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Hydrostatic equilibrium and stellar structure in f(R)-gravity

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Outlines

- Hydrostatic equilibrium of stellar structures
- The Newtonian limit of f(R)-gravity
- Stellar hydrostatic equilibrium in f(R)-gravity
- Solution of the standard and modified Lanè-Emden equations
- Jeans criterion for gravitational instability in f(R)-gravity
- The Jeans mass limit in f(R)-gravity
- Discussion and conclusions
- Next steps

Setting the problem

- Several open questions in modern Astrophysics ask for new paradigms.
- No final evidence of Dark Energy and Dark Matter at fundamental level (LHC, astroparticle physics, ground based experiments, LUX…).
- Such problems could be framed extending GR at infrared scales
- GR does not work at ultraviolet scales (no quantum gravity theory up to now).
- f(R)-gravity as minimal extension but other modifications are possible (Starobinsky inflation).
- Several stellar structures cannot be addressed by the standard theory of stellar evolution (magnetars, variable stars, etc..)
- Big issue: Is it possible to revise stellar theory in view of extended gravity?

Hydrostatic equilibrium of stellar structures

Conditions for hydrostatic equilibrium in Newtonian dynamics are

$$\frac{dp}{dr} = \frac{d\Phi}{dr}\rho$$

- p is the pressure,
- Φ is the gravitational potential,
- ρ is the density

The Poisson equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = -4\pi G\rho$$

Since we are taking into account only static and stationary situations, here we consider only time independent solutions

In general, the temperature τ appears and the density ρ satisfies an equation of state of the form

$$\rho = \rho(p, \tau)$$

R. Kippenhahn and A. Weigert, Stellar Structures and Evolution (Springer-Verlag, Berlin, 1990).

Hydrostatic equilibrium of stellar structures

A polytropic relation between p and ρ exists

$$p = K\rho^{\gamma}$$

- K is the polytropic constant that can be obtained by a combination of fundamental constants
- The constant Υ is the polytropic index determining the stellar fluid.
- Note that Φ > 0 is in the interior of the model, since we define the gravitational potential as -Φ

Inserting the polytropic equation of state, we obtain

$$\frac{d\Phi}{dr} = \gamma K \rho^{\gamma-2} \frac{d\rho}{dr}$$

Hydrostatic equilibrium of stellar structures

For $\gamma \neq 1$, the above equation can be integrated giving

$$\frac{\gamma K}{\gamma - 1}\rho^{\gamma-1} = \Phi \rightarrow \rho = \left[\frac{\gamma - 1}{\gamma K}\right]^{1/(\gamma-1)} \Phi^{1/(\gamma-1)} \doteq A_n \Phi^n$$

- We have chosen the integration constant to give $\Phi = 0$ at surface ($\rho = 0$)
- $n = \frac{1}{\gamma - 1}$ is the polytropic index

Inserting the above relation into the Poisson equation, we obtain a differential equation for the gravitational potential

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = -4\pi G A_n \Phi^n$$

Hydrostatic equilibrium of stellar structures

Defining now the dimensionless variables:

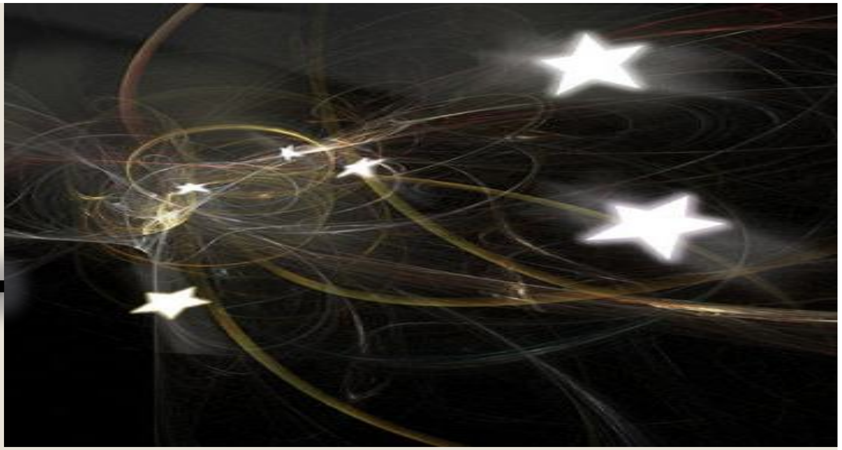
$$z = |\mathbf{x}|\sqrt{\frac{\chi A_n \Phi_c^{n-1}}{2}} \qquad w(z) = \frac{\Phi}{\Phi_c} = \left(\frac{\rho}{\rho_c}\right)^{1/n}$$

- The subscript c refers to the center of the star and the relation between ρ and Φ
- At the center ($r = 0$), we have $z = 0$, $\Phi = \Phi_c$, $\rho = \rho_c$ and therefore $w = 1$

Then we obtain the standard Lané-Emden equation describing the hydrostatic equilibrium of stellar structures in the Newtonian theory

$$\boxed{\frac{d^2 w}{dz^2} + \frac{2}{z}\frac{dw}{dz} + w^n = 0}$$

Solutions of the standard Lanè-Emden equations



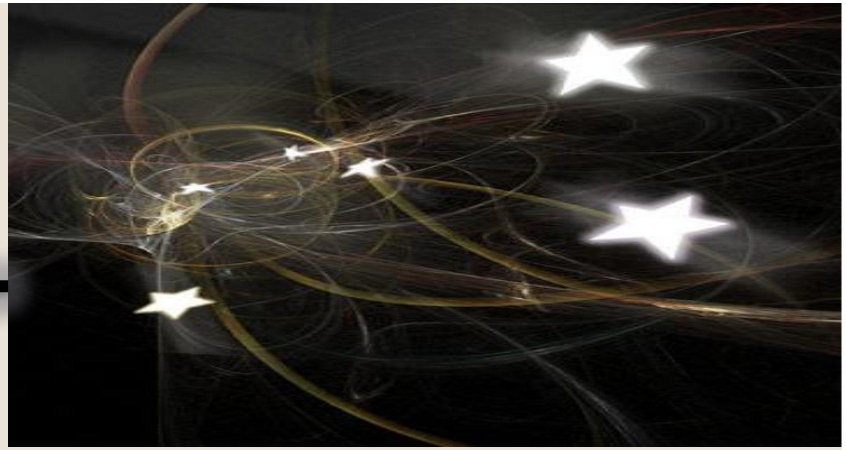
Only for three values of **n**, solutions have analytical expressions

$$n = 0 \longrightarrow w^{(0)}_{\mathrm{GR}}(z) = 1 - \frac{z^2}{6}$$

$$n = 1 \longrightarrow w^{(1)}_{\mathrm{GR}}(z) = \frac{\sin z}{z}$$

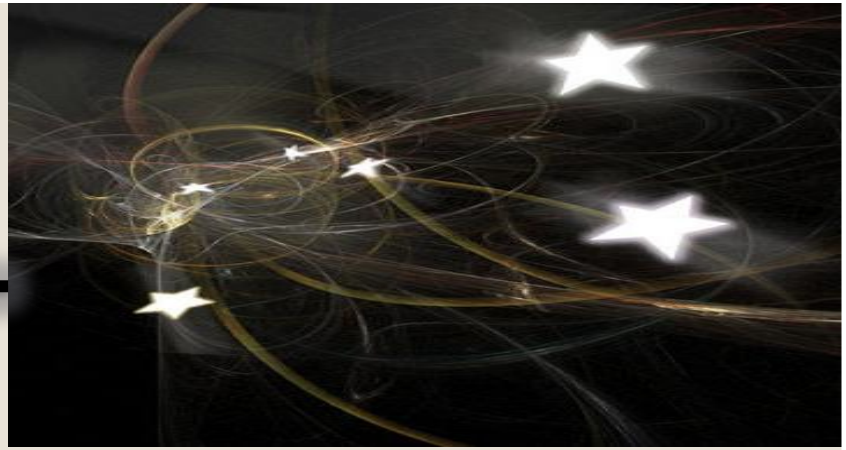
$$n = 5 \longrightarrow w^{(5)}_{\mathrm{GR}}(z) = \frac{1}{\sqrt{1 + \frac{z^2}{3}}}.$$

Solutions of the standard Lanè-Emden equations



- The surface of the polytrope of index n is defined by the value $z = z^{(n)}$. where $\rho = 0$ and thus $w = 0$
- For $n = 0$ and $n = 1$, the surface is reached for a finite value of $z = z^{(n)}$
- The case $n = 5$ gives rise to a model of infinite radius
- It can be shown that for $n<5$ the radius of polytropic models is finite; for $n>5$ they have infinite radius
- One finds $z^{(0)}_{GR} = \sqrt{6}$, $z^{(1)}_{GR} = \pi$, $z^{(5)}_{GR} = \infty$
- A general property of the solutions is that $z^{(n)}$ grows monotonically with the polytropic index **n**

Solutions of the standard Lanè-Emden equations



Apart from the three cases where analytic solutions are known, the standard Lane´-Emden can be solved numerically, considering the neighborhood of the stellar center, i.e.

$$w_{\mathrm{GR}}^{(n)}(z) = \sum_{i=0}^{\infty} a_i^{(n)} z^i$$

at lowest orders, solutions can be classified by the index **n**, that is

$$w_{\mathrm{GR}}^{(n)}(z) = 1 - \frac{z^2}{6} + \frac{n}{120} z^4 + \dots$$

- The case Υ=5/3 and n= 3/2 is the non-relativistic limit; the case γ=4/3 and n = 3 is the relativistic limit of a completely degenerate gas.

The Newtonian limit of f(R) - gravity

Let us start with a general class of Extended Theories of Gravity (ETG) given by the action

$$\mathcal{A} = \int d^4x\sqrt{-g}[f(R) + \mathcal{X}\mathcal{L}_m],$$

Varying the action with respect to the metric we obtain the field equations (standard GR is recovered for f(R)=R)

$$f'R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - f_{;\mu\nu} + g_{\mu\nu}\Box f' = \mathcal{X}T_{\mu\nu}$$

$$3\Box f' + f'R - 2f = \mathcal{X}T,$$

S. Capozziello, M. De Laurentis Phys. Rep. 509, 167-321 (2011)
S. Capozziello , M. Francaviglia, Gen. Relativ. Gravit. 40, 357 (2007)

The Newtonian limit of f(R) - gravity

In order to achieve the Newtonian limit of the theory the metric tensor has to be approximated as follows:

$$g_{\mu\nu} \sim \begin{pmatrix} 1 - 2\Phi(t, \mathbf{x}) + \mathcal{O}(4) & \mathcal{O}(3) \\ \mathcal{O}(3) & -\delta_{ij} + \mathcal{O}(2) \end{pmatrix},$$

The Ricci scalar formally becomes

$$R \sim R^{(2)}(t, \mathbf{x}) + \mathcal{O}(4).$$

The **n**-th derivative of Ricci function can be developed as

$$f^n(R) \sim f^n(R^{(2)} + \mathcal{O}(4)) \sim f^n(0) + f^{n+1}(0)R^{(2)} + \mathcal{O}(4)$$

here **R.^{n}** denotes a quantity of order **O(n)**

S. Capozziello, A. Stabile, and A. Troisi, Phys. Rev. D 76, 104019 (2007)

The Newtonian limit of f(R) - gravity

Field equations at O (2)-order, that is at the Newtonian level, are

$$R_{tt}^{(2)} - \frac{R^{(2)}}{2} - f''(0) \triangle R^{(2)} = \mathcal{X} T_{tt}^{(0)}$$

$$-3f''(0) \triangle R^{(2)} - R^{(2)} = \mathcal{X} T^{(0)},$$

- Δ is the Laplacian in the flat space $R_{tt} = \Delta \Phi$ and, for the sake of simplicity, we set f' (0) = 1

We recall that the energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu - p g_{\mu\nu}$$

- p is the pressure and ε is the energy density

The Newtonian limit of f(R) - gravity

Being the pressure contribution negligible in the field equations in the Newtonian approximation, we have

modified Poisson equations

$$\begin{cases} \triangle\,\Phi + \dfrac{R^{(2)}}{2} + f''(0)\,\triangle\,R^{(2)} = -\mathcal{X}\rho \\ 3f''(0)\,\triangle\,R^{(2)} + R^{(2)} = -\mathcal{X}\rho, \end{cases}$$

- ρ is now the mass density

For f''(R) = 0 we have the standard Poisson equation

$$\triangle\Phi = -4\pi G\rho$$

This means that as soon as the second derivative of f (R) is different from zero, deviations from the Newtonian limit of GR emerge

Stellar hydrostatic equilibrium in f(R) - gravity

From the Bianchi identity we have

$$T^{\mu\nu}{}_{;\mu} = 0 \rightarrow \frac{\partial p}{\partial x^k} = -\frac{1}{2}(p+\epsilon)\frac{\partial \ln g_{tt}}{\partial x^k}.$$

- *If the dependence on the temperature is negligible, this relation can be introduced into field equations, which becomes a system of three equations for p, Φ and R (2) and can be solved without the other structure equations.*

Let us suppose that matter still satisfies a polytropic equation

$$p = K\rho^{\gamma}$$

Stellar hydrostatic equilibrium in f(R)-gravity

We obtain an integro-differential equation for the gravitational potential , that is

$$\triangle\,\Phi(\mathbf{x}) + \frac{2\mathcal{X}A_n}{3}\Phi(\mathbf{x})^n = -\frac{m^2\mathcal{X}A_n}{6}\int d^3\mathbf{x}'\,\mathcal{G}(\mathbf{x},\mathbf{x}')\Phi(\mathbf{x}')^n$$

- ★ $\mathcal{G}(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi}\frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$ *is the Green function*

- ★ $m^2 = -\frac{1}{3f''(0)}$ *that is an effective mass related to the form of f (R)*

Stellar hydrostatic equilibrium in f(R)-gravity

Adopting again the dimensionless variables

$$z = \frac{|\mathbf{x}|}{\xi_0} \qquad w(z) = \frac{\Phi}{\Phi_c}$$

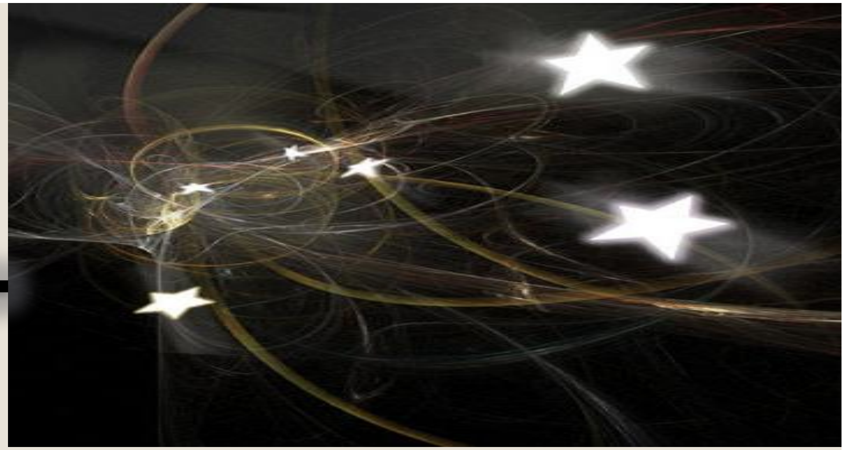
$$\xi_0 \doteq \sqrt{\frac{3}{2\mathcal{X}A_n\Phi_c^{n-1}}}$$

is a characteristic length linked to stellar radius ξ

The f(R)-gravity Lanè-Emden equation is

$$\boxed{\begin{aligned}&\frac{d^2w(z)}{dz^2} + \frac{2}{z}\frac{dw(z)}{dz} + w(z)^n \\ &\quad = \frac{m\xi_0}{8}\frac{1}{z}\int_0^{\xi/\xi_0} dz' z' \left\{e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|}\right\} w(z')^n\end{aligned}}$$

Solutions of the modified Lanè-Emden equations



For the modified Lane´-Emden, we have an exact solution for n = 0, in fact

$$w^{(0)}_{f(R)}(z) = 1 - \frac{z^2}{8} + \frac{(1+m\xi)e^{-m\xi}}{4m^2\xi_0^2}\left[1 - \frac{\sinh m\xi_0 z}{m\xi_0 z}\right],$$

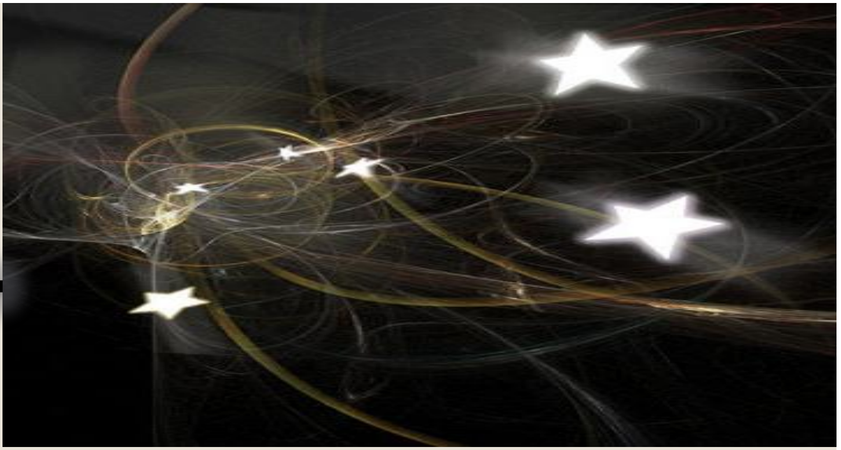
Where the boundary conditions **w(0) = 1** and **w'(0) = 0** are satisfied

A comment on the GR limit (that is **f(R) → R**) of above solution is necessary.

In fact, when we perform the limit m → ∞ we do not recover exactly $w^{(0)}_{GR}(z)$.
The difference is in the definition of quantity ξ_0

In GR it is $\xi_0 = \sqrt{\frac{2}{\chi A_n \Phi_c^{n-1}}}$

Solutions of the modified Lanè-Emden equations



The point $z^{(0)}_{f(R)}$ is calculated by imposing $W^{(0)}_{f(R)}(z^{(0)}_{f(R)}) = 0$ and by considering the Taylor expansion

$$\frac{\sinh m\xi_0 z}{m\xi_0 z} \sim 1 + \frac{1}{6}(m\xi_0 z)^2 + \mathcal{O}(m\xi_0 z)^4$$

We obtain $z^{(0)}_{f(R)} = \frac{2\sqrt{6}}{\sqrt{3+(1+m\xi)e^{-m\xi}}}$

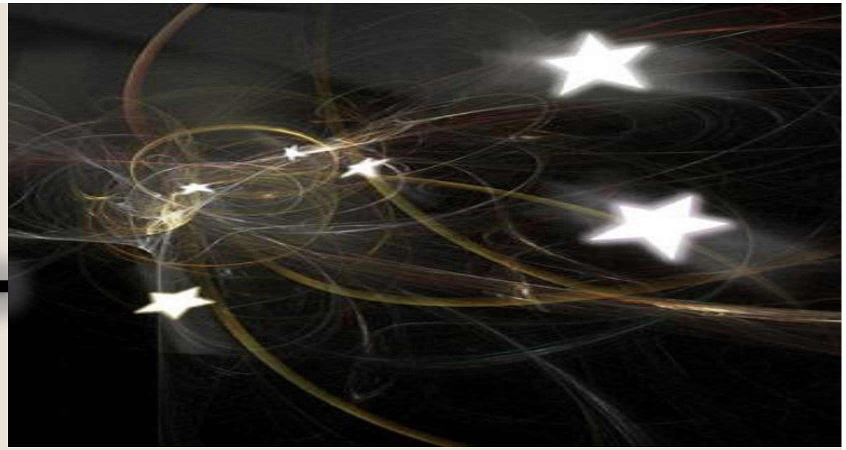
Since the stellar radius ξ is given by definition $\xi = \xi_0 \; z^{(0)}_{f(R)}$ we obtain

$$\xi = \sqrt{\frac{3\Phi_c}{2\pi G}} \frac{1}{\sqrt{1 + \frac{1+m\xi}{3} e^{-m\xi}}}.$$

By solving numerically the constraint, we find the modified expression of the radius

If $\mathbf{m} \rightarrow \infty$ we have the standard expression valid for the Newtonian limit of GR

Solutions of the modified Lanè-Emden equations



In the f(R)-gravity case, for n=0, the radius is smaller than in GR

In the case n= 1 we obtain
$$\frac{d^2\tilde{w}(z)}{dz^2} + \tilde{w}(z) = \frac{m\xi_0}{8}\int_0^{\xi/\xi_0} dz' \times \left\{e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|}\right\}\tilde{w}(z'),$$

$$\tilde{w} = zw$$

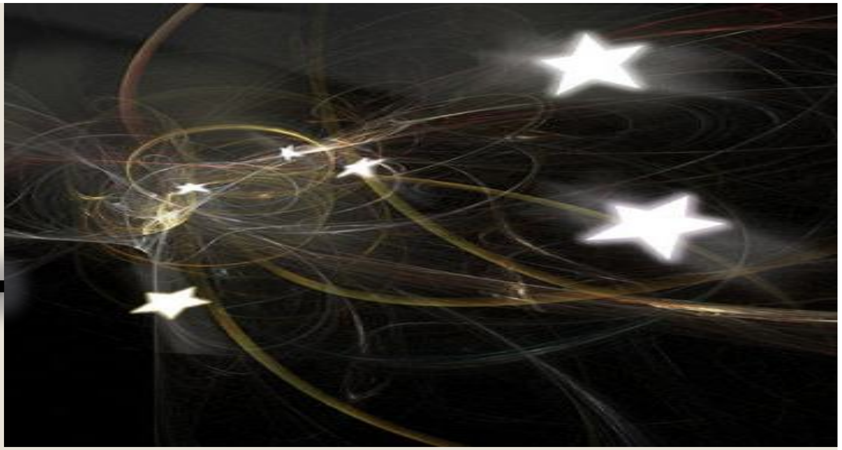
If we perturb this equations we have $\tilde{w}^{(1)}_{f(R)}(z) \sim \tilde{w}^{(1)}_{\mathrm{GR}}(z) + e^{-m\xi}\Delta\tilde{w}^{(1)}_{f(R)}(z).$

The coefficient $e^{-m\xi} < 1$ is the parameter with respect to which we perturb

And then

$$\frac{d^2\Delta\tilde{w}^{(1)}_{f(R)}(z)}{dz^2} + \Delta\tilde{w}^{(1)}_{f(R)}(z) = \frac{m\xi_0 e^{m\xi}}{8}\int_0^{\xi/\xi_0} dz'\left\{e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|}\right\}\tilde{w}^{(1)}_{\mathrm{GR}}(z')$$

Solutions of the modified Lanè-Emden equations



And the solutions is easily found to be

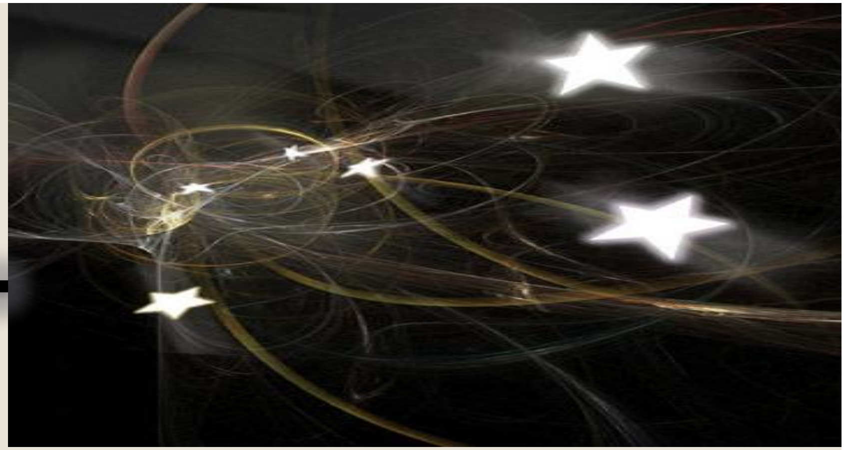
$$w^{(1)}_{f(R)}(z) \sim \frac{\sin z}{z}\left\{1 + \frac{m^2\xi_0^2}{8(1+m^2\xi_0^2)}\left[1 + \frac{2e^{-m\xi}}{1+m^2\xi_0^2}\right.\right.$$
$$\left.\left.\times\left(\cos\xi/\xi_0 + m\xi_0\sin\xi/\xi_0\right)\right]\right\}$$
$$-\frac{m^2\xi_0^2}{8(1+m^2\xi_0^2)}\left[\frac{2e^{-m\xi}}{1+m^2\xi_0^2}\right.$$
$$\left.\times\left(\cos\xi/\xi_0 + m\xi_0\sin\xi/\xi_0\right)\frac{\sinh m\xi_0 z}{m\xi_0 z} + \cos z\right].$$

Also in this case, for **m→∞**, we do not recover exactly $w^{(1)}{}_{GR}(z)$

The reason is the same of the previous n = 0 case

Analytical solutions for other values of n are not available

Solutions of the modified Lanè-Emden equations



Gravitational potential profiles generated by spherically symmetric sources of uniform mass with radius ξ can be achieved

We can impose a mass density of the form $$\rho = \frac{3M}{4\pi\xi^3}\Theta(\xi - |\mathbf{x}|),$$

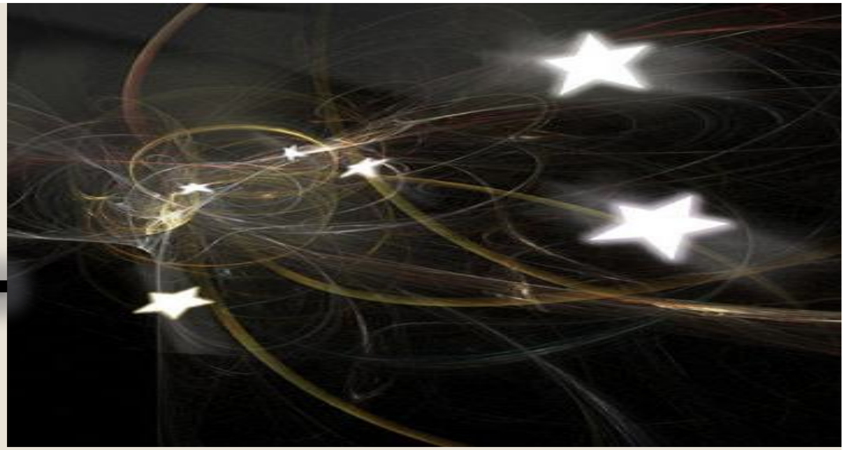
- Θ is the Heaviside function and M is the mass

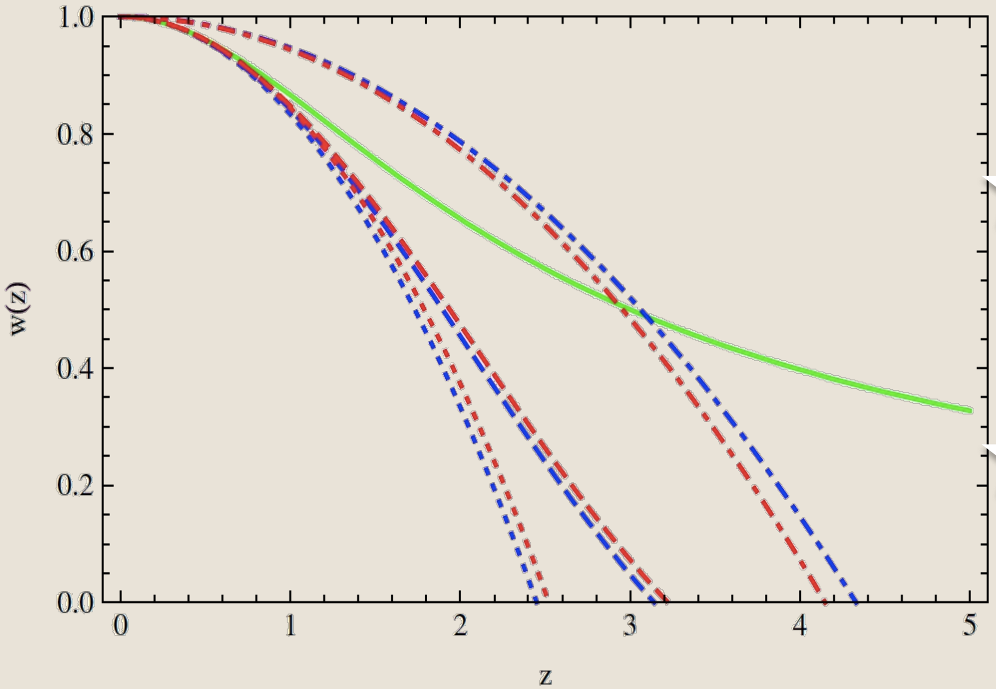
By solving field equations inside the star and considering the boundary conditions **w.(0) = 1** and w'(0)=0, we get

$$w_{f(R)}(z) = \left[\frac{3}{2\xi} + \frac{1}{m^2\xi^3} - \frac{e^{-m\xi}(1+m\xi)}{m^2\xi^3}\right]^{-1}\left[\frac{3}{2\xi} + \frac{1}{m^2\xi^3} - \frac{\xi_0^2 z^2}{2\xi^3} - \frac{e^{-m\xi}(1+m\xi)}{m^2\xi^3}\,\frac{\sinh m\xi_0 z}{m\xi_0 z}\right].$$

In the limit **m→**∞ we recover the GR case $w_{\mathrm{GR}}(z) = 1 - \frac{\xi_0^2 z^2}{3\xi^2}$

Solutions of the standard and modified Lanè-Emden equations





- Plot of solutions (blue lines) of standard Lane´-Emden: $w^{(0)}{}_{GR}(z)$ (dotted line) and $w^{(1)}{}_{GR}(z)$ (dashed line). The green line corresponds to $w^{(5)}{}_{GR}(z)$
- The red lines are the solutions of modified Lane´-Emden: $w^{(0)}{}_{f(R)}(z)$ (dotted line) and $w^{(1)}{}_{f(R)}(z)$ (dashed line).
- The blue dashed-dotted line is the potential derived from GR $w_{GR}(z)$ and the red dashed dotted line is the potential derived from f(R) gravity for a uniform spherically symmetric mass distribution
- From a rapid inspection of these plots, the differences between GR and **f(R)** gravitational potentials are clear and the tendency is that at larger radius **z** they become more evident.

Dust- dominated self-gravitating systems

The collapse of self-gravitational collisionless systems can be dealt with the introduction of coupled collisionless Boltzmann and Poisson equations

$$\frac{\partial f(\vec{r},\vec{v},t)}{\partial t} + (\vec{v}\cdot\vec{\nabla}_r)f(\vec{r},\vec{v},t) - (\vec{\nabla}\Phi\cdot\vec{\nabla}_v)f(\underbrace{\vec{r},\vec{v}},t) = 0$$

three-dimensional vectors in the spatial manifold

$$\vec{\nabla}^2\Phi(\vec{r},t) = 4\pi G\int f(\vec{r},\vec{v},t)d\vec{v},$$

A self-gravitating system at equilibrium is described by a time-independent distribution function $f_o(x, v)$ and a potential $\Phi_0(x)$ that are solutions of above equations

J. Binney and S. Tremaine, Galactic Dynamics (Princeton University Press, Princeton, NJ, 1994).

Dust- dominated self-gravitating systems

Considering a small perturbation to this equilibrium:

$$f(\vec{r}, \vec{v}, t) = f_0(\vec{r}, \vec{v}) + \epsilon f_1(\vec{r}, \vec{v}, t),$$

$$\Phi(\vec{r}, t) = \Phi_0(\vec{r}) + \epsilon \Phi_1(\vec{r}, t),$$

★ *Where* $\varepsilon \ll 1$ *and*

by substituting in Boltzmann and Poisson equations and by linearizing, one obtains:

$$\frac{\partial f_1(\vec{r}, \vec{v}, t)}{\partial t} + \vec{v} \cdot \frac{\partial f_1(\vec{r}, \vec{v}, t)}{\partial \vec{r}} - \vec{\nabla}\Phi_1(\vec{r}, t) \cdot \frac{\partial f_0(\vec{r}, \vec{v})}{\partial \vec{v}} - \vec{\nabla}\Phi_0(\vec{r}) \cdot \frac{\partial f_1(\vec{r}, \vec{v}, t)}{\partial \vec{v}} = 0,$$

$$\vec{\nabla}^2 \Phi_1(\vec{r}, t) = 4\pi G \int f_1(\vec{r}, \vec{v}, t) d\vec{v}.$$

Dust- dominated self-gravitating systems

*Since the equilibrium state is assumed to be homogeneous and time-independent, one can set f_o (**x**,**v**,t) = f(**v**), and so-called* ***Jeans "swindle" to set $\Phi_o = 0$***

In Fourier components

$$-i\omega f_1 + \vec{v}\cdot(i\vec{k}f_1) - (i\vec{k}\Phi_1)\cdot\frac{\partial f_0}{\partial \vec{v}} = 0,$$

$$-k^2\Phi_1 = 4\pi G\int f_1 d\vec{v}.$$

By combining these equations, we obtain the ***dispersion relation***

$$1 + \frac{4\pi G}{k^2}\int \frac{\vec{k}\cdot\frac{\partial f_0}{\partial \vec{v}}}{\vec{v}\cdot\vec{k} - \omega}\, d\vec{v} = 0$$

Dust- dominated self-gravitating systems

In the case of stellar systems, by assuming a Maxwellian distribution function for f_o we have

$$f_0 = \frac{\rho_0}{(2\pi\sigma^2)^{(3/2)}} e^{-(v^2/2\sigma^2)}$$

$$1 - \frac{2\sqrt{2\pi}G\rho_0}{k\sigma^3} \int \frac{v_x e^{-(v_x^2/2\sigma^2)}}{kv_x - \omega} dv_x = 0.$$

By setting ω = 0, the limit for instability is obtained: $k^2(\omega = 0) = \frac{4\pi G\rho_0}{\sigma^2} = k_J^2,$

by which it is possible to define the Jeans mass (M_J) as the mass originally contained within a sphere of diameter λ_J:

$$M_J = \frac{4\pi}{3}\rho_0\left(\frac{1}{2}\lambda_J\right)^3,$$

Where $\lambda_J^2 = \frac{\pi\sigma^2}{G\rho_0}$ is the Jeans length

....and then we can write $M_J = \frac{\pi}{6}\sqrt{\frac{1}{\rho_0}}\cdot\left(\frac{\pi\sigma^2}{G}\right)^3$

Dust- dominated self-gravitating systems

In order to evaluate the integral in the dispersion relation, we have to study the singularity at $\omega = k\, v_x$. To this end, it is useful to write the dispersion relation as

$$1 - \frac{k_J^2}{k^2} W(\beta) = 0,$$

defining $$W(\beta) \equiv \frac{1}{\sqrt{2\pi}} \int \frac{x e^{-(x^2/2)}}{x - \beta} dx,$$

- Where $\beta = \frac{\omega}{k\sigma}$ and $x = \frac{v_x}{\sigma}$
- We set also $\omega = i\omega_I$ and $Re[W(\frac{\omega}{k\sigma})] = 0$ because we are interested in the unstable modes

These modes appear when the imaginary part of ω is greater than zero and in this case the integral in the dispersion relation can be resolved just with previous prescriptions.

Dust- dominated self-gravitating systems

In order to study unstable models, we replace the following identities

$$\int_0^\infty \frac{x^2 e^{-x^2}}{x^2+\beta^2}\,dx = \frac{1}{2}\sqrt{\pi} - \frac{1}{2}\pi\beta e^{\beta^2}[1-\operatorname{erf}\beta],$$

$$\operatorname{erf}\beta(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$$

into the dispersion relation obtaining:

$$k^2 = k_J^2\left\{1 - \frac{\sqrt{\pi}\,\omega_I}{\sqrt{2}k\sigma} e^{(\omega_I/\sqrt{2}k\sigma}\left[1-\operatorname{erf}\left(\frac{\omega_I}{\sqrt{2}k\sigma}\right)\right]\right\}.$$

This is the standard dispersion relation describing the criterion to collapse for infinite homogeneous fluid and stellar systems

The Newtonian limit of f(R) - gravity

As discussed above, field equations in f(R)-gravity give rise to the modified Poisson equations.

$$R^{(2)} \simeq \frac{1}{2}\nabla^2 g^{(2)}_{00} - \frac{1}{2}\nabla^2 g^{(2)}_{ii}$$

that can be recast as $$R^{(2)} \simeq \nabla^2(\Phi - \Psi)$$

Ψ is the further gravitational potential related to the metric component $g^{(2)}_{ii}$

...and then the field equations assume this form

$$\nabla^2\Phi + \nabla^2\Psi - 2f''(0)\nabla^4\Phi + 2f''(0)\nabla^4\Psi = 2\mathcal{X}\rho$$

$$\nabla^2\Phi - \nabla^2\Psi + 3f''(0)\nabla^4\Phi - 3f''(0)\nabla^4\Psi = -\mathcal{X}\rho.$$

S. Capozziello, M. De Laurentis Phys. Rep. 509, 167-321 (2011)

S. Capozziello, M. De Laurentis Ann. Phys. 524, 545 (2012)

Jeans criterion for gravitational instability in f(R)-gravity

Let us assume the standard collisionless Boltzmann equation:

$$\frac{\partial f(\vec{r},\vec{v},t)}{\partial t}+(\vec{v}\cdot\vec{\nabla}_r)f(\vec{r},\vec{v},t)-(\vec{\nabla}\Phi\cdot\vec{\nabla}_v)f(\vec{r},\vec{v},t)=0,$$

Where, according to the Newtonian theory, only the potentia Φ *is present*

Considering the f(R) Poisson equations, also the potential Ψ *has to be considered so we obtain the coupled equations*

$$\nabla^2(\Phi+\Psi)-2\alpha\nabla^4(\Phi-\Psi)=16\pi G\int f(\vec{r},\vec{v},t)d\vec{v}$$

$$\nabla^2(\Phi-\Psi)+3\alpha\nabla^4(\Phi-\Psi)=-8\pi G\int f(\vec{r},\vec{v},t)d\vec{v}.$$

★ *We have replaced f'(0) with the greek letter* α

Jeans criterion for gravitational instability in f(R)-gravity

As in standard case, we consider small perturbations to the equilibrium and linearize the equations. In Fourier space, they become

$$-i\omega f_1 + \vec{v} \cdot (i\vec{k} f_1) - (i\vec{k}\Phi_1) \cdot \frac{\partial f_0}{\partial \vec{v}} = 0,$$

$$-k^2(\Phi_1 + \Psi_1) - 2\alpha k^4(\Phi_1 - \Psi_1) = 16\pi G \int f_1 d\vec{v},$$

$$k^2(\Phi_1 - \Psi_1) - 3\alpha k^4(\Phi_1 - \Psi_1) = 8\pi G \int f_1 d\vec{v}.$$

Jeans criterion for gravitational instability in f(R)-gravity

Combining the above equations we obtain a relation between Φ_1 *and* Ψ_1

$$\Psi_1 = \frac{3 - 4\alpha k^2}{1 - 4\alpha k^2}\Phi_1$$

And then the dispersion relation is

$$1 - 4\pi G\frac{1 - 4\alpha k^2}{3\alpha k^4 - k^2}\int\left(\frac{\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}}{\vec{v}\cdot\vec{k} - \omega}\right)d\vec{v} = 0.$$

As in standard case, one can write

$$1 + \frac{2\sqrt{2\pi}G\rho_0}{\sigma^3}\frac{1 - 4\alpha k^2}{3\alpha k^4 - k^2}\left[\int\frac{kv_x e^{-(v_x^2/2\sigma^2)}}{kv_x - \omega}dv_x\right] = 0.$$

By eliminating the higher-order terms (imposing $\alpha = 0$*),*
We obtain again the standard dispersion of Newton physics

Jeans criterion for gravitational instability in f(R)-gravity

In order to compute the integral in the dispersion relation , we consider the same approach used in the classical case, and finally we obtain:

$$1 + \mathcal{G}\,\frac{1-4\alpha k^2}{3\alpha k^4-k^2}\left[1-\sqrt{\pi}\,x e^{x^2}\left(1-\mathrm{erf}[x]\right)\right] = 0,$$

Where $x = \frac{\omega_I}{\sqrt{2}k\sigma}$ and $\mathcal{G} = \frac{4G\pi\rho_0}{\sigma^2}$

To compare the modified and classical dispersion relation we normalize the equation to the classical Jeans length by fixing the parameter of f(R)- gravity, that is

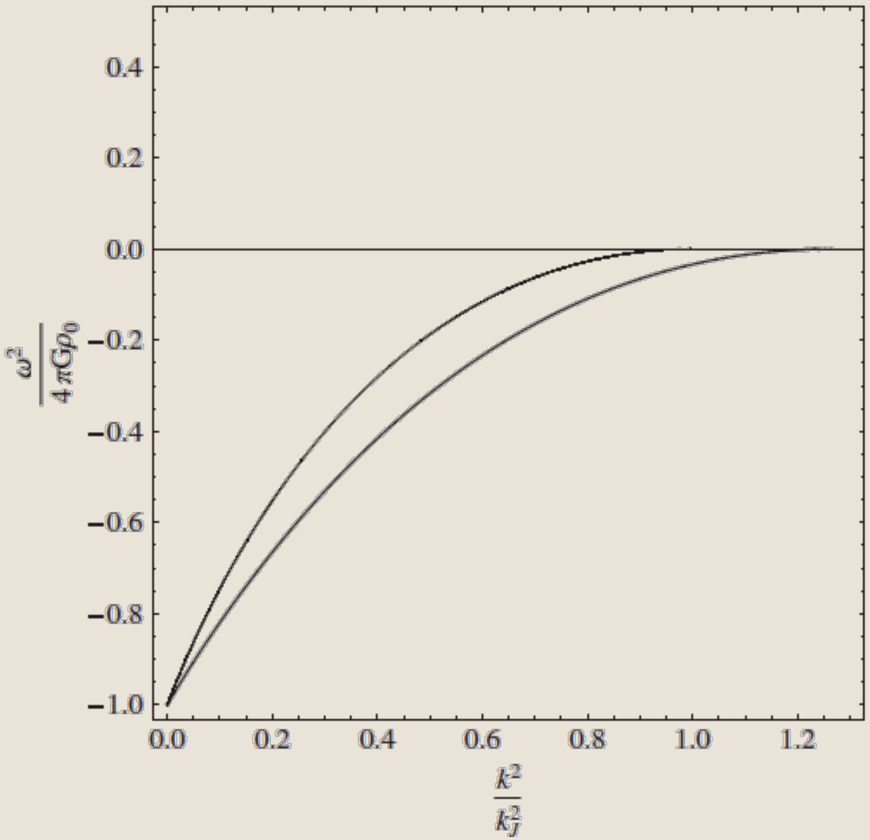
$$\alpha = -\frac{1}{k_j^2} = -\frac{\sigma^2}{4\pi G\rho_0}.$$

This parameterization is correct because the dimension (an inverse of squared length) allows us to parameterize as in standard case

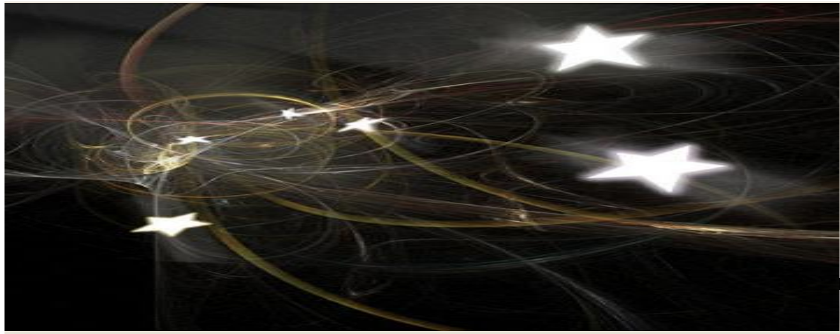
Jeans criterion for gravitational instability in f(R)-gravity

Finally we write and plot this relation

$$\frac{3k^4}{k_j^4}+\frac{k^2}{k_j^2}=\left(\frac{4k^2}{k_j^2}+1\right)[1-\sqrt{\pi}xe^{x^2}(1-\mathrm{erf}[x])]=0.$$



The bold line indicates the plot of the modified dispersion relation.
The thin line indicates the plot of the standard dispersion equation



The Jeans mass limit in f(R)-gravity

A numerical estimation of the f(R) instability length in terms of the standard Newtonian one can be achieved

By solving numerically the above equation with the condition $\omega = 0$, we obtain that the collapse occurs for

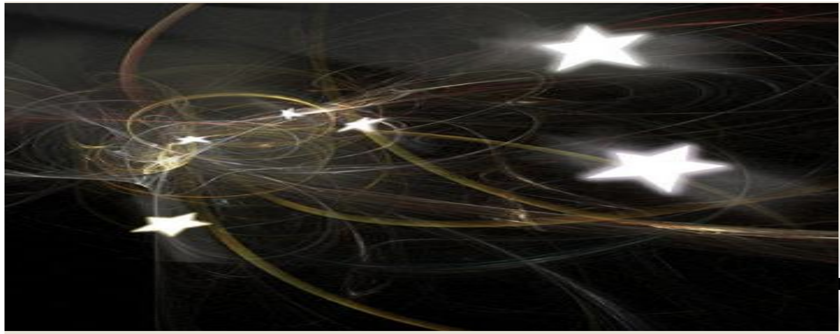
$$k^2 = 1.2637 k_J^2$$

However we can estimate also analytically the limit for the instability

In order to evaluate the Jeans mass limit in f(R)- gravity, we set $\omega = 0$

$$3\sigma^2 \alpha k^4 - (16\pi G\rho_0\alpha + \sigma^2)k^2 + 4\pi G\rho_0 = 0.$$

The additional condition $\alpha < 0$ discriminates the class of viable f(R) models: in such a case We obtain stable cosmological solution and positively defined massive states



The Jeans mass limit in f(R)-gravity

The condition $\alpha<0$ selects the physically viable models allowing to solve the above equation for real values of k.

In particular, the above numerical solution can be recast as $k^2 = \frac{2}{3}(3+\sqrt{21})\pi\frac{G\rho}{\sigma^2}$.

The relation to the Newtonian value of the Jeans instability is $k^2 = \frac{1}{6}(3+\sqrt{21})k_J^2$.

Now, we can define the new Jeans mass as $\tilde{M}_J = 6\sqrt{\frac{6}{(3+\sqrt{21})^3}}M_J$

which is proportional to the standard Newtonian value

These specific solutions can be confronted with some observed structures.

The M_J – T relation

One can deal with the star formation problem in two ways:

- We can take into account the formation of individual stars and
- We can discuss the formation of the whole star system starting from interstellar clouds

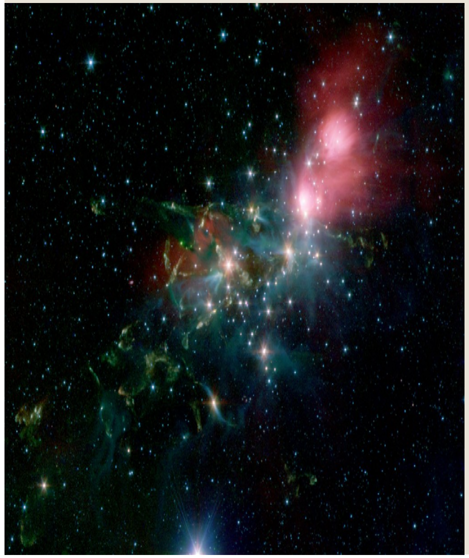
To answer these problems it is very important to study then interstellar medium (ISM) and its properties

The ISM physical conditions in the galaxies change in a very wide range, from hot X-ray emitting plasma to cold molecular gas, so it is very complicated to classify the ISM by its properties

The M_J – T relation

However, we can distinguish, in the first approximation, between

- **Diffuse hydrogen clouds**. The most powerful tool to measure the properties of these clouds is the 21 cm line emission of HI. They are cold clouds so the temperature is in the range 10 ÷ 50 K, and their extension is up to 50 ÷ 100 kpc from galactic center



- **Diffuse molecular clouds** are generally self-gravitating, magnetized, turbulent fluids systems, observed in sub-mm. The most of the molecular gas is H_2, and the rest is CO. Here, the conditions are very similar to the HI clouds but in this case, the cloud can be more massive. They have, typically, masses in the range $3 \div 100\ M_\odot$, temperature in $15 \div 50$ K and particle density in $(5 \div 50) \times 10^8\ m^{-3}$.

The M_J – T relation

- **Giant molecular clouds** are very large complexes of particles (dust and gas), in which the range of the masses is typically $10^5 \div 10^6 M_\odot$ but they are very cold. The temperature is ≈15 K, and the number of particles is $(1 \div 3) \times 10^8$ m^{-3}. However, there exist also small molecular clouds with masses $M < 10^4$ $M_\odot$. They are the best sites for star formation, despite the mechanism of formation does not recover the star formation rate that would be $250 M_\odot$ yr^{-1}

The $M_J - T$ relation

- **HII regions.** They are ISM regions with temperatures in the range $10^3 \div 10^4$ K, emitting primarily in the radio and IR regions. At low frequencies, observations are associated to free-free electron transition (thermal Bremsstrahlung). Their densities range from over a million particles per cm^3 in the ultracompact H II regions to only a few particles per cm^3 in the largest and most extended regions. This implies total masses between 10^2 and 10^5 $M_\odot$

- **Bok globules** are dark clouds of dense cosmic dust and gas in which star formation sometimes takes place. Bok globules are found within H II regions, and typically have a mass of about 2 to 50 $M_\odot$ contained within a region of about a light year.

The M_J – T relation

Using very general conditions, we want to show the difference in the Jeans mass Value between standard and f(R)- gravity.

Let us take into account $$M_J = \frac{\pi}{6}\sqrt{\frac{1}{\rho_0}\left(\frac{\pi\sigma^2}{G}\right)^3},$$

- in which ρ_o is the ISM density and σ is the velocity dispersion of particles due to the temperature

These two quantities are defined as $\rho_0 = m_H n_H \mu,$ and $\sigma^2 = \frac{k_B T}{m_H}$

Where $\mathbf{n_H}$ is the number of particles measured in $\mathbf{m}^{-3}$, is the mean molecular Weight, $\mathbf{k_B}$ is the Boltzmann constant and $\mathbf{m_H}$ is the proton mass

By using these relations, we are able to compute the Jeans mass for interstellar clouds and to plot its behavior against the temperature

The M_J – T relation

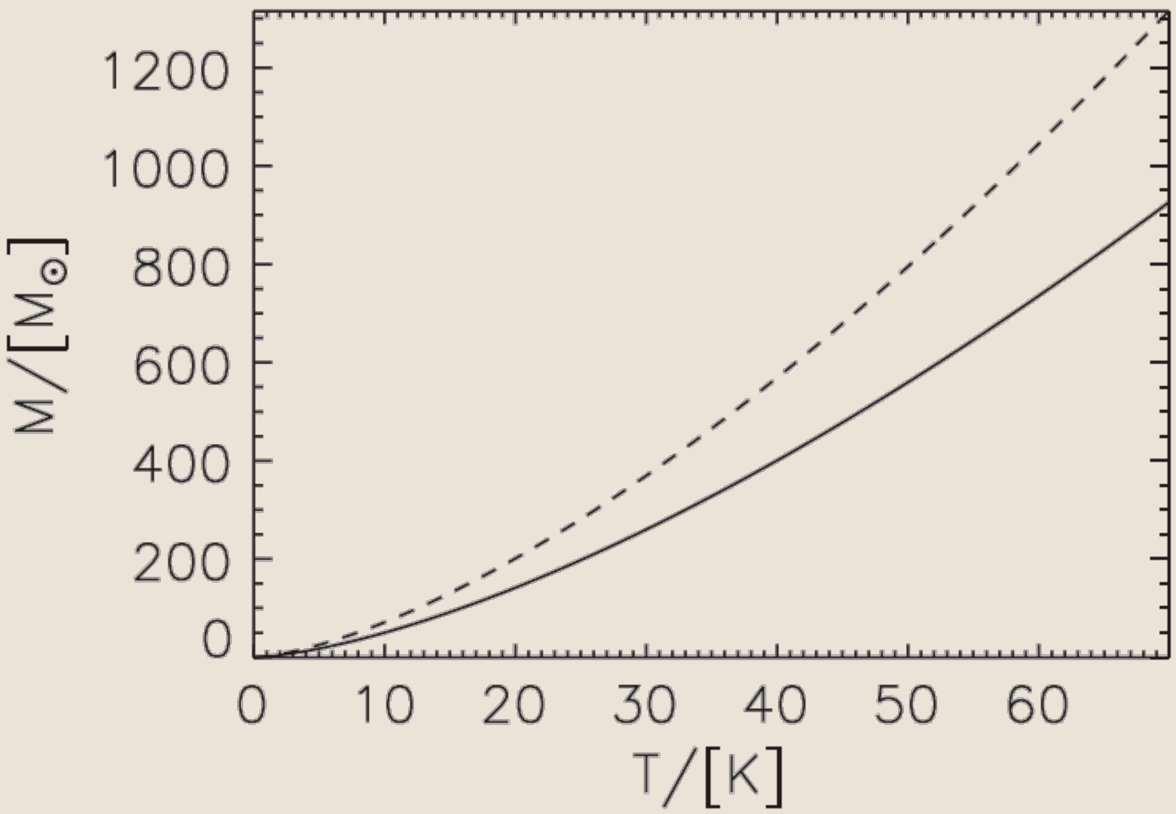
Any astrophysical system reported in Table is associated to a particular (M_J – T)-region.

Subject	T (K)	n (10^8 m^{-3})	μ	M_J ($M_\odot$)	$\tilde{M}_J$ ($M_\odot$)
Diffuse hydrogen clouds	50	5.0	1	795.13	559.68
Diffuse molecular clouds	30	50	2	82.63	58.16
Giant molecular clouds	15	1.0	2	206.58	145.41
Bok globules	10	100	2	11.24	7.91

Differences between the two theories for any self-gravitating system are clear

The $M_J - T$ relation

Dashed-line indicates the Newtonian Jeans mass behavior with respect to the temperature.
Continue-line indicates the same for *f*(R)-gravity Jeans mass.



The M_J – T relation

By referring to the catalog of molecular clouds in Roman-Duval et al., Astrophys. J. 723, 492 (2010), we have calculated the Jeans mass in the Newtonian and *f*(R) cases.

In all cases we note a substantial difference between the classical and *f*(R) value.

Subject	T K	n (10^8 m^{-3})	M_J ($M_\odot$)	$\tilde{M}_J$ ($M_\odot$)
GRSMC G 053.59 + 00.04	5.97	1.48	18.25	12.85
GRSMC G 049.49 − 00.41	6.48	1.54	21.32	15.00
GRSMC G 018.89 − 00.51	6.61	1.58	22.65	15.94
GRSMC G 030.49 − 00.36	7.05	1.66	22.81	16.06
GRSMC G 035.14 − 00.76	7.11	1.89	28.88	20.33
GRSMC G 034.24 + 00.14	7.15	2.04	29.61	20.84
GRSMC G 019.94 − 00.81	7.17	2.43	29.80	20.98
GRSMC G 038.94 − 00.46	7.35	2.61	31.27	22.01
GRSMC G 053.14 + 00.04	7.78	2.67	32.06	22.56
GRSMC G 022.44 + 00.34	7.83	2.79	32.78	23.08
GRSMC G 049.39 − 00.26	7.90	2.81	35.64	25.09
GRSMC G 019.39 − 00.01	7.99	2.87	35.84	25.23
GRSMC G 034.74 − 00.66	8.27	3.04	36.94	26.00
GRSMC G 023.04 − 00.41	8.28	3.06	38.22	26.90
GRSMC G 018.69 − 00.06	8.30	3.62	40.34	28.40
GRSMC G 023.24 − 00.36	8.57	3.75	41.10	28.93
GRSMC G 019.89 − 00.56	8.64	3.87	41.82	29.44
GRSMC G 022.04 + 00.19	8.69	4.41	47.02	33.10
GRSMC G 018.89 − 00.66	8.79	4.46	47.73	33.60
GRSMC G 023.34 − 00.21	8.87	4.99	48.98	34.48
GRSMC G 034.99 + 00.34	8.90	5.74	50.44	35.50
GRSMC G 029.64 − 00.61	8.90	6.14	55.41	39.00
GRSMC G 018.94 − 00.26	9.16	6.16	55.64	39.16
GRSMC G 024.94 − 00.16	9.17	6.93	56.81	39.99
GRSMC G 025.19 − 00.26	9.72	7.11	58.21	40.97
GRSMC G 019.84 − 00.41	9.97	11.3	58.52	41.19

Discussion and Conclusions

- The hydrostatic equilibrium of a stellar structure in the framework of $f(R)$ gravity has been considered.
- Adopting a polytropic equation of state relating the mass density to the pressure, we derive the modified Lane´-Emden equation and its solutions for $n = 0,1$ which can be compared to the analogous solutions coming from the Newtonian limit of GR
- When we consider the limit $f(R) \rightarrow R$, we obtain the standard hydrostatic equilibrium theory coming from GR
- A peculiarity of $f(R)$ gravity is the non-viability of the Gauss theorem, and then the modified Lane´-Emden equation is an integro-differential equation Where the mass distribution plays a crucial role
- The correlation between two points in the star is given by a Yukawa-like term of the corresponding Green function

Discussion and Conclusions

- We have analyzed the Jeans instability mechanism, adopted for star formation, considering the Newtonian approximation of f(R) gravity

- The related Boltzmann-Vlasov system leads to modified Poisson equations depending on the f(R) model

- In particular, it is possible to get a new dispersion relation where instability criterion results modified

- The leading parameter is α, i.e. the second derivative of the specific f(R) model. Standard Newtonian Jeans instability is immediately recovered for α=0 corresponding to the Hilbert-Einstein Lagrangian of GR.

- A new condition for the gravitational instability is derived, showing unstable modes with faster growth rates.

Discussion and Conclusions

- We can observe the instability decreases in $f(R)$- gravity: such decrease is related to a larger Jeans length and then to a lower Jeans mass
- We have also compared the behavior with the temperature of the Jeans mass for various types of interstellar molecular clouds
- In our model the limit (in unit of mass) to start the collapse of an interstellar cloud is lower than the classical one advantaging the structure formation.
- Real solutions for the Jean mass can be achieved only for $\alpha < 0$ and this result is in agreement with cosmology
- In particular, the condition $\alpha<0$ is essentials to set a well formulated and well-posed Cauchy problem in $f(R)$- gravity
- It is worth noticing that the Newtonian value is an upper limit for the Jean mass coinciding with $f(R) = R$
- Stellar structure can give a FUNDAMENTAL tool against Dark Side! See S. Capozziello and M. De Laurentis Ann. Der. Phys. 524 (2012) 545

Next Steps

- A next step is to derive self-consistent numerical solutions of the modified Lane´-Emden equation and build up realistic star models where further values of the polytropic index n and other physical parameters, e.g. temperature, opacity, transport of energy, are considered.

- These models are a challenging task, since, up to now, there is no self-consistent, final explanation for compact objects (e.g. neutron stars) with masses larger than Volkoff mass, while observational evidence widely indicates these objects. (e.g.- magnetars, variable stars, etc..)

- From an observational point of view, reliable constraints can be achieved by a careful analysis of the proto-stellar phase taking into account magnetic fields, turbulence and collisions.

- Final step: the f(R)-Hertzsprung-Russell diagram !