Alternative gravity model with a non-minimal coupling between matter and curvature

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Action functional

$$S = \int \left[\frac{1}{2} f_1(R) + [1 + f_2(R)] \mathcal{L}_m \right] \sqrt{-g} d^4 x \qquad (1)$$

- GR: $f_1(R) = 2\kappa R$, $f_2(R) = 0$, $\kappa = c^4/16\pi G$

Modified Einstein field equations

$$(F_1 + 2F_2 \mathcal{L}_m) R_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} = \Delta_{\mu\nu} (F_1 + 2F_2 \mathcal{L}_m) + (1 + f_2) T_{\mu\nu} \qquad (2)$$

O. Bertolami, C. Boehmer, T. Harko and F. Lobo (2007)

- $\Delta_{\mu\nu} = \nabla_\mu \nabla_\nu - g_{\mu\nu} \Box$, $F_i(R) \equiv f_i'(R)$
- Energy-momentum tensor: $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}}$

Trace of Einstein field eqs.

$$\left(F_1 + 2F_2\mathcal{L}_m\right)R - 2f_1 = -3\Box\left(F_1 + 2F_2\mathcal{L}_m\right) + \left(1 + f_2\right)T \quad (3)$$

- Differential, not algebraic equation
- Bianchi identities, $\nabla^\mu G_{\mu\nu} = 0$ imply

Non-(covariant) conservation law

$$\nabla^\mu T_{\mu\nu} = \frac{F_2}{1 + f_2}\left(g_{\mu\nu}\mathcal{L}_m - T_{\mu\nu}\right)\nabla^\mu R \quad (4)$$

Koivisto (2006),O. Bertolami, C. Boehmer, T. Harko and F. Lobo (2007)

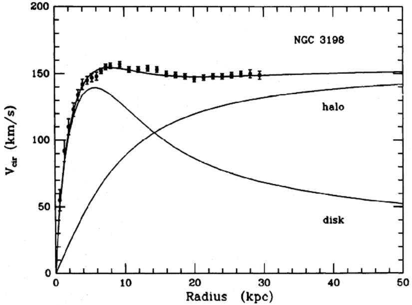
- Analogy with scalar fields for non-trivial $f_1(R), f_2(R)$
- Energy exchange matter $\leftrightarrow$ scalar fields

O. Bertolami and JP (2008)

- Essential (cannot be gauged away for general $\mathcal{L}$)

T. P. Sotiriou and V. Faraoni (2008)

Motivation



- Galactic rotation puzzle
  - Missing matter to account rotation velocity of stars
  - $\rightarrow$ Dark matter!
  - Or "Dark gravity"?
    - Maybe a non-minimal coupling of matter with geometry?

O. Bertolami, JP (2010)

- Cluster dark matter

O. Bertolami, P. Frazão, JP (2012)

Mechanism

- Trivial $f_1(R) = 2\kappa R$ and power-law $f_2(R) = \left(\frac{R}{R_n}\right)^n$
- Dark matter dominates at large distances
  - low density $\rightarrow$ low $R \rightarrow n < 0$
- Visible matter content: dust
  - Perfect fluid with $p = 0$, $T_{\mu\nu} = \rho U_{\mu\nu} U_\nu$
  - $\mathcal{L} = -\rho$ bare Lagrangean density

O. Bertolami, F. S. N. Lobo and JP (2008)

Trace of modified Einstein eqs.

$$R = \frac{1}{2\kappa}\left[1 + (1-2n)\left(\frac{R}{R_n}\right)^n\right]\rho - \frac{3n}{\kappa}\Box\left[\left(\frac{R}{R_n}\right)^n \frac{\rho}{R}\right] \quad (5)$$

Neglect linear ρ term $\rightarrow$ "static" solution

$$R = R_n\left[(1-2n)\frac{\rho}{\rho_0}\right]^{1/(1-n)} \rightarrow \quad (6)$$

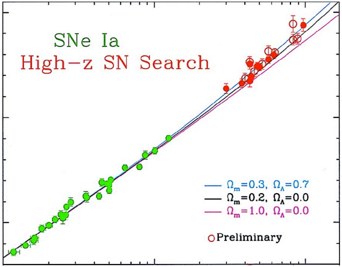
$$\rho_{DM} \sim \rho^{1/(1-n)}$$

- Interpretation
  - At large distances, $R \propto \rho_{dm} \propto \rho^{1/(1-n)}$!
  - Tully-Fisher law: $M \sim M_{dm} \propto v^{2(1-n)}$!
- Substitute into modified Einstein eqs. $\rightarrow$ mimicked DM...
  - is dragged by visible matter (same four-velocity U^μ)
  - has non-vanishing pressure $p_{dm} = \frac{n}{1-4n}2\kappa R$
  - Since $n < 0$, EOS parameter $\omega = n/(1-n) < 0$
    - Hint at unification with dark energy?
    - Dark matter matches cosmological background $\sim 100\ kpc$!

Motivation

- Accelerated expansion of the Universe
  - Missing energy with negative pressure $\rightarrow$ Dark energy!
  - Or "Dark gravity"?
    - Multi-scalar-tensor analogy $\rightarrow$ two-field quintessence

M. C. Bento, O. Bertolami and N. M. C. Santos (2002)



- Similar to galactic rotation puzzle:
  - spherically symmetric $g_{\mu\nu}(r) \leftrightarrow$ FRW $g_{\mu\nu}(t)$
  - GR at small distances $\leftrightarrow$ earlier times
  - Dark Universe at large distances $\leftrightarrow$ late times $\rightarrow n < 0$

O. Bertolami, P. Frazão and J. Páramos (2010)

Mechanism

- Flat $k = 0$, isotropic and homogeneous Universe

FRW metric

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{\sqrt{1 - kr^2}} + d\Omega^2 \right) \quad (7)$$

- Dust filled Universe, $T^{\mu\nu} = \rho U^\mu U^\nu = (\rho, 0, 0, 0)$
- Constant deceleration parameter $\rightarrow a(t) = a_0 (t/t_0)^\beta$
  - Expanding Universe $\rightarrow \beta > 0$, accelerating $\beta > 1$

Quantities

$$\begin{aligned} H &\equiv \frac{\dot{a}}{a} = \frac{\beta}{t} \\ R &\equiv 6 \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] = \frac{6\beta}{t^2} (2\beta - 1) \\ q &\equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1}{\beta} - 1 \end{aligned} \quad (8)$$

Covariant energy conservation

- Energy is conserved!

Non-(covariant) conservation law, $\nu = 0$

$$\nabla^\mu T_{\mu 0} = \frac{F_2}{1+f_2}\left(g_{\mu 0}\mathcal{L}_m - T_{\mu 0}\right)\nabla^\mu R = -\frac{F_2}{1+f_2}\left(\mathcal{L}_m + T_{00}\right)\dot{R} = 0 \quad (9)$$

Matter density

$$\dot{\rho} + 3H\rho = 0 \rightarrow \rho(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^3 \quad (10)$$

Modified dynamics

Modified Friedmann and Raychaudhuri Eqs.

$$H^2 + \frac{k}{a^2} = \frac{1}{6\kappa}(\rho_m + \rho_c) \quad (11)$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{12\kappa}\left[\rho_m + \rho_c + 3(p_m + p_c)\right]$$

- Curvature pressure p_c, density ρ_c depend on $f_1(R), f_2(R)$

Modified dynamics

- Use trivial $f_1(R) = 2\kappa R$ and power-law $f_2(R) = (R/R_n)^n$

Curvature density and pressure

$$\begin{aligned} \rho_c &\approx -6\rho_0\beta \frac{1-2\beta+n(5\beta+2n-3)}{\left(\frac{t}{t_0}\right)^{3\beta}\left(\frac{t}{t_n}\right)^{2n}\left[6\beta(2\beta-1)\right]^{1-n}} \\ p_c &\approx -2\rho_0 n \frac{2+4n^2-\beta(2+3\beta)+n(8\beta-6)}{\left(\frac{t}{t_0}\right)^{3\beta}\left(\frac{t}{t_n}\right)^{2n}\left[6\beta(2\beta-1)\right]^{1-n}} \end{aligned} \tag{12}$$

- $t_n \equiv R_n^{-1/2}$ marks onset of accelerated phase
- Solve Friedmann Eq. $\rightarrow \beta(n) = 2(1-n)/3$

- Deceleration parameter

$$n = 1 - \frac{3}{2(1+q)} \quad , \quad q = -1 + \frac{3}{2(1-n)} \tag{13}$$

- EOS parameter $p_c = \omega \rho_c$

$$\omega = \frac{n}{1-n} \tag{14}$$

  - Same as in DM scenario
  - $n \rightarrow \infty$, $\omega \rightarrow -1$ and $q \rightarrow -1$
    - Cosmological constant Λ
    - Unobtainable with $f_2(R)$: matter term is not constant!

Fitting $q(z)$ profiles

- Numerically solve Friedmann Eq. for fixed n
- Fit t_n to available $q(z)$ curve

Y. G. Gong and A. Wang (2007)

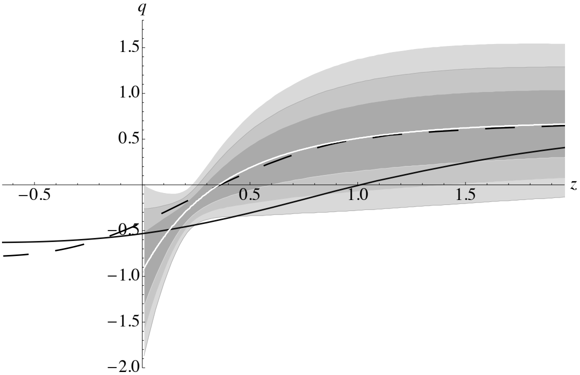


Figure: $q(z)$ for $n = -4$, $t_2 = t_0/4$ (full) and $n = -10$, $t_2 = t_0/2$ (dashed); 1σ, 2σ and 3σ regions shaded, best fit (white)

Equivalence with a multiscalar-tensor theory

- $f(R)$ theories $\leftrightarrow$ Jordan-Brans-Dicke theory with $\omega = 0$

P. Teyssandier and P. Tourranc (1983), H. Schmidt (1990) and D. Wands (1994)

$f(R)$ action

$$S = \int \left(f(R) + \mathcal{L} \right) \sqrt{-g} \, d^4x \qquad (15)$$

JBD with $\omega = 0$ action

$$S = \int \left(F(\phi)R - V(\phi) + \mathcal{L} \right) \sqrt{-g} \, d^4x \qquad (16)$$

- $F(\phi) = f'(\phi) \quad , \quad V(\phi) = \phi F(\phi) - f(\phi)$
- Varying action (40) w.r.t. ϕ yields $\phi = R$

Non-minimal coupling: two scalar fields required

$$\varphi^1 \propto \log[F_1(R) + F_2(R)\mathcal{L}] \quad , \quad \varphi^2 = R \tag{17}$$

- Conformal transformation from Jordan frame ($F(\phi)R$ term) to Einstein frame (R uncoupled from ϕ):

$$g_{\mu\nu} \to g^*_{\mu\nu} = A^{-2}(\varphi_1) g_{\mu\nu} \quad , \quad A(\varphi_1) = \exp\left(-\frac{\varphi_1}{\sqrt{3}}\right)$$

T. Damour and G. Esposito-Farese (1992)

Multi-scalar-tensor model

$$S = \int \left[R^* - 2g^{*\mu\nu}\sigma_{ij}\varphi^i_{,\mu}\varphi^j_{,\nu} - 4U + f_2(\varphi^2)\mathcal{L}^*\right]\sqrt{-g}d^4x$$

$$\mathcal{L}^* = A^4(\varphi_1)\mathcal{L} \quad , \quad U = \frac{1}{4}A^2(\varphi_1)\left[\varphi^2 - A^2(\varphi_1) f_1(\varphi^2)\right]$$

- Kinetic term $g^{*\mu\nu}\sigma_{ij}\varphi^i_{,\mu}\varphi^j_{,\nu}$

Field metric

$$\sigma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{18}$$

- Only φ^1 is a dynamical field

Non-conservation law

$$\nabla^\mu T_{\mu\nu} = -\frac{\sqrt{3}}{3} T\varphi^1_{,\nu} + \frac{F_2}{f_2}\left(g_{\mu\nu}\mathcal{L} - T_{\mu\nu}\right)\nabla^\mu\varphi^2 \simeq \alpha_i T\varphi^i_{,\nu}$$

$$\text{with} \quad \alpha_i \equiv \frac{\partial \log A}{\partial \varphi^i} \quad \rightarrow \quad \alpha_1 = -\frac{1}{\sqrt{3}} \quad , \quad \alpha_2 = 0$$

Solar System Impact

First order expansion around cosmological background $R \sim R_0$

Outside spherical body

$$\nabla^2 R - m^2 R = 0$$
$$m^2 \equiv \frac{1}{3}\left[\frac{F_1(R_0) + F_2(R_0)\rho_c}{F_1'(R_0) - 2F_2'(R_0)\rho_c} - R_0 + \frac{6\rho_c \Box F_2'(R_0) - 3\Box\left(F_1'(R_0) - 2F_2'(R_0)\rho_c\right)}{F_1'(R_0) - 2F_2'(R_0)\rho_c}\right] \tag{19}$$

Long-range force, $mr \ll 1$:

$$\gamma = \frac{1}{2}\left[\frac{1 + f_2(R_0) + 4F_2(R_0)R_0 + 12\Box F_2(R_0)}{1 + f_2(R_0) + F_2(R_0)R_0 + 3\Box F_2(R_0)}\right]$$

- For $\gamma \sim 1 \neq 1/2$, requires fine-tuning
- Valid for newtonian stars in perturbative regime $R \sim R_0$!

Chiba *et al.* 2007 , O. Bertolami, R. March, JP 2013

Short range force

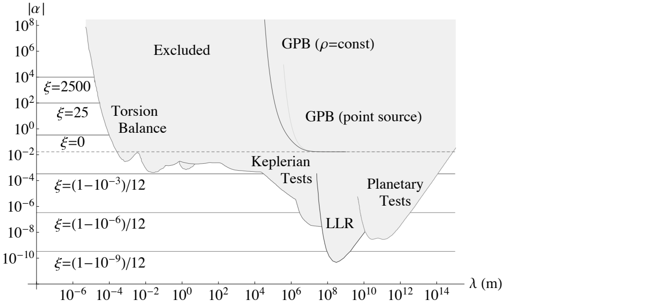
If $mr \gg 1$, then one may disregard cosmological contribution

$$f_1(R) = 2\kappa \left(R + \frac{R^2}{6m^2} \right) \quad , \quad f_2(R) = 1 + 2\xi \frac{R}{m^2}$$

Yukawa contribution

$$U(r) = -\frac{GM}{r} \left(1 + \alpha e^{-mr} \right) \quad , \quad \alpha = \frac{1}{3} - 4\xi$$

Naf, Jetzer *et al.* 2010 , N. Castel-Branco, R. March, JP 2014



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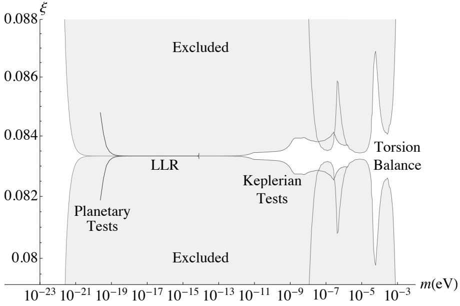


- Starobinsky inflation: $m \sim 10^{-6} M_{Pl}$ unconstrained
  - Also with NMC preheating, $1 \lesssim \xi \lesssim 10^4$
- Medium range: $f(R)$ forbidden, NMC allowed ($\xi \sim 1/12$)
  - $\sim$ characteristic masses: fine-tuning or common origin?

Other Applications

Stellar Structure (perturbative)

O. Bertolami and JP (2007, 2014)

Gravitational Collapse (strong; finite density singularity!)

C. Bastos and JP (2012)

Post-Inflationary Preheating

O. Bertolami, P. Frazão and JP (2011)

Cosmology:

- Localized Cosmological Constant bubbles

  O. Bertolami and JP (2011)

- Cosmological Perturbations

  O. Bertolami, P. Frazão and JP (2013)

- Modifications of Friedmann Equation

  O. Bertolami and JP (2014)

Summary

Non-minimal coupling between matter and curvature

- Wide phenomenology, distinctive features
- Description of Dark Matter (galaxies and clusters)...
  - ...and Dark Energy (and (p)reheating)!
- Equivalent to a two-scalar-tensor-theory
  - Long range force must escape $\gamma = 1/2$
  - Short range force allowed (weak force $\rightarrow \xi \sim 1/12$)
- Specific n for different regimes hint at Laurent expansion

$$f_2(R) = \sum_n \left(\frac{R}{R_n} \right)^n$$

- Clear signature of Equivalence Principle breaking $\rightarrow$ search in violent / time evolving phenomena

Thank you!

Strong coupling between curvature and kitten

Galactic dark matter

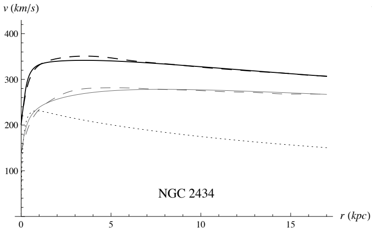
- Observed density profiles:
  - DM: $\rho_{dm} \propto r^{-3}$ (NFW) , $\rho_{dm} \propto r^{-2}$ (IS)
  - Visible: $\rho \propto r^{-4}$ (Hernquist)
- $\rho_{dm} \propto \rho^{1/(1-n)} \rightarrow \quad n_{NFW} = -1/3 \quad , \quad n_{IS} = -1$
- Tully-Fisher law: $M \sim M_{dm} \propto v^{2(1-n)} = v^{8/3} \quad \wedge \quad v^4$
- Sample of seven $E0$ galaxies (approx. spherical)

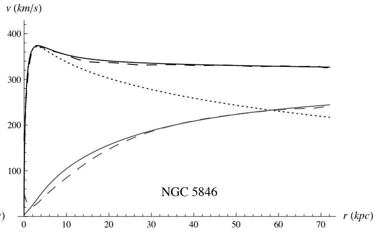
A. Kronawitter, R. P. Saglia, O. Gerhard and R. Bender (2000, 2001)

S. M. Faber, G. Wegner, D. Burstein and R. L. Davies (1989)

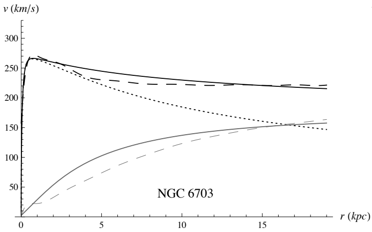
H. W. Rix *et al.* (1997)

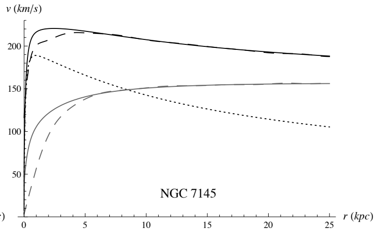
Results





Rotation curve (observed: dash, mimicked: full) = visible (dot) + DM (observed: dash grey, mimicked: full grey)





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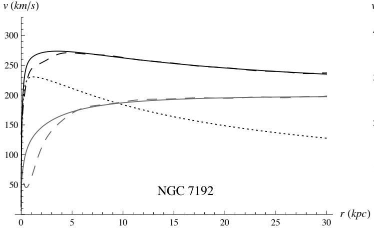

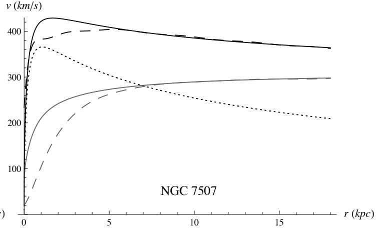

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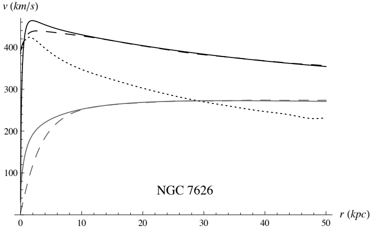


Results

NGC	r_1 (Gpc)	r_3 $(10^5\ Gpc)$	$r_{\infty\ 1}$ (kpc)	$r_{\infty\ 3}$ (kpc)
2434	∞	0.9	0	33.1
5846	37	∞	138	0
6703	22	∞	61.2	0
7145	22.3	47.3	60.9	14.2
7192	14.8	24	86.0	18.3
7507	4.9	2.9	178	31.1
7626	28	9.6	124	42.5

- Average $\bar{r}_1 = 21.5\ Gpc$, s.d. $\sigma_1 = 10.0\ Gpc$
- Average $\bar{r}_3 = 1.69 \times 10^6\ Gpc$, s.d. $\sigma_3 = 1.72 \times 10^6\ Gpc$

- Deviation from sphericity
- Other choices:
  - $f_1(R)$ term; simplistic $f_2(R)$ coupling (Laurent series...)
  - Visible matter density ρ
  - NFW or IS ρ_{dm} (different n)
  - $\mathcal{L} = -\rho$

Inflationary (P)reheating

Inflation: early phase of fast expansion of the Universe

- Driven by scalar field slow-rolling down suitable potential
  - Or non-trivial curvature term $f_1(R)$
  - Formal equivalence with scalar tensor theory
  - Starobinsky inflation: $f_1(R) = 2\kappa R + R^2/6M^2$

A. A. Starobinsky (1980)

- Problem: at the end of inflation, Universe is too cold!
- "Old reheating": scalar field oscillates around minimum
  - Decays into particles and reheats the Universe
  - Problem: fine tuning of parameters, overproduction
- Solution: preheating

Dolgov and Kirilova (1990) , Traschen and Brandenberger (1990)

Kofman *et al.* (1994) ; Shtanov *et al.* (1995)

Preheating

Quantum field χ with mass m coupled to scalar curvature:

Lagrangean density

$$\mathcal{L}_\chi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} m^2 \chi^2 - \frac{1}{2} \xi R \chi^2 \tag{20}$$

- Spacetime dependent effective mass $m_{eff}^2 = m^2 + \xi R$

Fourier decomposition

$$\chi(t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left[a_k \chi_k(t) e^{-i\mathbf{k}\cdot\mathbf{x}} + a_k^\dagger \chi_k^*(t) e^{i\mathbf{k}\cdot\mathbf{x}} \right] \tag{21}$$

- Particle creation/annihilation with mass m, momentum $\mathbf{k}$

Parametric resonance

During oscillatory phase

$$\ddot{\chi}_k + \left(\frac{k^2}{a^2} + m^2 + \xi R - \frac{9}{4} H^2 - \frac{3}{2} \dot{H} \right) \chi_k = 0 \rightarrow \qquad (22)$$

$$\ddot{\chi}_k + \left(\frac{k^2}{a^2} + m^2 - \frac{4M\xi}{t - t_o} \sin\left[M(t - t_o) \right] \right) \chi_k \simeq 0$$

- Varying frequency → parametric resonance → explosive particle production

Equivalent to Mathieu equation

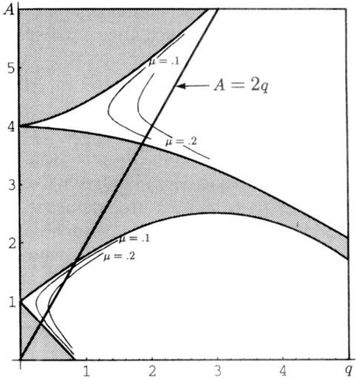
$$\frac{d^2 \chi_k}{dz^2} + \left[A_k - 2q \cos(2z) \right] \chi_k \simeq 0 \qquad (23)$$

Parametric resonance

Mathieu equation

$$\frac{d^2\chi_k}{dz^2} + \left[A_k - 2q\cos(2z)\right]\chi_k \simeq 0 \qquad (24)$$

- Flouquet chart shows resonance bands



Linear non-minimal coupling

Generalize coupling with curvature

$$\begin{aligned} \mathcal{L}_\chi &= -\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}m^2\chi^2 - \frac{1}{2}\xi R\chi^2 \rightarrow \\ \mathcal{L}_\chi &= -\left(1+2\xi\frac{R}{M^2}\right)\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \frac{1}{2}m^2\chi^2\right) \end{aligned} \quad (25)$$

- Linear $f_2(R)$ couples with all matter contributions
  - radiation, ultra-relativistic...
- Subdominant during slow-roll inflation: $1 < \xi < 10^4$

Linear non-minimal coupling

During oscillatory phase

$$\ddot{\chi}_k + \left(\frac{k^2}{a^2} + m^2 + \xi R - \frac{9}{4}H^2 - \frac{3}{2}\dot{H}\right)\chi_k = 0 \rightarrow (26)$$

$$\ddot{\chi}_k + \left(3H + 2\xi\frac{\dot{R}}{M^2}\right)\dot{\chi}_k + \left(\frac{k^2}{a^2} + m^2\right)\chi_k = 0$$

- $X_k \equiv a^{3/2} f_2^{1/2} \chi_k$: friction term transforms into mass term
- Also leads to parametric resonance!

O. Bertolami, P. Frazão and JP (2011)

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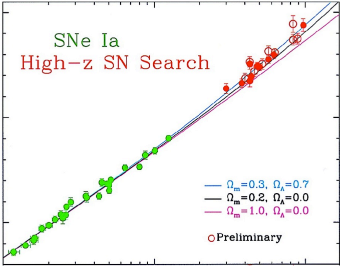

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- Dust filled Universe, $T^{\mu\nu} = \rho U^\mu U^\nu = (\rho, 0, 0, 0)$
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Quantities

$$\begin{aligned} H &\equiv \frac{\dot{a}}{a} = \frac{\beta}{t} \\ R &\equiv 6\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right] = \frac{6\beta}{t^2}(2\beta - 1) \\ q &\equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1}{\beta} - 1 \end{aligned} \qquad (28)$$

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- Energy is conserved!

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$$\nabla^\mu T_{\mu 0} = \frac{F_2}{1+f_2}\left(g_{\mu 0}\mathcal{L}_m - T_{\mu 0}\right)\nabla^\mu R = \frac{F_2}{1+f_2}\left(\mathcal{L}_m + T_{00}\right)\dot{R} = 0 \rightarrow \quad (29)$$

$$\dot{\rho} + 3H\rho = 0$$

Matter density

$$\rho(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^3 \quad (30)$$

Modified dynamics

Modified Friedmann and Raychaudhuri Eqs.

$$H^2 + \frac{k}{a^2} = \frac{1}{6\kappa}(\rho_m + \rho_c) \quad (31)$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{12\kappa}\left[\rho_m + \rho_c + 3(p_m + p_c)\right]$$

- Curvature pressure p_c, density ρ_c depend on $f_1(R), f_2(R)$

Modified dynamics

- Use trivial $f_1(R) = 2\kappa R$ and power-law $f_2(R) = (R/R_n)^n$

Curvature density and pressure

$$\rho_c \approx -6\rho_0\beta \frac{1-2\beta+n(5\beta+2n-3)}{\left(\frac{t}{t_0}\right)^{3\beta}\left(\frac{t}{t_n}\right)^{2n}\left[6\beta(2\beta-1)\right]^{1-n}} \quad (32)$$

$$p_c \approx -2\rho_0 n \frac{2+4n^2-\beta(2+3\beta)+n(8\beta-6)}{\left(\frac{t}{t_0}\right)^{3\beta}\left(\frac{t}{t_n}\right)^{2n}\left[6\beta(2\beta-1)\right]^{1-n}}$$

- $t_n \equiv R_n^{-1/2}$ marks onset of accelerated phase
- Solve Friedmann Eq. $\rightarrow \beta(n) = 2(1-n)/3$

- Deceleration parameter

$$n = 1 - \frac{3}{2(1+q)} \quad , \quad q = -1 + \frac{3}{2(1-n)} \tag{33}$$

- EOS parameter $p_c = \omega \rho_c$

$$\omega = \frac{n}{1-n} \tag{34}$$

  - Same as in DM scenario
  - $n \to \infty$, $\omega \to -1$ and $q \to -1$
    - Cosmological constant Λ
    - Unobtainable with $f_2(R)$: matter term is not constant!
- Previous R_1 and R_3 for $n = -1$ (IS) and $n = -1/3$ (NFW)
  - $r_1, r_3 \ll r_H$ Hubble radius
  - No cosmological role

Fitting $q(z)$ profiles

- Numerically solve Friedmann Eq. for fixed n
- Fit t_n to available $q(z)$ curve

Y. G. Gong and A. Wang (2007)

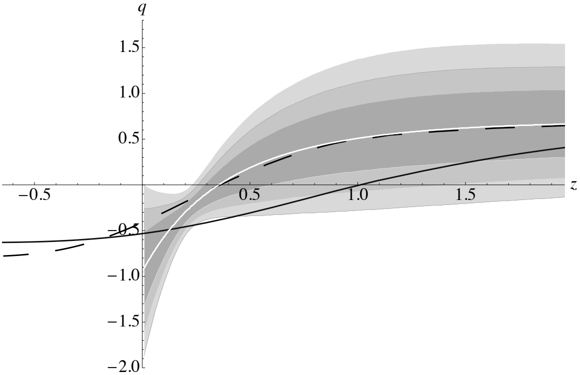


Figure: $q(z)$ for $n = -4$, $t_2 = t_0/4$ (full) and $n = -10$, $t_2 = t_0/2$ (dashed); 1σ, 2σ and 3σ regions shaded, best fit (white)

Linear coupling

- Probe $f_2(R) = R/R_1$
  - Where is curvature is high, but not too much? The Sun!
  - Perturbative treatment
  - Observable: central temperature

O. Bertolami and JP (2008)

- Birkhoff theorem: spherically symmetric, static $g_{\mu\nu}$

Tolman-Oppenheimer-Volkoff equation

$$p' + G(\rho + p)\frac{m_e + 4\pi p r^3}{r^2 - 2Gm_e r} = a\left[\left(\left[\frac{5}{8}p'' - 4\pi G p\rho\right] r - \frac{p'}{4}\right)\rho + p\rho'\right] . \quad (35)$$

  - $a \equiv 16\pi G/R_1$, $[a] = M^{-4}$
  - Suitably defined effective mass m_e
  - 2^{nd}, not 1^{st} order ODE

- Newtonian limit

Modified hydrostatic equilibrium

$$p' + \frac{G m_e \rho}{r^2} = a \left[\left(\left[\frac{5}{8} p'' - 4\pi G p \rho \right] r - \frac{p'}{4} \right) \rho + p \rho' \right] \quad . \quad (36)$$

Polytropic equation of state

$$p = K \rho_B^{(n+1)/n} \quad (37)$$

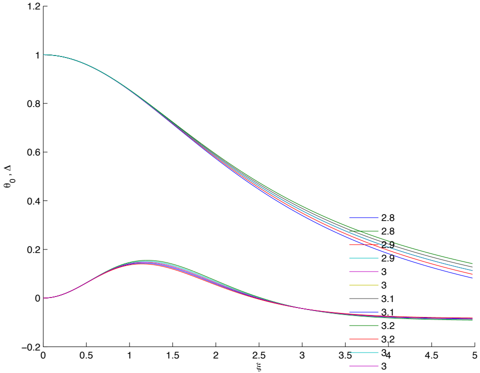
- n polytropic index
  - $n = -1$: isobaric
  - $n = 0$: isometric
  - $n \to +\infty$: isothermal
  - $n = 1/(\gamma - 1)$: adiabatic ($\gamma \equiv c_p/c_V$)
  - $n = 1.5$: giant planets, white/brown dwarfs, red giants
  - $n = 5$: boundless system
  - $n = 3$: first solar model (A. Eddington)
- K polytropic constant

- $\rho = \rho_c \theta^n(\xi) \quad , \quad p = p_c \theta^{n+1}(\xi)$
- $\xi = r/R_n \quad , \quad R_n^2 \equiv (n+1)p_c/4\pi G \rho_c^2$
- $\rho_c = 1.622 \times 10^5 \; kg/m^3 \quad , \quad p_c = 2.48 \times 10^{16} \; Pa$

Modified Lane-Emden equation

$$\frac{1}{\xi^2}\left[\xi^2\theta'\left(1 + \frac{3n-1}{4(n+1)} + A_c\theta^n\left[\left\{\frac{5}{8}\left(\theta'' + n\frac{\theta'^2}{\theta}\right\} - N_c\theta^{n+1}\right]\frac{\xi}{\theta'}\right]\right)\right]' = -\theta^n\left[1 + A_c\left(\frac{3}{8}\left[\theta'' + n\frac{\theta'^2}{\theta}\right] + \frac{\theta'}{4\xi} - \frac{\theta^n}{2}\right)\right] \quad (38)$$

- 3^{rd} order ODE! Linearize...

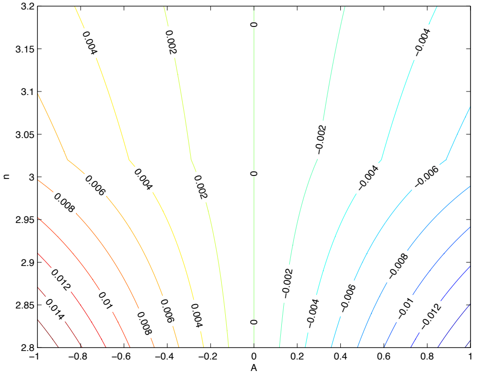


Unperturbed solution $\theta_0(\xi)$ and perturbation

Central temperature bound

$$\left| \frac{T_c}{T_{c0}} - 1 \right| < 6\% \rightarrow |R_1| > \left(1.53 \times 10^{-17}\ eV\right)^2 \sim 10^{-90} M_P^2$$

- Not very interesting...



Relative central temperature T_c deviation

Central temperature bound

$$\left| \frac{T_c}{T_{c0}} - 1 \right| < 6\% \rightarrow |R_1| > \left(1.53 \times 10^{-17}\, eV\right)^2 \sim 10^{-90} M_P^2$$

- Not very interesting...

Parameterized Post-Newtonian formalism

- Expand metric as sum of all potentials down to $O(c^{-4})$
- 10 PPN parameters related to fundamental properties
  - β: nonlinearity in the superposition law for gravity
  - γ: space-curvature produced by unit rest mass
  - General Relativity: $\beta = \gamma = 1$, other parameters vanish
- Momentum conservation, no preferred-frame/location $\rightarrow$

PPN metric

$$g_{00} = -1 + 2U - 2\beta U^2 \quad , \quad g_{ij} = (1 + 2\gamma U)\delta_{ij}$$

C. Will (2006)

Equivalence with a multiscalar-tensor theory

- $f(R)$ theories $\leftrightarrow$ Jordan-Brans-Dicke theory with $\omega = 0$

P. Teyssandier and P. Tourranc (1983), H. Schmidt (1990) and D. Wands (1994)

$f(R)$ action

$$S = \int \left(f(R) + \mathcal{L}\right) \sqrt{-g}\, d^4x \tag{39}$$

JBD with $\omega = 0$ action

$$S = \int \left(F(\phi)R - V(\phi) + \mathcal{L}\right) \sqrt{-g}\, d^4x \tag{40}$$

- $F(\phi) = f'(\phi) \quad , \quad V(\phi) = \phi F(\phi) - f(\phi)$
- Varying action (40) w.r.t. ϕ yields $\phi = R$

Non-minimal coupling: two scalar fields required

$$\varphi^1 \propto \log[F_1(R) + F_2(R)\mathcal{L}] \quad , \quad \varphi^2 = R \qquad (41)$$

- Conformal transformation from Jordan frame ($F(\phi)R$ term) to Einstein frame (R uncoupled from ϕ):

$$g_{\mu\nu} \to g^*_{\mu\nu} = A^{-2}(\varphi_1) g_{\mu\nu} \quad , \quad A(\varphi_1) = \exp\left(-\frac{\varphi_1}{\sqrt{3}}\right)$$

T. Damour and G. Esposito-Farese (1992)

Multi-scalar-tensor model

$$S = \int \left[R^* - 2g^{*\mu\nu}\sigma_{ij}\varphi^i_{,\mu}\varphi^j_{,\nu} - 4U + f_2(\varphi^2)\mathcal{L}^*\right]\sqrt{-g}d^4x$$

$$\mathcal{L}^* = A^4(\varphi_1)\mathcal{L} \quad , \quad U = \frac{1}{4}A^2(\varphi_1)\left[\varphi^2 - A^2(\varphi_1)f_1(\varphi^2)\right]$$

- Kinetic term $g^{*\mu\nu}\sigma_{ij}\varphi^i_{,\mu}\varphi^j_{,\nu}$

Field metric

$$\sigma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (42)$$

- Only φ^1 is a dynamical field
- Solar System: perturbative coupling $\rightarrow F_2/f_2 \sim 0$

Non-conservation law

$$\nabla^\mu T_{\mu\nu} = -\frac{\sqrt{3}}{3} T\varphi^1_{,\nu} + \frac{F_2}{f_2}\left(g_{\mu\nu}\mathcal{L} - T_{\mu\nu}\right)\nabla^\mu\varphi^2 \simeq \alpha_i T\varphi^i_{,\nu}$$

$$\text{with} \quad \alpha_i \equiv \frac{\partial \log A}{\partial \varphi^i} \quad \rightarrow \quad \alpha_1 = -\frac{1}{\sqrt{3}} \quad , \quad \alpha_2 = 0$$

Parameterized Post-Newtonian formalism

- Use field metric to raise/lower latin indices: $\alpha^i \equiv \sigma^{ij}\alpha_j$

$$\alpha^2 \equiv \alpha_i \alpha^i = \sigma^{ij}\alpha^i\alpha_j = 0 \quad , \quad \alpha_{i,j} \equiv \frac{\partial \alpha_j}{\partial \varphi^i} = 0$$

Since $\alpha_2 = 0$

$$\beta = 1 + \frac{1}{2}\left[\frac{\alpha^i\alpha^j\alpha_{j,i}}{(1+\alpha^2)^2}\right]_\infty = 1$$

$$\gamma = 1 - 2\left[\frac{\alpha^2}{1+\alpha^2}\right]_\infty = 1$$

- Same as in GR!
- If perturbative effects are considered, $\beta \sim 1$ and $\gamma \sim 1$
  - Choose $f_2(R)$, solve Einstein field eqs., expand metric

Choice of Lagrangian density of a perfect fluid

Non-(covariant) conservation law

$$\nabla^\mu T_{\mu\nu} = \frac{F_2}{1+f_2}\left(g_{\mu\nu}\mathcal{L}_m - T_{\mu\nu}\right)\nabla^\mu R \quad (43)$$

- In GR, $\mathcal{L}_m$ serves to obtain $T_{\mu\nu}$ only
- If $f_2(R) \neq 0$, $\mathcal{L}_m$ appears in eqs. motion!

Perfect fluid

$$T_{\mu\nu} = (\rho + p)U_{\mu\nu}U_\nu + p g_{\mu\nu} \quad (44)$$
$$p \equiv n\frac{\partial\rho}{n} - \rho$$

- U^μ: four-velocity
- n: particle number density
- $J^\mu = \sqrt{-g}nU^\mu$: flux vector of particle number density n

Action in GR

$$S_m = \int d^4x \left[-\sqrt{-g}\, \rho(n,s) + J^\mu \phi_\mu \right] \qquad (45)$$

J. D. Brown (1993)

- ϕ_μ: contains thermodynamical potentials
  - particle number conservation
  - entropy exchange
  - definition of temperature
  - chemical free energy
- Equivalent Lagrangean densities:
  - Begin with $\mathcal{L}_0 = -\rho$
  - Substitute eqs. motion back into action Eq. (45)
  - Read "on-shell" $\mathcal{L}_i$:
    - $\mathcal{L}_1 = p$
    - $\mathcal{L}_2 = -na \quad , \quad a(n,T) = \rho(n)/n - sT$

- How to couple $f_2(R)$ to a perfect fluid?

Modified action

$$S_m = \int d^4x \left[-\sqrt{-g}\,[1+f_2(R)]\,\rho(n,s) + J^\mu \phi_\mu \right] \qquad (46)$$

Equivalent to on-shell Lagrangian?

$$S_m = \int d^4x \sqrt{-g}\,[1+f_2(R)]\,p \qquad (47)$$

- Yes, but...

Redefined thermodynamical quantities, *e.g.*

$$T = \frac{1}{n}\frac{\partial \rho}{\partial s}\bigg|_n = \frac{1}{1+f_2(R)}\theta_{,\mu}U^\mu \qquad (48)$$

O. Bertolami, F. S. N. Lobo and JP (2008)