The functional RG and the C-function

Carlo Pagani

SISSA, Trieste

arxiv: 1312.7097

with Alessandro Codello and Giulio d'Odorico.

First FLAG meeting "The Quantum and Gravity"

28-30 May, Bologna.

Recent growing interest in general theorems for RG flows. In 4-d the "a-theorem" has been proved and the "c-theorem" recovered.

Komargodski and Schwimmer JHEP 1112 (2011) 099; Komargodski JHEP 1207 (2012) 069; Luty, Polchinski and Rattazzi JHEP 1301 (2013) 152.

It is difficult to find an explicit form for these functions. We will focus on 2-d problem.

It generally requires non-perturbative tools. Typically one considers the relation between the entanglement entropy and the c-anomaly.

Our main goal: explore the c-function by means of the Functional Renormalization Group.

Brief review of Effective Average Action (EAA) and Exact Renormalization Group Equation (ERGE)

Introduce the modified generating functional of connected Green's functions:

$$e^{W_k[J]} = \int \mathcal{D}\chi e^{-S[\chi]-\Delta_k S[\chi]+\int dx \chi J}$$

cutoff action which suppresses modes $p^2 < k^2$

$$\Delta_k S\left[\chi\right] = \frac{1}{2} \int \chi R_k \left[-\nabla^2\right] \chi$$

Brief review of Effective Average Action (EAA) and Exact Renormalization Group Equation (ERGE)

Introduce the modified generating functional of connected Green's functions:

$$e^{W_k[J]} = \int \mathcal{D}\chi e^{-S[\chi]-\Delta_k S[\chi]+\int dx \chi J}$$

cutoff action which suppresses modes $p^2 < k^2$

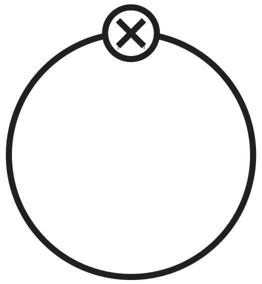
$$\Delta_k S\left[\chi\right] = \frac{1}{2}\int \chi R_k\left[-\nabla^2\right]\chi$$

Let $\tilde{\Gamma}_k[\phi]$ be the Legendre transform of $W_k[J(\phi)]$. The Effective Average Action is defined:

$$\Gamma_k[\phi] \equiv \tilde{\Gamma}_k[\phi] - \Delta_k S[\phi] \qquad \phi = \langle\chi\rangle$$

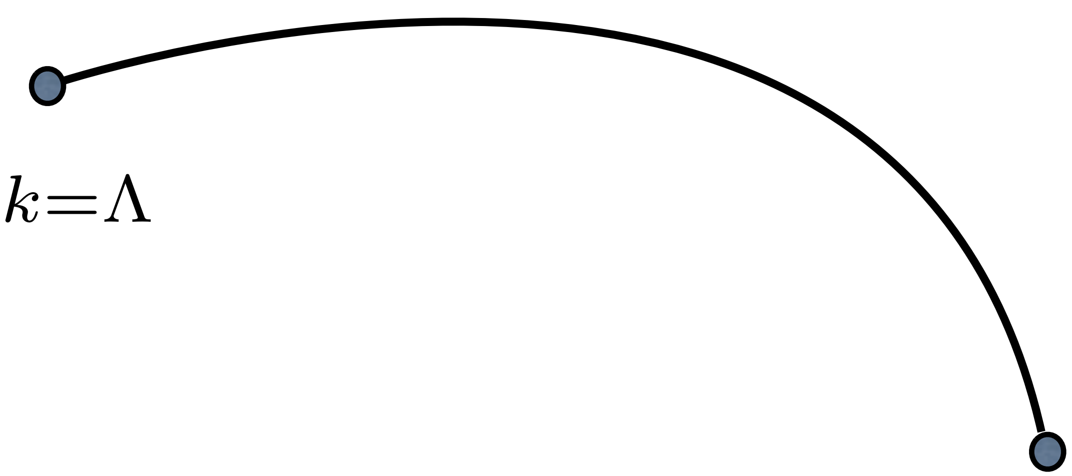
The scale dependence of Γ_k is given by the ERGE:

$$k\frac{\partial}{\partial k}\Gamma_k = \frac{1}{2}\mathrm{Tr}\left[\frac{k\partial_k R_k}{\Gamma_k^{(2)}+R_k}\right] = \frac{1}{2}$$



It is an exact equation which can be seen as a renormalization group improvement of its 1-loop analogue.

It smoothly interpolates between the microscopic action



$\Gamma_{k=\Lambda}$

$\Gamma_{k=0}$ and the Effective Action.

Let us consider the Wess-Zumino action:

(P.O. Mazur and E. Mottola, Phys. Rev. D 64 (2001) 104022)

$$\Gamma_{UV}\left(e^{2\sigma}g_{\mu\nu}, e^{w\sigma}\phi\right) - \Gamma_{UV}\left(g_{\mu\nu},\phi\right) = -\tfrac{c_{UV}}{24\pi}\int\sqrt{g}\left[\sigma\Delta\sigma + \sigma R\right]$$

$k = 0$

$$\Gamma_{IR}\left(e^{2\sigma}g_{\mu\nu}, e^{w\sigma}\phi\right) - \Gamma_{IR}\left(g_{\mu\nu},\phi\right) = -\tfrac{c_{IR}}{24\pi}\int\sqrt{g}\left[\sigma\Delta\sigma + \sigma R\right]$$

Let us consider the Wess-Zumino action:

(P.O. Mazur and E. Mottola, Phys. Rev. D 64 (2001) 104022)

$$\Gamma_{UV}\left(e^{2\sigma}g_{\mu\nu}, e^{w\sigma}\phi\right) - \Gamma_{UV}\left(g_{\mu\nu}, \phi\right) = -\frac{c_{UV}}{24\pi}\int\sqrt{g}\left[\sigma\Delta\sigma + \sigma R\right]$$

$$\Gamma_k\left(e^{2\sigma}g_{\mu\nu}, e^{w\sigma}\phi\right) - \Gamma_k\left(g_{\mu\nu}, \phi\right) = -\frac{c_k}{24\pi}\int\sqrt{g}\left[\sigma\Delta\sigma + \sigma R\right] - \int\sqrt{g}\sigma k^{d_i}\tilde{\beta}_i\mathcal{O}_i$$

$k = 0$

$$\Gamma_{IR}\left(e^{2\sigma}g_{\mu\nu}, e^{w\sigma}\phi\right) - \Gamma_{IR}\left(g_{\mu\nu}, \phi\right) = -\frac{c_{IR}}{24\pi}\int\sqrt{g}\left[\sigma\Delta\sigma + \sigma R\right]$$

EAA and the 3 clues

1. Derivative expansion.

2. The scale anomaly.

3. The conformal anomaly.

1. Derivative expansion.

Expand the EAA in a basis of operators compatible with the symmetries of the system:

$$\Gamma_k = \sum_i \int d^2x \sqrt{g}\, g_i \mathcal{O}_i$$

Good approximation to describe critical properties!

Clue 1:

$$\Gamma_k = \int d^2x \sqrt{g}\, \left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + V_k(\phi)\right]$$

↑ Local Potential Approximation (LPA)

2. Scale anomaly

Scale symmetry is broken at the quantum level (flat space).

$$S_0 = \int d^4x \sqrt{g} \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{g}{4!} \phi^4 \right]$$

$$\delta_\sigma S_0 = 0, \qquad g_{\mu\nu} \to \Omega^2 g_{\mu\nu},\ \phi \to \Omega^{-1} \phi$$

but

$$\delta_\sigma \left(S_0 + \Gamma_{1l} \right) = \delta_\sigma \left(-\frac{1}{2} \int_{\frac{\varepsilon}{\mu^2}}^{\infty} \frac{ds}{s} \mathrm{Tr} \left[e^{-s(\Delta+U)} \right] \right) = - \int_{\frac{\varepsilon}{\mu^2}}^{\infty} ds \sigma \mathrm{Tr} \left[(\Delta + U) \, e^{-s(\Delta+U)} \right]$$

$$= \int_{\frac{\varepsilon}{\mu^2}}^{\infty} ds \sigma \frac{d}{ds} \mathrm{Tr} e^{-s(\Delta+U)} = - \mathrm{Tr} \left[\sigma e^{-s(\Delta+U)} \right]$$

$$\to - \frac{1}{16\pi^2} \int \sqrt{g} \, b_4(\Delta + U) = -\frac{1}{4!} \left(\frac{3g^2}{16\pi^2} \right) \phi^4 = -\frac{1}{4!} \beta_g \phi^4$$

3. Conformal anomaly

Massless scalar field:

$$S_0 = \int d^2x \sqrt{g} \frac{1}{2} \phi \Delta \phi$$

$$\delta\Gamma = -\frac{1}{(4\pi)} \int \sqrt{g} b_2 = -\frac{1}{(24\pi)} \int \sqrt{g} R$$

The Effective Action reads:

$$\Gamma = \int \sqrt{g} \frac{1}{2} \phi \Delta \phi - \frac{c}{96\pi} \int \sqrt{g} R \frac{1}{\Delta} R$$

↑ can be found using ERGE plus non-local Heat Kernel (A. Codello Ann. Phys. 325 (2010) 1727)

An ansatz satisfying the requirments

Suitable extension of the LPA

$$\Gamma_k = \int \sqrt{g}\left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + V_k(\phi)\right] + \cdots$$

Using:

$$\delta_\sigma\left(\tfrac{1}{2\Delta}R\right) = \tfrac{1}{2\Delta}R + \sigma$$

$$\Gamma_k = \int \sqrt{g}\left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \textstyle\sum_i \lambda_i\mathcal{O}_i\right] \longleftarrow \text{Clue 1}$$

$$-\int \sqrt{g}\beta_i\mathcal{O}_i\frac{1}{2\Delta}R \longleftarrow \text{Clue 2}$$

$$-\frac{c}{96\pi}\int \sqrt{g}R\frac{1}{\Delta}R \longleftarrow \text{Clue 3}$$

Functional equation for the c-function

Structure of the EAA:

$$\begin{aligned}\Gamma_k &= \int d^2x\sqrt{g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V_k(\phi)\right.\\ &- \left.\partial_t V_k(\phi)\frac{1}{2\Delta}R - \frac{c_k}{96\pi}R\frac{1}{\Delta}R\right]\end{aligned}$$

Choose a convenient background

$$e^{w\tau}\phi, \quad e^{2\tau}\delta_{\mu\nu}$$

$$-\int \tau k^{d_i}\tilde{\beta}_i\mathcal{O}_i, \qquad -\frac{c_k}{96\pi}(2\Delta\tau)\frac{1}{\Delta}(2\Delta\tau) = -\frac{c_k}{24\pi}\tau\Delta\tau$$

Functional equation for the c-function

Structure of the EAA:

$$\begin{aligned}\Gamma_k &= \int d^2x\sqrt{g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V_k(\phi)\right.\\ &- \left.\partial_t V_k(\phi)\frac{1}{2\Delta}R - \frac{c_k}{96\pi}R\frac{1}{\Delta}R\right]\end{aligned}$$

Flow equation for the c-function:

$$\partial_t c_k = \partial_t\left[-12\pi\frac{d}{dp^2}\left(\frac{\delta\Gamma_k\left(e^{w\tau}\phi, e^{2\tau}\delta_{\mu\nu}\right)}{\delta\tau(p)\delta\tau(-p)}\right)\right]_{\phi=0}$$

Exact checks: massive scalar field

Add a mass term to the Gaussian fixed-point action:

$$S_* = \int d^2x\sqrt{g}\,\frac{1}{2}\phi\Delta\phi \longrightarrow S = \int d^2x\sqrt{g}\left[\frac{1}{2}\phi\Delta\phi + \frac{1}{2}m^2\phi^2\right]$$

$$c_{UV} = 1 \longrightarrow c_{IR} = 0$$

Improve using the general structure of the ansatz:

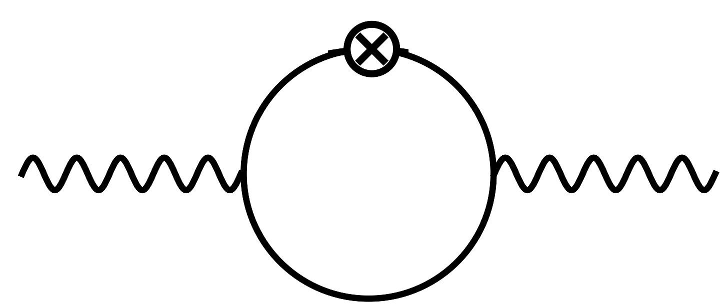
$$\begin{aligned}\Gamma_k &= \int d^2x\sqrt{g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V_k(\phi)\right.\\ &- \left.\partial_t V_k(\phi)\frac{1}{2\Delta}R - \frac{c_k}{96\pi}R\frac{1}{\Delta}R\right]\end{aligned}$$

Rescaling both the field and the metric we can compute $\langle \tau\tau \rangle \Big|_{p^2}$ in flat space:

$$\Gamma_k = \int d^2x \sqrt{g} \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{c_k}{96\pi} R \frac{1}{\Delta} R \right]$$
$$= \int d^2x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}e^{2\tau}m^2\phi^2 - \frac{c_k}{24\pi} \tau\Delta\tau \right]$$

Using the ERGE for the functional derivatives of the EAA:

$$\partial_t \left(\frac{\delta^2\Gamma_k}{\delta\tau(p)\delta\tau(-p)} \right)\Big|_{p^2} =$$



We solve the equation with the boundary condition $c_\infty = 1$

$$\partial_t c_k = \frac{4ak^2m^2}{(ak^2+m^2)^3} \qquad c_k = 1 - \frac{m^4}{(ak^2+m^2)^2}$$

The Sine-Gordon model

Perturb the Gaussian fixed-point action:

$$S = \int \sqrt{g} \left[\frac{1}{2} \phi \Delta \phi - \frac{m}{\beta^2} \left(\cos\left(\beta \phi\right) - 1 \right) \right]$$

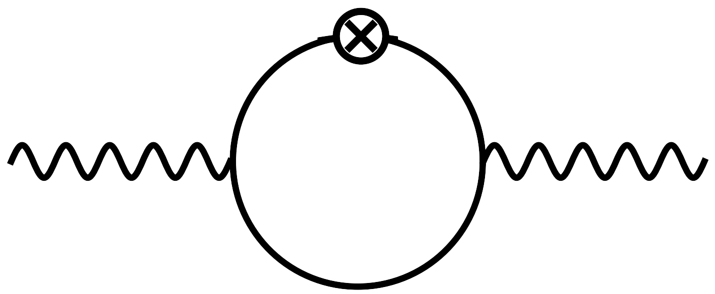
$$\Delta c = 1$$

The beta functions are:

$$\partial_t \tilde{m} = \frac{\tilde{m}}{4\pi} \left(\frac{\beta^2 - 8\pi(1+\tilde{m})}{1+\tilde{m}} \right)$$

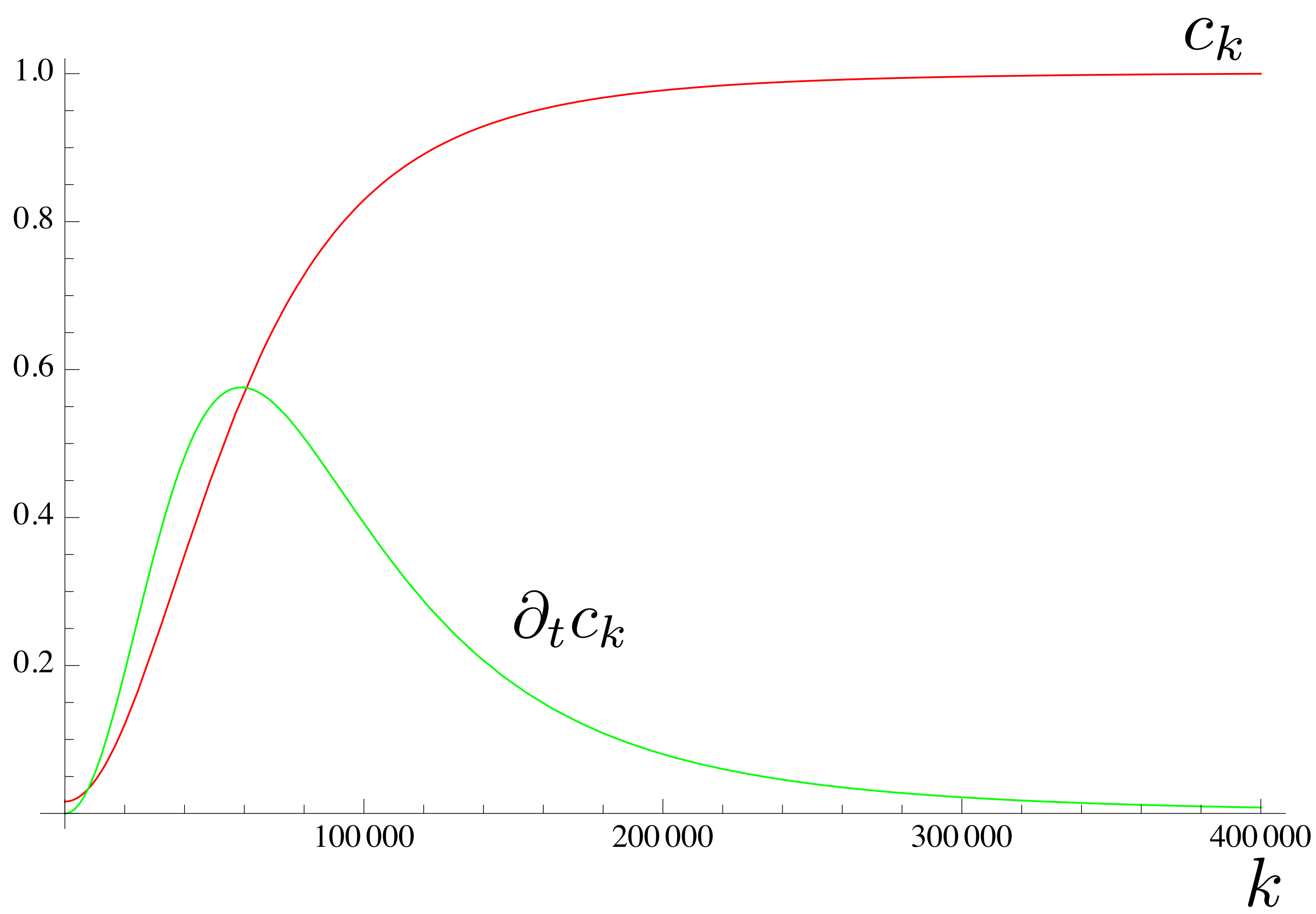
$$\partial_t \beta = -\frac{3\tilde{m}\beta^3}{8\pi(1+\tilde{m})^3}$$

Using the extended ansatz we find:



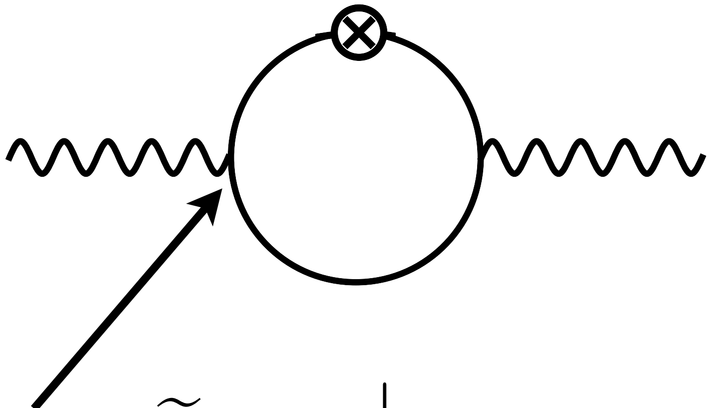
$$\partial_t c_k = \frac{\tilde{m}^2 \left(\beta^2 - 8\pi(1+\tilde{m})\right)^2}{16\pi^2(1+\tilde{m})^5}$$

Numerical integration gives satisfactory agreement:



Loop expansion and the C-function

Up to now we limited ourselves to 1 loop results:



$\sim \partial_t \widetilde{V}_k'' (\phi)\Big|_{\phi=0}$

For instance:

$$V_k(\phi) = \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 + \cdots \quad \Rightarrow \quad \text{vertex} \sim \tilde{\beta}_{m^2}$$

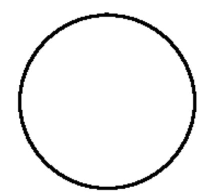
What about the others beta functions? $\partial_t c_k = G_{ij} \tilde{\beta}_i \tilde{\beta}_j$

Loop expansion from the EAA

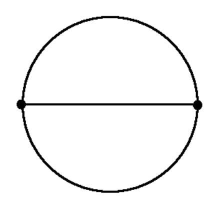
It is possible to recover the loop expansion of the EA from the EAA.

(See: D. F. Limit and J. M. Pawlowski, PRD 66, 025030 (2002), hep-th/0202188 and A. Codello, M. Demmel and O. Zanusso arXiv: 1310.7625).

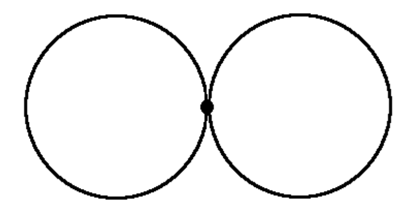
$$\frac{1}{2}\mathrm{Tr}\log\left(\Delta + V''\right) \longleftrightarrow$$



$$\Gamma_{2loop} \longleftrightarrow -\frac{1}{12}$$

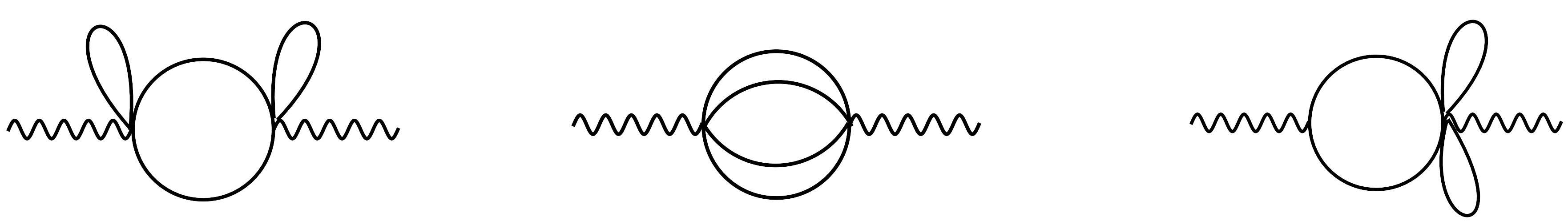


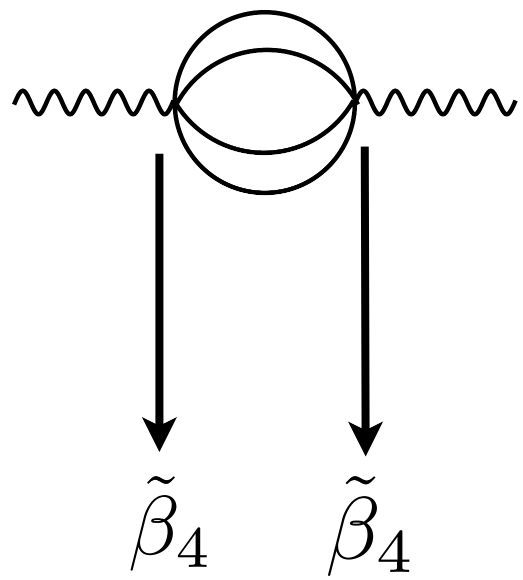
$$+\frac{1}{8}$$



$$\vdots$$

This expansion is generalized to a loop expansion of the EAA.



$\partial_t c_k \sim$ 

$\longleftrightarrow$ $\partial_t c_k = G_{ij}\tilde{\beta}_i\tilde{\beta}_j$

The calculation has to be repeated for all the relevant diagrams to build an estimate for the Zamolodchikov's metric.

Non unitary theories: Yang-Lee model

$$V(\phi) = ih\phi + ig\phi^3$$

change in the sign of the entries.

Conclusions

We investigated a general form of the EAA which takes into account scale and conformal anomalies.

We set up a non perturbative functional equation for the c-function.

From this generalized ansatz we can find non-trivial flow for the c-function. Both from 2-point function and loop expansion.

Strategy may be generalized to 4 dimensions.

Conclusions

We investigated a general form of the EAA which takes into account scale and conformal anomalies.

We set up a non perturbative functional equation for the c-function.

From this generalized ansatz we can find non-trivial flow for the c-function. Both from 2-point function and loop expansion.

Strategy may be generalized to 4 dimensions.

Thank you!!