-loop calculations**

- For instance, the two-loop factorizable FDR integral

$$\int \frac{[d^4 q_1]}{(\bar{q}_1^2 - m_1^2)^\alpha} \times \int \frac{[d^4 q_2]}{(\bar{q}_2^2 - m_2^2)^\beta}$$

  is simply the product of two one-loop FDR integrals

- That **is not** the case in DR, where further expanding in ϵ is required

TEST3: $\Gamma(\mathbf{H} \to \mathbf{gg})$

R. P., arXiv:1307.0705 [hep-ph]

- FDR is used to compute the NLO QCD corrections to $\mathbf{H} \to \mathbf{gg}$ in the large top mass limit
- The well known fully inclusive result

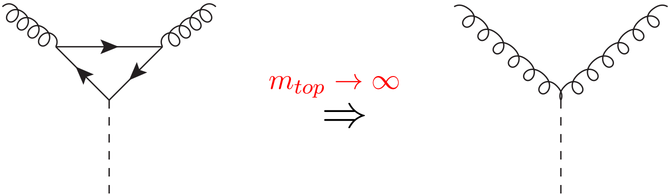
$$\Gamma(\mathbf{H} \to \mathbf{gg}) = \Gamma^{(0)}(\alpha_S(M_H^2)) \left[1 + \frac{95}{4} \frac{\alpha_S}{\pi} \right]$$

is re-derived, where

$$\Gamma^{(0)}(\alpha_S(M_H^2)) = \frac{G_F \alpha_S^2(M_H^2)}{36\sqrt{2}\pi^3} M_H^3$$

- **UV**, **IR** and **CL** divergences, besides α_S **renormalization**

The Model

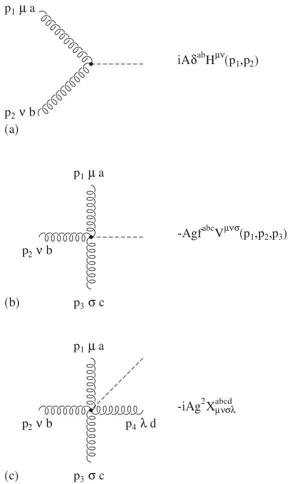


$$\mathcal{L}_{\text{eff}} \;=\; -\frac{1}{4} A H G^{a}_{\mu\nu} G^{a,\mu\nu}$$

$$A \;=\; \frac{\alpha_S}{3\pi v}\left(1 + \frac{11}{4}\frac{\alpha_S}{\pi}\right)$$

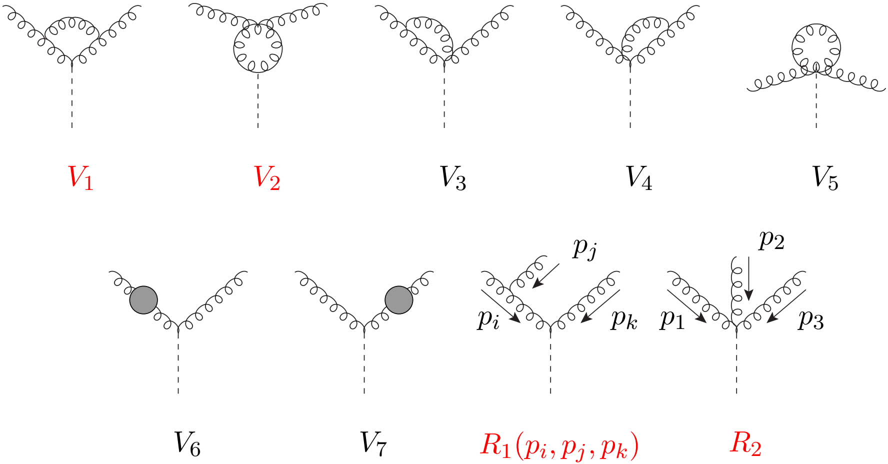
where v is the vacuum expectation value, $v^2 = (G_F\sqrt{2})^{-1}$

Generated Feynman Rules



V and X as in QCD and $\mathrm{H}^{\mu\nu}(\mathrm{p}_1, \mathrm{p}_2) = \mathrm{g}^{\mu\nu}\mathrm{p}_1 \cdot \mathrm{p}_2 - \mathrm{p}_1^{\nu}\mathrm{p}_2^{\mu}$

Contributing Diagrams



FDR vs CL/UV Virtual Infinities

- CL/UV singularities also regulated by μ^2, e.g.

$$B^{\mathrm{FDR}}(p^2=0,0,0) = \int [d^4q] \frac{1}{\bar{q}^2((q+p)^2-\mu^2)} = \mathbf{0}!$$

- **Due to a cancellation between** CL **and** UV **regulators**

$$B^{\mathrm{FDR}}(p^2,0,0) = -i\pi^2 \lim_{\mu\to 0} \int_0^1 dx \, \left[\ln(\mu^2 - p^2 x(1-x)) - \ln(\mu^2)\right]$$

- **Should be matched in the treatment of the Reals**

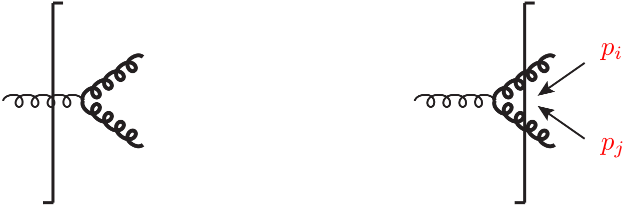
The Virtual Part

- Overlapping CL/IR infinities **also regulated by μ^2**. If $\bar{D}_i = (q+p_i)^2 - \mu^2$ with $p_i^2 = 0$:

$$
\begin{aligned}
C(s) &= \int [d^4q] \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2} = \lim_{\mu \to 0} \int d^4q \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2} \\
&= \frac{i\pi^2}{s} \left[\frac{\ln^2(\mu_0) - \pi^2}{2} + i\,\pi \ln(\mu_0) \right] \\
s &= M_H^2 = -2(p_1 \cdot p_2) \quad \text{with} \quad (\mu_0 = \mu^2/s)
\end{aligned}
$$

$$
\Gamma_V(\mathbf{H} \to \mathbf{gg}) = -3 \frac{\alpha_S}{\pi} \Gamma^{(0)}(\alpha_S)\, M_H^2\, \mathcal{R}e \left[\frac{C(M_H^2)}{i\pi^2} \right]
$$

The Real Part (off-shell momenta)



$$\frac{1}{2(p_i \cdot p_j)} \to \frac{1}{(p_i + p_j)^2} = \frac{1}{s_{ij}} \quad \text{with } p^2_{i,j,k} = \mu^2 \to 0 \ (\mu\text{−massive PS})$$

- The matrix element squared reads (diagrams R_1 and R_2)

$$
\begin{aligned}
|M|^2 \quad = \quad & 192\,\pi\alpha_S A^2 \Bigg[\frac{s_{23}^3}{s_{12}s_{13}} + \frac{s_{13}^3}{s_{12}s_{23}} + \frac{s_{12}^3}{s_{13}s_{23}} + \frac{2(s_{13}^2+s_{23}^2)+3s_{13}s_{23}}{s_{12}} \\
& + \frac{2(s_{12}^2+s_{23}^2)+3s_{12}s_{23}}{s_{13}} + \frac{2(s_{12}^2+s_{13}^2)+3s_{12}s_{13}}{s_{23}} \\
& + 6(s_{12}+s_{13}+s_{23}) \Bigg]
\end{aligned}
$$

- To be integrated over the μ-massive 3-body PS

$$\int d\Phi_3 = \frac{\pi^2}{4s} \int ds_{12}ds_{13}ds_{23}\,\delta(s - s_{12} - s_{13} - s_{23} + 3\mu^2)$$

- $\frac{1}{s_{ij}s_{jk}}$ generate $\ln^2(\mu^2)$ terms of IR/CL origin

  $\frac{1}{s_{ij}}$ collinear $\ln(\mu^2)$s

- By introducing the dimensionless variables $(x+y+z=1)$

$$x = \frac{s_{12}}{s} - \mu_0\,, \quad y = \frac{s_{13}}{s} - \mu_0\,, \quad z = \frac{s_{23}}{s} - \mu_0$$

$$I(s) = \int_R dx dy \, \frac{1}{(x+\mu_0)(y+\mu_0)}\,, \quad J_p(s) = \int_R dx dy \, \frac{x^p}{(y+\mu_0)}$$

- Then $(\mu_0 = \mu^2/s)$

$$\begin{aligned} I(s) &\sim \frac{\ln^2(\mu_0) - \pi^2}{2} \\ J_p(s) &\sim -\frac{1}{p+1}\ln(\mu_0) - \frac{1}{p+1}\left[\frac{1}{p+1} + 2\sum_{n=1}^{p+1}\frac{1}{n}\right] \end{aligned}$$

- Finally

$$\Gamma_R(\mathbf{H} \to \mathbf{ggg}) = 3\frac{\alpha_S}{\pi}\,\Gamma^{(0)}(\alpha_S) \times \left[\frac{1}{4} + I(M_H^2) - \frac{3}{2}J_0(M_H^2) - J_2(M_H^2)\right]$$

and

$$\begin{aligned}\Gamma(\mathbf{H} \to \mathbf{gg}) &= \Gamma_V(\mathbf{H} \to \mathbf{gg}) + \Gamma_R(\mathbf{H} \to \mathbf{ggg}) \\ &= \Gamma^{(0)}(\alpha_S)\left[1 + \frac{\alpha_S}{\pi}\left(\frac{95}{4} - \frac{11}{2}\ln\frac{M_H^2}{\mu^2}\right)\right]\end{aligned}$$

α_S Renormalization

- The residual μ^2 is a universal dependence on the renormalization scale $(\mu = \mu_R)$
- $\ln(\mu_R^2)$ can be reabsorbed in the gluonic running of the strong coupling constant **(Finite Renormalization)**

$$\begin{aligned} \Gamma^{(0)}(\alpha_S) &\to \Gamma^{(0)}(\alpha_S(\mu_R^2)) \\ \alpha_S(M_H^2) &= \frac{\alpha_S(\mu_R^2)}{1 + \frac{\alpha_S}{2\pi}\frac{11}{2}\ln\frac{M_H^2}{\mu_R^2}} \end{aligned}$$

$$\Gamma(\mathbf{H} \to \mathbf{gg}) = \Gamma^{(0)}(\alpha_S(M_H^2))\left[1 + \frac{95}{4}\frac{\alpha_S}{\pi}\right]$$

quod erat demostrandum

Non-renormalizable QFTs

Extending the FDR framework to a non-renormalizable QFTs described by a Lagrangian $\mathcal{L}_{NR}$:

1. $\mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m, \ln(\mu_R))$
2. Combinations of observables in which μ_R disappears unambiguously predicted by $\mathcal{L}_{NR}$. E. g. (at one loop)

$$
\begin{aligned}
\mathcal{O}^{\rm TH}_{m+1} &= \alpha \ln(\mu_R) + k_1 \\
\mathcal{O}^{\rm TH}_{m+2} &= \beta \ln(\mu_R) + k_2 \\
\mathcal{O}^{\rm TH}_{\rm Predictable} &= \frac{\mathcal{O}^{\rm TH}_{m+1}}{\alpha} - \frac{\mathcal{O}^{\rm TH}_{m+1}}{\beta} = \frac{k_1}{\alpha} - \frac{k_2}{\beta}
\end{aligned}
$$

3. In principle *just one* additional measurement needed to fix μ_R, by solving

$$\mathcal{O}^{\mathrm{TH}}_{m+2}(\bar{p}_1, \ldots, \bar{p}_m, \ln(\mu'_R)) = \mathcal{O}^{\mathrm{EXP}}_{m+2}$$

   and setting $\mu_R = \mu'_R$ in $\mathcal{O}^{\mathrm{TH}}_{m+1}$

4. **Predictivity** restored in the infinite loop limit

Important facts

1. It is crucial that, in FDR, the original cut-off $\mu \to 0$ is traded with an adjustable scale μ_R

2. One has to assume that the solution for μ'_R still allows a perturbative treatment, i.e.

$$|g^2 \ln \mu'_R| < 1$$

where g is the coupling constant of the theory

3. Strategy NOT verified in practice: more investigation needed

4. Meaning of the extra measurement: disentangling the effects of the unknown UV completion of $\mathcal{L}_{NR}$ - parametrized with a logarithmic dependence on μ_R - from the physical spectrum

Conclusions

1. Based on the FDR classification of the UV infinities a new interpretation of the renormalization procedure is possible
2. One subtracts the divergences directly at the level of the *integrand* (order by order re-definition of the vacuum)
3. Results of renormalizable QFTs reproduced (but only **finite** and **global** renormalization left, with $\mathcal{L}$ untouched)
4. It is postulated that in non-renormalizable QFTs ONE additional measurement can fix the theory, which becomes predictive *without modifying the original Lagrangian*
5. Focus moved from occurrence of UV infinities to consistency of the QFT at hand (does it reproduce data?)
6. Working in four dimensions enhances potential of numerical approaches (in progress)

Thank you!

Backup slides

"Gauge invariance implies a tight interplay between the numerator of an integrand and its denominator. Changing either of the two will generally destroy gauge invariance."

Veltman (1974)

Shift invariance (one-loop example)

Given

$$\begin{aligned}\bar{D} &= q^2 - M^2 - \mu^2 \\ \bar{D}_p &= (q+p)^2 - M^2 - \mu^2\end{aligned}$$

and

$$\begin{aligned}I^{(0)} &= \int [d^4q] \frac{1}{\bar{D}^2}\,, \qquad I_p^{(0)} = \int [d^4q] \frac{1}{\bar{D}_p^2} \\ I^{(2)} &= \int [d^4q] \frac{1}{\bar{D}}\,, \qquad I_p^{(2)} = \int [d^4q] \frac{1}{\bar{D}_p}\end{aligned}$$

I prove that

$$I^{(0)} = I_p^{(0)} \quad \text{and} \quad I^{(2)} = I_p^{(2)}$$

$$I^{(0)} = I_p^{(0)}$$

From the FDR defining expansions one obtains

$$\frac{1}{\bar{D}^2} = \left[\frac{1}{\bar{q}^4}\right] + J_F^{(0)}$$
$$\frac{1}{\bar{D}_p^2} = \left[\frac{1}{\bar{q}^4}\right] + J_{F,p}^{(0)}$$

Then

$$I^{(0)} = \lim_{\mu\to 0}\int d^n q \left(\frac{1}{\bar{D}^2} - \frac{1}{\bar{q}^4}\right) = \lim_{\mu\to 0}\int d^n q \left(\frac{1}{\bar{D}_p^2} - \frac{1}{\bar{q}^4}\right) = I_p^{(0)}$$

$$I^{(2)} = I_p^{(2)}$$

From the FDR defining expansions one obtains

$$\begin{aligned}
\frac{1}{\bar{D}} &= \left[\frac{1}{\bar{q}^2}\right] + M^2\left[\frac{1}{\bar{q}^4}\right] + J_F^{(2)} \\
\frac{1}{\bar{D}_p} &= \left[\frac{1}{\bar{q}^2}\right] + (M^2 - p^2)\left[\frac{1}{\bar{q}^4}\right] - 2p^\alpha\left[\frac{q_\alpha}{\bar{q}^4}\right] + 4p^\alpha p^\beta\left[\frac{q_\alpha q_\beta}{\bar{q}^6}\right] + J_{F,p}^{(2)}
\end{aligned}$$

Then

$$I^{(2)} = \lim_{\mu\to 0}\int d^n q\left(\frac{1}{\bar{D}} - \frac{1}{\bar{q}^2} - \frac{M^2}{\bar{q}^4}\right)$$

and

$$I_p^{(2)} = I^{(2)} + \underbrace{\int d^n q\left(\frac{p^2}{\bar{q}^4} + 2\frac{(q\cdot p)}{\bar{q}^4} - 4\frac{(q\cdot p)^2}{\bar{q}^6}\right)}_{=0}$$

This is because

$$\int d^n q \frac{1}{q^2-\mu^2} = \int d^n q \frac{1}{(q+p)^2-\mu^2} =$$

$$\int d^n q \frac{1}{q^2-\mu^2}\left[1 - \Big(\underbrace{\frac{p^2+2(q\cdot p)}{\bar{q}^2} - 4\frac{(q\cdot p)^2}{\bar{q}^4}}_{\propto\ p^2\ \text{when integrated}}\Big) + \mathcal{O}(p^3)\right]$$

Then

$$\int d^n q \left(\frac{p^2}{\bar{q}^4} + 2\frac{(q\cdot p)}{\bar{q}^4} - 4\frac{(q\cdot p)^2}{\bar{q}^6}\right) = 0$$

which can also be tested by a direct computation

Naive treatment of scaleless integrals in DR $(n = 4 + \epsilon)$

$$B^{\mathrm{DR}}(p^2,0,0) = \int d^n q \frac{1}{q^2 (q+p)^2} \qquad (p^2 = 0)$$

$$\begin{aligned} \frac{1}{(q+p)^2} &= \frac{1}{q^2 - M^2} - \left(\frac{1}{q^2 - M^2} - \frac{1}{(q+p)^2} \right) \\ &= \frac{1}{q^2 - M^2} - \frac{M^2 + 2(q \cdot p)}{(q^2 - M^2)(q+p)^2} \end{aligned}$$

$$B^{\mathrm{DR}}(p^2,0,0) = \underbrace{\int d^n q \frac{1}{q^2 (q^2 - M^2)}}_{\text{defined if } \epsilon<0} - \underbrace{\int d^n q \frac{M^2 + 2(q \cdot p)}{q^2 (q^2 - M^2)(q+p)^2}}_{\text{defined if } \epsilon>0}$$

They cancel but do they define $B^{\mathrm{DR}}(p^2,0,0)$?

(no value of ϵ exists where they are defined simultaneously)