2. Reaction models (advanced)

1. Overview of scattering theories
2. The CDCC method (Continuum Discretized Coupled Channel)
3. Technical aspects: R-matrix, Lagrange meshes
4. The Eikonal method

1. Introduction

Main goal in nuclear physics: understanding the structure of exotic nuclei

Available data:

- Elastic (inelastic) scattering, Breakup, Fusion, etc.

Role of the theory: how to interpret these scattering data?

projectile
→ structure??

target
- heavy nucleus
- considered as inert

Combination of two ingredients

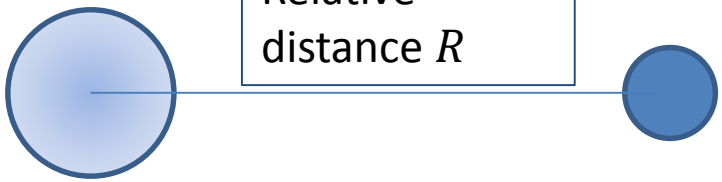
1. **Description of the scattering process**
   - Low energies (around the Coulomb barrier): optical model, CDCC
   - High energies (typically ~50-100 MeV/u): eikonal (+variants)
2. **Description of the projectile**
   - 2-body, 3-body, A-body,….
   - Microscopic, non-microscopic
   - Need of bound states and scattering states (difficult for 3-body systems)

2. Overview of scattering theories

Schrödinger equation: $H\Psi(\boldsymbol{\xi_i}, \boldsymbol{R}) = E\Psi(\boldsymbol{\xi_i}, \boldsymbol{R})$

$\boldsymbol{\xi_i}$: set of internal coordinates in the projectile

$\boldsymbol{R}$:describes the relative motion between the target and the projectile



General expression for the Hamiltonian H: common to all models

$$H = H_0(\xi_i) + T_R + \sum_i V(\xi_i, R)$$

projectile

Relative kinetic energy

Interaction between target and projectile

Projectile + heavy target

even at low energies: many open channels → absorption in $V(\xi_i, R)$

2. Overview of scattering theories

Various scattering theories

$$H = H_0(\xi_i) + T_R + \sum_i V(\xi_i, R)$$

1. **Coupled-channel methods**:
   - No internal coordinates , $H_0(\xi_i) = 0$
   - Simple but limited (no halo structure, no continuum effects)

   - Folding procedures:

$$V(R) = \int \rho_1(r)\rho_2(s)v(R - r + s)drds$$

   With $\rho_1(r), \rho_2(s) =$ densities of nuclei 1 and 2 (from experiment or from theory)
   $v(x) =$nucleon-nucleon interaction (M3Y, DDM3Y, JLM, etc.)

   - Can be extended to multichannel calculations
   - Example: S. Ohkuko, Y. Hirabayashi, Phys. Lett. B684 (2010) 127
     investigate ^{16}O condensate states in a multichannel model
     ($\alpha +$ $^{12}C(0_1^+, 0_2^+, 2_1^+, 2_2^+, 3^-)$ with ^{12}C RGM densities (from M. Kamimura)

2. Overview of scattering theories

2. CDCC: Continuum Discretized Coupled Channel

Valid at low energies (partial-wave expansion of $\Psi(\boldsymbol{\xi_i}, \boldsymbol{R})$)

G. Rawitscher, Phys. Rev. C 9, 2210 (1974)

N. Austern et al., Phys. Rep. 154 (1987) 126

- First step: solutions of $H_0(\xi_i)\phi_k(\xi_i) = E_{0k}\phi_k(\xi_i)$
  $E_{0k} < 0$:physical state(s), $E_{0k} > 0$: pseudostates (=approximation of the continuum)

- Second step: expansion of the total wave function

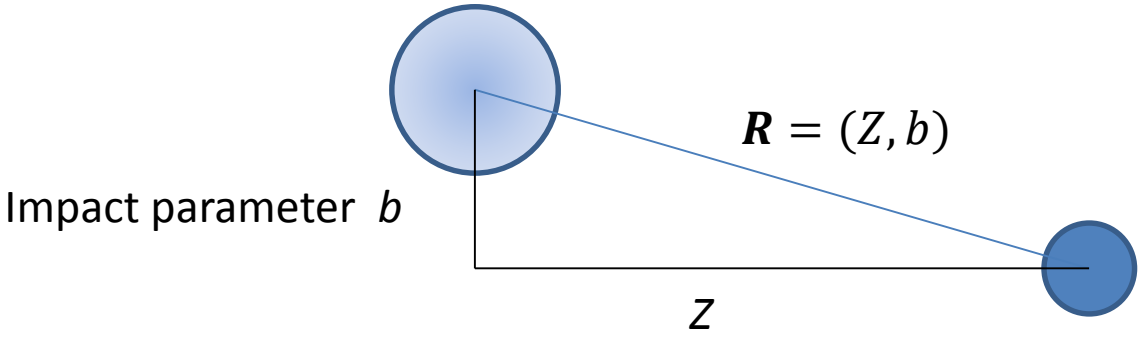
$$\Psi(\boldsymbol{\xi_i}, \boldsymbol{R}) = \sum_k \phi_{0k}(\boldsymbol{\xi_i}) u_k(\boldsymbol{R})$$

→ system of coupled equations

- Third step : solving the system (positive energy E)
  → phase shifts, scattering cross sections
  - Two-body projectiles: (d=p+n, ^{7}Li=α+t, ^{11}Be=^{10}Be+n, etc): relatively simple
  - Three-body projectiles (^{6}He=α+n+n, ^{9}Be=α+α+n): more complicated!
  - Recent developments: A-nucleon projectile (cluster approximation: ^{7}Li)

3. Eikonal method

- Valid at high energies (k large)
- Wave function factorized as $\Psi(\boldsymbol{\xi_i}, \boldsymbol{R}) = \exp(ikZ) \times \widehat{\Psi}(\boldsymbol{\xi_i}, \boldsymbol{R})$



- For k large: the SE is simplified → allows more sophisticated descriptions of the projectile
- *Standard eikonal method*: only ground state of the projectile (=adiabatic approximation: excitation time long compared to the interaction time)
- *Dynamic eikonal*: excited states are included (D. Baye et al., PRL 95 (2005) 082502)
- *Eikonal CDCC*: the projectile is described over a basis (~dynamic eikonal) (Hashimoto et al., Phys. Rev. C 83, 054617 (2011))
- *Extensions possible to relativistic energies* (K. Ogata, C. Bertulani, PTP 123 (2010) 701)
- *Microscopic eikonal* (E.C. Pinilla, P. Descouvemont, Phys. Lett. B 686(2010) 124)

3. The CDCC method

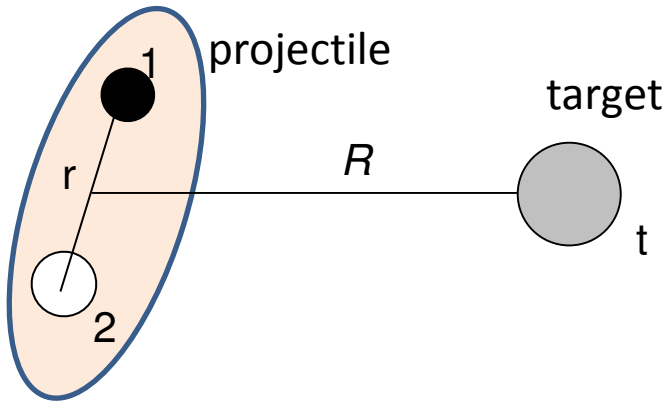
CDCC=Continuum Discretized Coupled Channels

3. The CDCC method

Goal of CDCC (Continuum Discretized Coupled Channels) theory

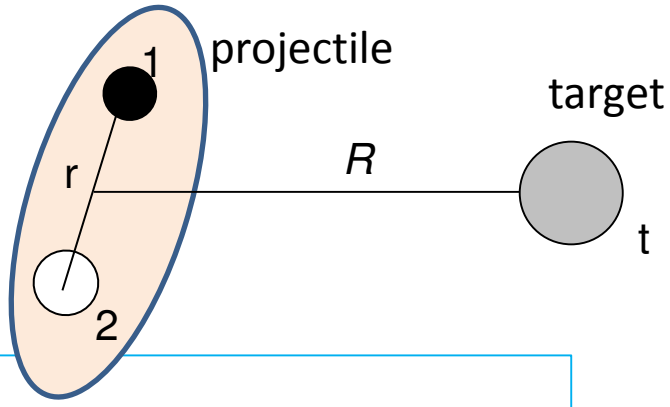
Solution of 3-body (4-body) scattering problem

G. Rawitscher, Phys. Rev. C 9, 2210 (1974)
M. Kamimura et al, Prog. Theor. Phys. Suppl. 89 (1986) 1
N. Austern et al., Phys. Rep. 154 (1987) 126



- Projectile assumed to be described by two clusters
- Introduced for deuteron-induced reactions (breakup important)
  → well adapted to exotic nuclei
- Various cross sections: elastic, breakup inelastic, fusion, etc.
  (**Energy near the Coulomb barrier**)
- Recently extended to 3-body projectiles (^{6}He=α+n+n)

3. The CDCC method



Hamiltonian of the system

$$H = H_0(\boldsymbol{r}) - \frac{\hbar^2}{2\mu}\Delta_{\boldsymbol{R}} + V_{t1}\left(\boldsymbol{R} + \frac{A_2}{A_p}\boldsymbol{r}\right) + V_{t2}\left(\boldsymbol{R} - \frac{A_1}{A_p}\boldsymbol{r}\right)$$

V_{t1}, V_{t2}= optical potentials (typically: Woods-Saxon)
from elastic scattering between particules 1 or 2 and the target
➔ need for elastic-scattering data!

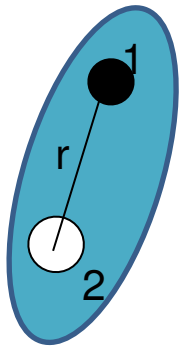
Two steps:

1. Diagonalize H_0 :
   $H_0\Phi^{jm}(\boldsymbol{r}) = E^j\Phi^{jm}(\boldsymbol{r})$ → wave function of the (2-body) projectile $\Phi^{jm}(\boldsymbol{r})$

2. Expand the total wave function over this basis
   → wave function of the system projectile+target $\Psi^{JM\pi}(\boldsymbol{R},\boldsymbol{r})$
   → system of coupled differential equations

3. The CDCC method

First step: diagonalization of the **projectile** hamiltonian H_0



$H_0\Phi_n^{jm}(\boldsymbol{r}) = E_n^j\Phi_n^{jm}(\boldsymbol{r})$, with $\Phi_n^{jm}(\boldsymbol{r}) = \varphi_n^j(r)Y_j^m(\Omega_r)$

j=spin of the projectile (ground-state + excited states + breakup channels)
n=excitation level in partial wave j

- Expand $\varphi_n^j(r)$ on a basis (N functions):

$$\varphi_n^j(r) = \sum_{k=1}^{N} c_{nk} f_k(r)$$

example: gaussians $f_k(r) = \exp(-(r/a_i)^2)$

- Solve the eigenvalue problem:

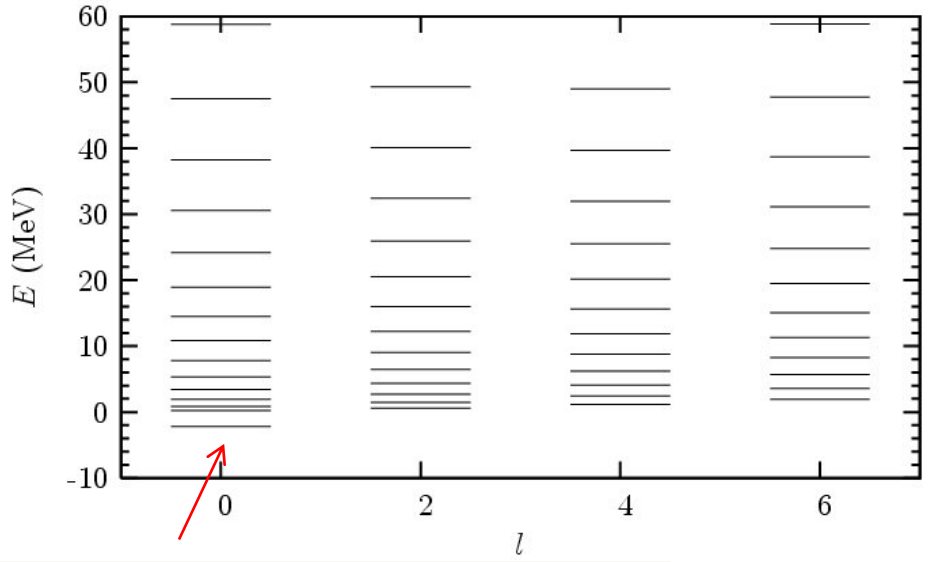
$$\sum_{k'} \left(< f_k|H|f_{k'} > - E_n^j < f_k \middle| f_{k'} > \right) c_{nk'} = 0$$

with $< f_k\,|f_{k'} > = \int f_k(r) f_{k'}(r) r^2 dr$

- Solutions with $E_n^j < 0$: physical states
  Solutions with $E_n^j > 0$: pseudostates: simulate breakup effects

3. The CDCC method

Example: deuteron=p+n



- Pseudostates : E_n>0
- Simulate breakup effects
- Depend on the basis

- Ground state: E_n<0
- Independent of the basis

3. The CDCC method

2nd step:

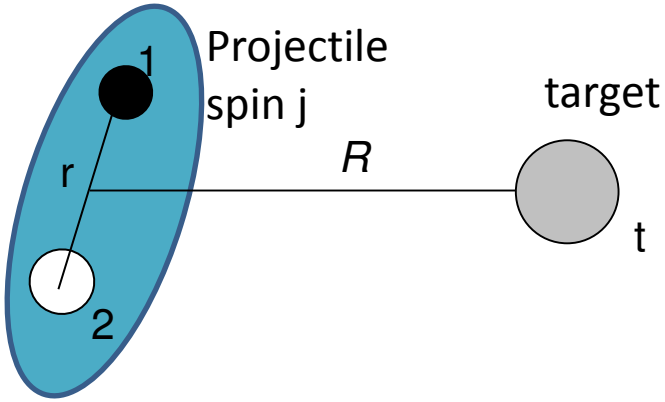
1. Expand the total wave function (projectile + target)

$$\Psi^{JM\pi}(\boldsymbol{R},\boldsymbol{r}) = \sum_{jLn} u_{jLn}^{J\pi}(R)\left[\Phi_n^j(\boldsymbol{r}) \otimes Y_L(\Omega_R)\right]^{JM}$$

To be determined

2-body wave functions

Angular functions



→ Multichannel wave function:

j=spin of the projectile

n=excitation level

L=projectile-target angular momentum

3. The CDCC method

2. Expand the potential in multipoles (numerical integration over the angle)

$$V_{t1}\left(\boldsymbol{R}+\frac{A_2}{A_p}\boldsymbol{r}\right)+V_{t2}\left(\boldsymbol{R}-\frac{A_1}{A_p}\boldsymbol{r}\right)=\sum_{\lambda}V_{\lambda}(r,R)P_{\lambda}(\cos\theta)$$

3. Insert both expansions in the Schrödinger associated with H

$$H=H_0(\boldsymbol{r})-\frac{\hbar^2}{2\mu}\Delta_{\boldsymbol{R}}+V_{t1}\left(\boldsymbol{R}+\frac{A_2}{A_p}\boldsymbol{r}\right)+V_{t2}\left(\boldsymbol{R}-\frac{A_1}{A_p}\boldsymbol{r}\right)$$

➔ System of coupled equations: provide the radial functions

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dR^2}-\frac{L(L+1)}{R^2}\right)+E_c-E\right]u_c^{J\pi}(R)+\sum_{c'}V_{cc'}^{J\pi}(R)u_{c'}^{J\pi}(R)=0$$

=standard coupled-channel system (general form common to most scattering theories)

In CDCC (2-body projectiles) the potentials are obtained from
$V_{cc'}^{J\pi}(R)\sim\int\varphi_n^j(r)V_{\lambda}(r,R)\varphi_n^j(r)dr$ (factors: 6j, Clebsch-Gordan, etc…)

Channel c= j: projectile quantum numbers
n: excitation level of the projectile [physical state (E<0) or PS (E>0)]
L : orbital angular momentum between projectile and target

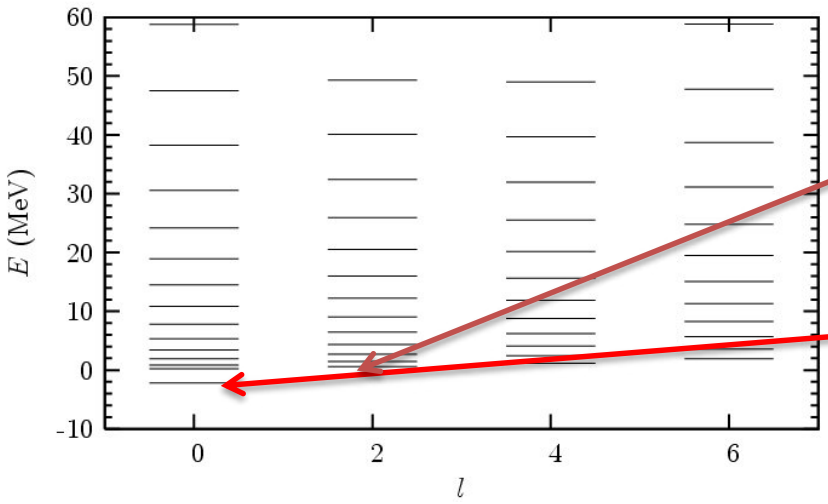
Number of c values: typically ~100-200

3. The CDCC method

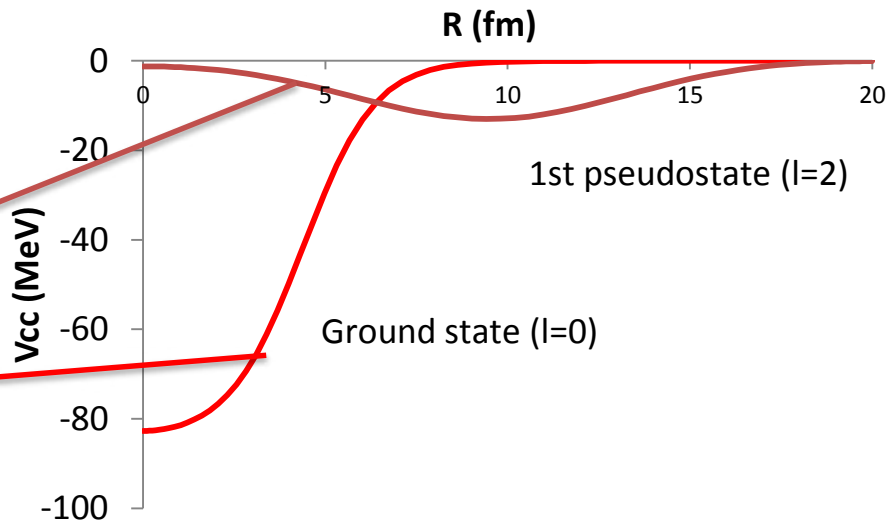
Test case: d+^{58}Ni at E_{lab}=80 MeV

- Standard benchmark in the literature
- p+n: gaussian potential (reproduces E_d=-2.22 MeV)
- p+^{58}Ni, n+^{58}Ni: optical potentials
- Elastic scattering and breakup

deuteron basis:
$l = 0,2,4,6$



d+^{58}Ni potentials (nuclear)



→ long range for PS

3. The CDCC method

Role of the continuum

Elastic scattering (entrance channel 1): element U_{11} needed

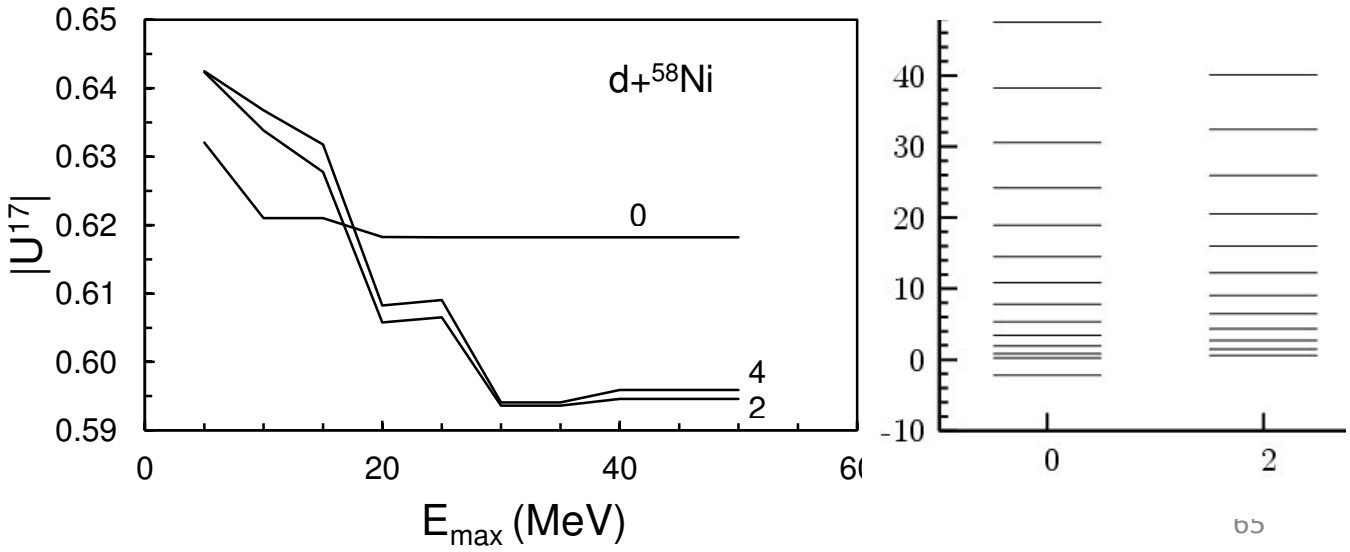
Single channel: matrix 1x1 $\begin{pmatrix} U_{11} \end{pmatrix}$ → Provides elastic cross sections in a **single**-channel calculation

Multichannel: matrix NxN $\begin{pmatrix} U_{11} & \cdots & U_{1N} \\ \vdots & \ddots & \vdots \\ U_{N1} & \cdots & U_{NN} \end{pmatrix}$

Provides elastic c.s. in a **multi**-channel calculation

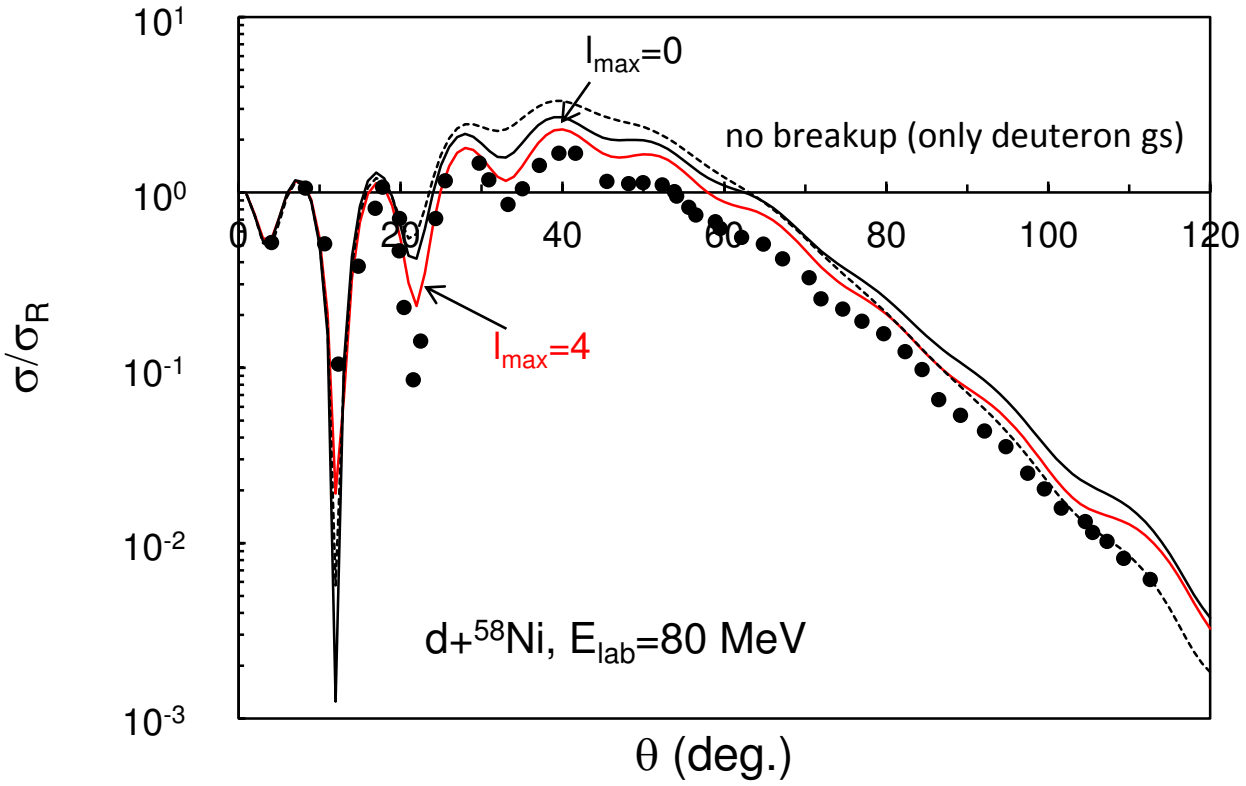
Couplings to the continuum: **modify U_{11}**

Example: d+^{58}Ni, J=17



3. The CDCC method

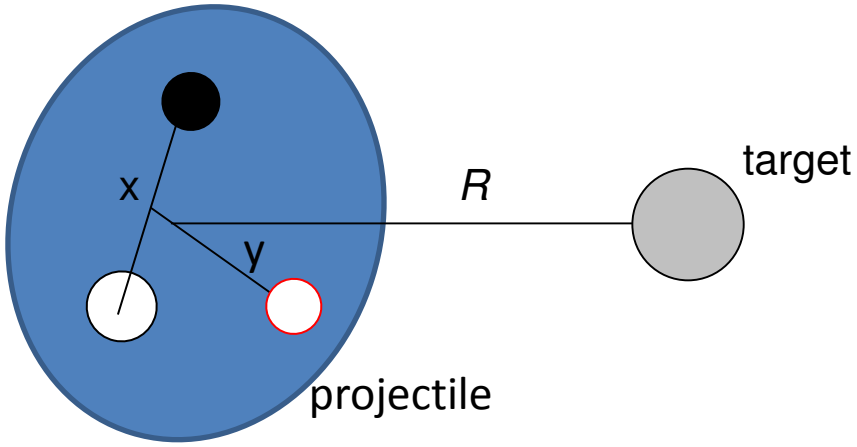
Convergence with the p+n angular momentum



➔ importance of p+n breakup channels

3. The CDCC method

Extension to three-body projectiles: $H(x, y, R) = H_0(x, y) + T_R + V_{t1} + V_{t2} + V_{t3}$



- Previous calculations
  T. Matsumoto et al., Phys. Rev. C70 (2004) 061601
  M. Rodriguez-Gallardo et al., Phys. Rev. C77 (2008) 064609

- Projectile can be described in the hyperspherical harmonics (ex:^{6}He=α+n+n, ^{9}Be=α+α+n)

- Same principle:
  - Eigenfunctions of the projectile Hamiltonian $H_0(x, y)$
  - Expansion of the total wave function

- Coupling potentials more complicated

- Similar coupled-channel system

Technical aspects associated with CDCC

- Solving the coupled-channel equations → R-matrix method
- Basis to expand the wave functions → Lagrange meshes

4. Technical aspects associated with CDCC: the R-matrix method

- Goal: to solve a coupled-channel problem for positive energies

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}\right)u_c^{J\pi}(r)+\sum_{c'}V_{c,c'}^{J\pi}(r)u_{c'}^{J\pi}(r)=(\boldsymbol{E}-E_c)u_c^{J\pi}(r)$$

c= channel
$u_c^{J\pi}(r)$= wave function in channel c

- Equation common to many problems (only the potentials $V_{c,c'}^{J\pi}(r)$ are different)
- In general many states to simulate the continuum → many coupled equations
- Negative energies $\boldsymbol{E}$ (bound states): variational methods
- Positive energies $\boldsymbol{E}$ (scattering states, resonances): more difficult

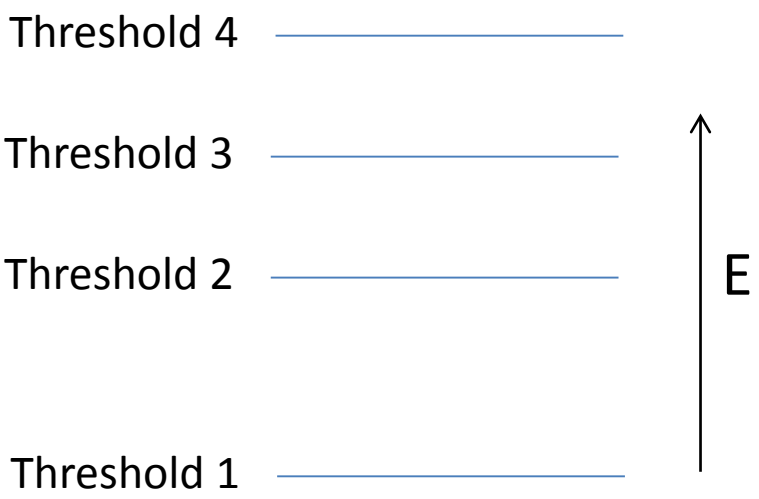
4. Technical aspects associated with CDCC: the R-matrix method

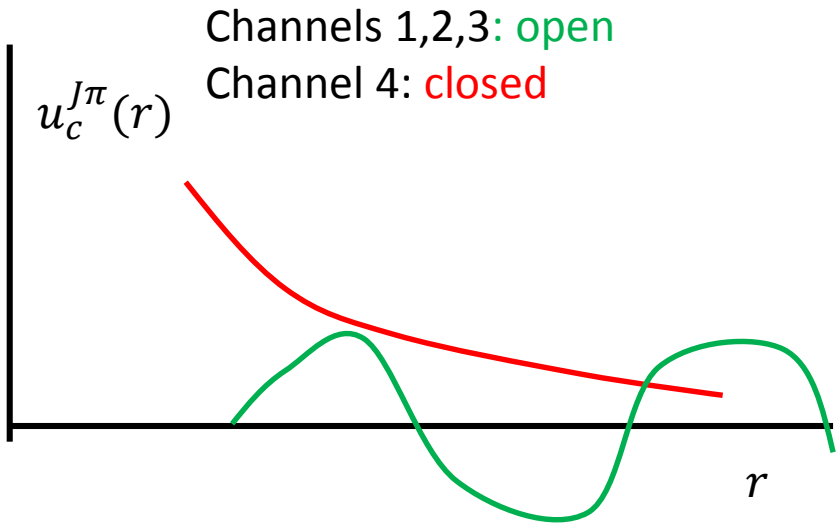
Two options:

1. discretization method (FRESCO)
2. **R-matrix** (adopted here)

R-matrix or discretization method?

- Still an open question…
- Should provide the same results → tests
- Obvious advantage of the R-matrix: treatment of closed channels





Closed channels: $u_c^{J\pi}(r) \to 0$,

- Unstable in discretization methods
- Automatic in the R-matrix

4. Technical aspects associated with CDCC: the R-matrix method

Introduction – framework

- Presented for 2 structureless particles
  one channel (simple presentation)
  potential model → $(\boldsymbol{T+V-E})\boldsymbol{u(r)=0}$

- N basis functions $\phi_i(r)$: variational calculations: $u(r)=\sum_{i=1}^{N} c_i\phi_i(r)$
  tend to zero at large distances
  valid at short distances only

- Definition of 2 regions: radius a (**channel radius**)

internal region: $r \le a$

$$u_{int}(r)=\sum_{i=1}^{N} c_i\phi_i(r)$$

external region: $r \ge a$

$$u_{ext}(r)=I_\ell(r)-U_\ell O_\ell(r)$$

- Schrödinger equation + Matching at $r=a$
  → collision matrix U (~phase shift) and coefficients c_i (wave function)
  !! Channel radius a is not a parameter!!

4. Technical aspects associated with CDCC: the R-matrix method

Procedure

Step 1: Compute the matrix elements of the Hamiltonian over the internal region

$$H_{ij} = <\phi_i|\, H|\phi_j>_{int} = \int_0^a \phi_i(r)(T+V)\phi_j(r)dr$$

$$N_{ij} = <\phi_i|\phi_j>_{int} = \int_0^a \phi_i(r)\phi_j(r)dr$$

Problem: the kinetic energy is not hermitian over a finite interval [0,a]

$$\int_0^a \phi_i(r)\frac{d^2}{dr^2}\phi_j(r)dr \neq \int_0^a \phi_j(r)\frac{d^2}{dr^2}\phi_i(r)dr$$

→ Bloch operator

$$\mathcal{L}(L) = \frac{\hbar^2}{2\mu a}\delta(r-a)\left(\frac{d}{dr} - \frac{L}{r}\right)r$$

L=arbitrary constant (L=0 in most cases)

4. Technical aspects associated with CDCC: the R-matrix method

role of the surface operator $\mathcal{L}(L) = \frac{\hbar^2}{2\mu a}\delta(r-a)\left(\frac{d}{dr} - \frac{L}{r}\right)r$

1. makes $\mathrm{T} + \mathcal{L}(L)$ hermitian

$$< \phi_i | \, T + \mathcal{L}(L) | \phi_j >_{int} = < \phi_j | \, T + \mathcal{L}(L) | \phi_i >_{int}$$

2. The Schrödinger equation

$$(H - E)u_\ell = 0$$

is replaced by the Bloch-Schrödinger equation

$$(H - E + \mathcal{L}(L))u_{int} = \mathcal{L}(L)u_{int} = \mathcal{L}(L)u_{ext}$$

→the Bloch operator ensures $u'_{int}(a) = u'_{ext}(a)$ (for the exact solution)

« Old » use of the R-matrix theory

- No Bloch operator
- Basis states such that $\phi_i{}'(a) = 0$
  - →Hermiticity OK
  - →but the total wave function has always a derivative =0

4. Technical aspects associated with CDCC: the R-matrix method

Step 2: Determine the matrix elements

$$C_{ij}(E) = < \phi_i | \, H + \mathcal{L}(L) - E | \phi_j >_{int}$$

Step 3: Determine the R matrix (size 1x1 for single-channel problems) at energy E

$$R(E) = \frac{\hbar^2 a}{2\mu} \sum_{ij} \phi_i(a) \, (C^{-1})_{ij} \phi_j(a)$$

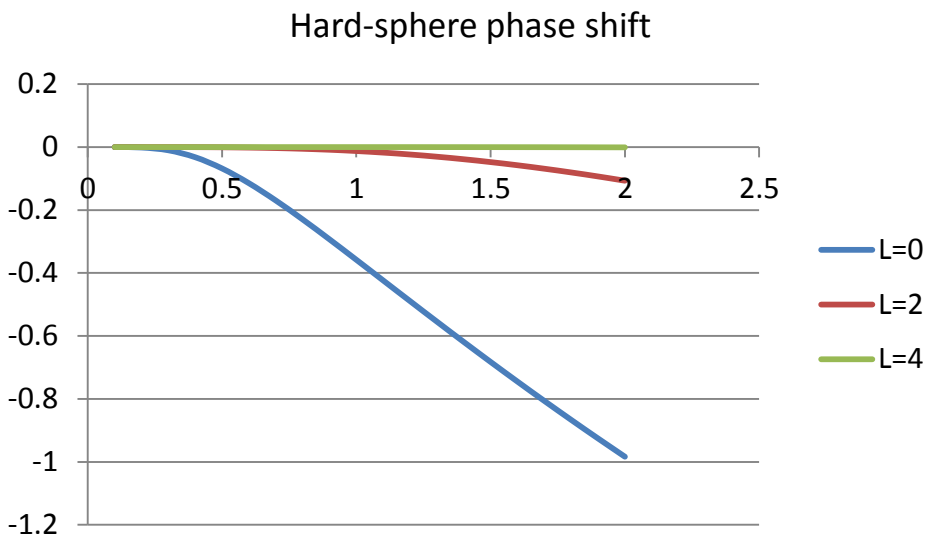
Step 4: Determine the collision matrix from the R matrix

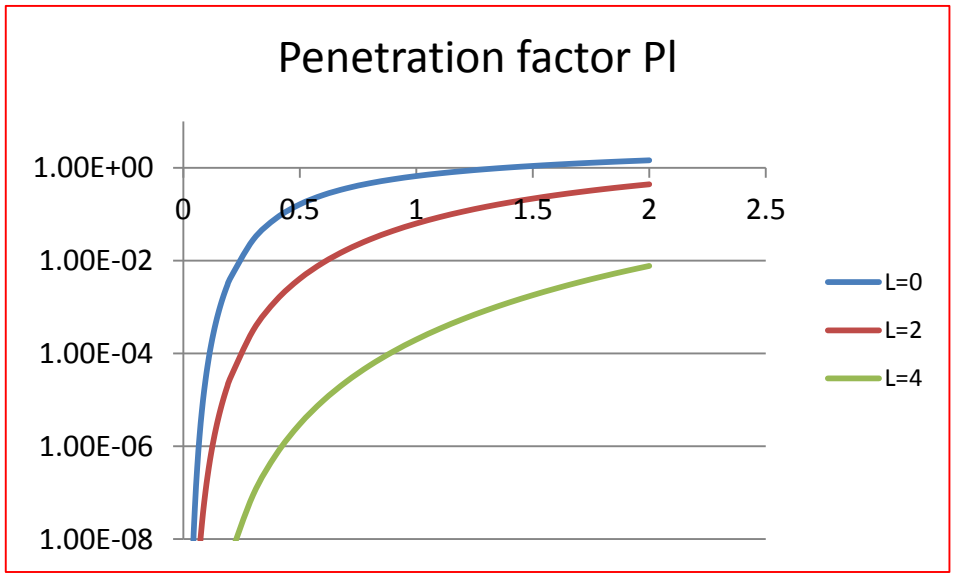
$$U(E) = \frac{I(ka)}{O(ka)} \frac{1 - L^* R(E)}{1 - L R(E)} = \exp(2i\delta) = \exp(2i\delta_{HS}) \exp(2i\delta_R)$$

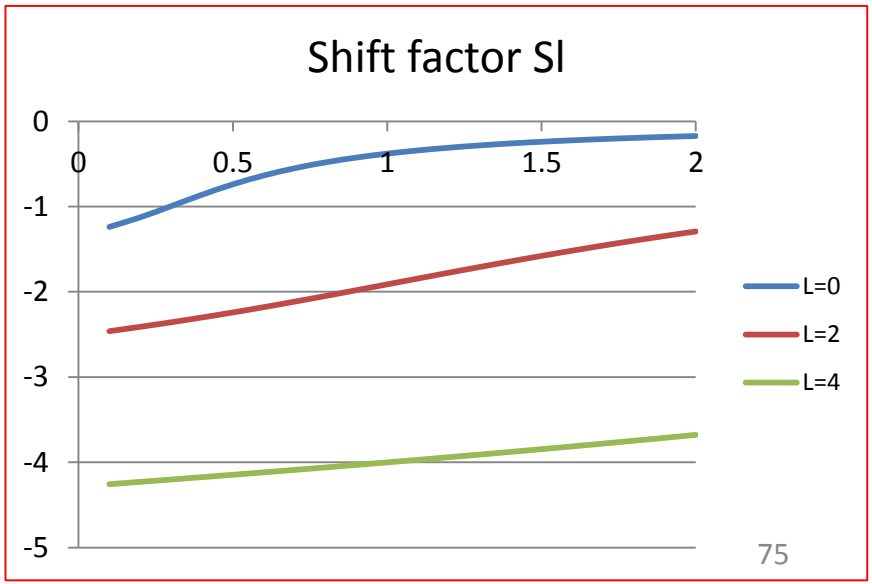
- Hard-sphere phase shift: $\exp(2i\delta_{HS}) = \frac{I(ka)}{O(ka)} \rightarrow \delta_{HS} = -\operatorname{atan} \frac{F(ka)}{G(ka)}$
- R-matrix phase shift: $\exp(2i\delta_R) = \frac{1 - L^* R(E)}{1 - L R(E)} \rightarrow \delta_R = \operatorname{atan} \frac{PR}{1 - SR}$
- Both depend on a but **the sum should not depend on a**
- $L = ka \frac{O'(ka)}{O(ka)} = S(E) + iP(E)$ (L is complex

4. Technical aspects associated with CDCC: the R-matrix method

Example: $\alpha+^3He$
(a=5 fm, Coulomb barrier ~1 MeV)







4. Technical aspects associated with CDCC: the R-matrix method

Extension to multichannel

Single channel: $C_{ij}(E) =< \phi_i | \, T + V + \mathcal{L}(L) - E | \phi_j >_{int}$

$$R(E) = \frac{\hbar^2 a}{2\mu} \sum_{ij} \phi_i(a) \, (C^{-1})_{ij} \phi_j(a)$$

Multichannel: $C_{ci,c'j}(E) =< \phi_i | \, (T + \mathcal{L}(L) + E_c - E)\delta_{cc'} + V_{cc'} | \phi_j >_{int}$

$$R_{cc'}(E) = \frac{\hbar^2 a}{2\mu} \sum_{ij} \phi_i(a) \, (C^{-1})_{ci,c'j} \phi_j(a)$$

size of the R-matrix= number of channels c (+angular momentum couplings)

- Typical number for CDCC ~100-200
- Typical number of basis states ϕ_i: ~30-40

→ inversion of matrix C requires long computer times (complex)

4. Technical aspects associated with CDCC: The Lagrange-mesh method

The Lagrange-mesh method

- Ref: D. Baye, Phys. Stat. Sol. 243 (2006) 1095)
- used in many different fields (convenient variational basis)

- Gauss approximation: $\int_0^a g(x)dx \approx \sum_{k=1}^{N} \lambda_k g(x_k)$
  - N= order of the Gauss approximation
  - x_k=roots of an orthogonal polynomial, λ_k=weights
  - If interval [0,a]: Legendre polynomials
    [0,∞]: Laguerre polynomials

- Lagrange functions for [0,1]: $f_i(x) \sim P_N(2x-1)/(x-xi)$
  - x_i are roots of $P_N(2x_i - 1) = 0$
  - with the Lagrange property: $f_i(x_j) = \lambda_i^{-1/2}\delta_{ij}$

4. Technical aspects associated with CDCC: The Lagrange-mesh method

- Matrix elements AT THE GAUSS APPROXIMATION

- Overlap $< f_i|f_j > = \int f_i(x) f_j(x) dx \approx \sum_{k=1}^{N} \lambda_k f_i(x_k) f_j(x_k) \approx \delta_{ij}$ (since $f_i(x_j) = \lambda_i^{-1/2} \delta_{ij}$)

- Kinetic energy $< f_i|T|f_j >$
  - More complicated
  - Analytical calculation
  - Must be performed once (independent of the potential)

- Potential

$$< f_i|V|f_j > = \int f_i(x) V(x) f_j(x) dx \approx \sum_{k=1}^{N} \lambda_k f_i(x_k) f_j(x_k) V(x_k) \approx V(x_i) \delta_{ij}$$

$\Rightarrow$ **no integral needed**
very simple!

- The Gauss approximation must be valid →regularization for Coulomb, centrifugal

$f_i(x) \to \left(\frac{x}{x_i}\right)^n f_i(x)$, with n=1/2, 1, etc.

4. Technical aspects associated with CDCC: The Lagrange-mesh method

Application to the R-matrix: Matrix elements

$$C_{ci,c'j}(E) = < \phi_i | \, (T + \mathcal{L}(L) + E_c - E)\delta_{cc'} + V_{cc'} | \phi_j >_{int}$$

- In general: integral over *R* (from *R=0* to *R=a*)
  Lagrange mesh: value of the potential at the mesh points ➔ very fast
- Typical sizes: ~100-200 channels c, N~30-40
- Test: collision matrix ***U*** does not depend on *N, a*
- Choice of *a*: compromize (*a* too small: R-matrix not valid, *a* too large: *N* large)

- Some remarks
  - ☺ Fast (no integral), accurate
  - ☺ The basis is defined by *N* and *a* only : no further parameter (≠ for gaussians)
  - ☹ Legendre functions: more zeros near *R=0*, and *R=a* → not optimal

4. Technical aspects associated with CDCC: The Lagrange-mesh method

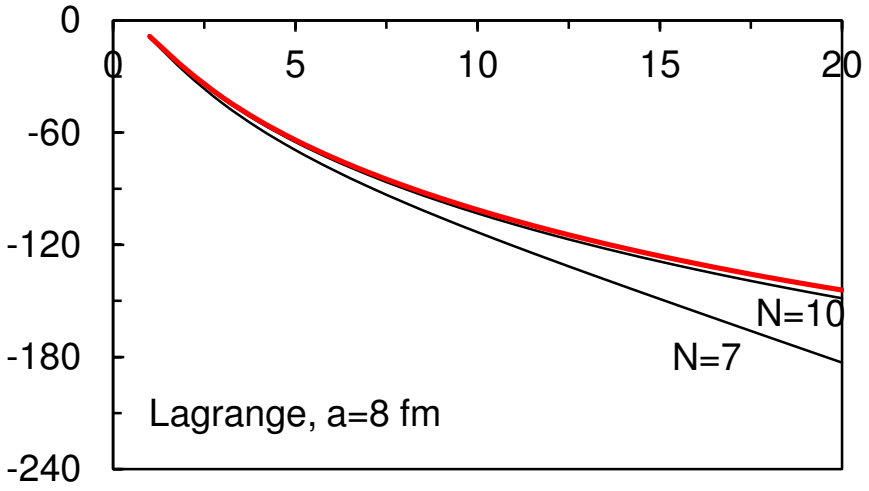
Simple example: potential α+^{3}He : gaussian: $V_N(r) = -66.1\ \exp(-(r/2.52)^2)$

2 bases:

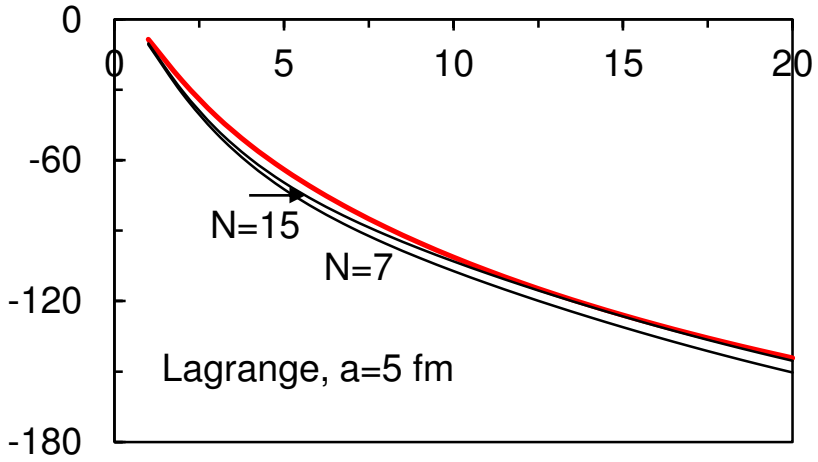
- Lagrange functions
- sine functions $u_i^\ell(r) = \sin\frac{\pi r}{a}(i - \frac{1}{2})$
  - Matrix elements very simple
  - Derivative $u_i'(a) = 0$

4. Technical aspects associated with CDCC: The Lagrange-mesh method

$l = 0$ phase shifts α+^{3}He



a=8 fm: Fast convergence



a=5 fm: too small (V_N not negligible)

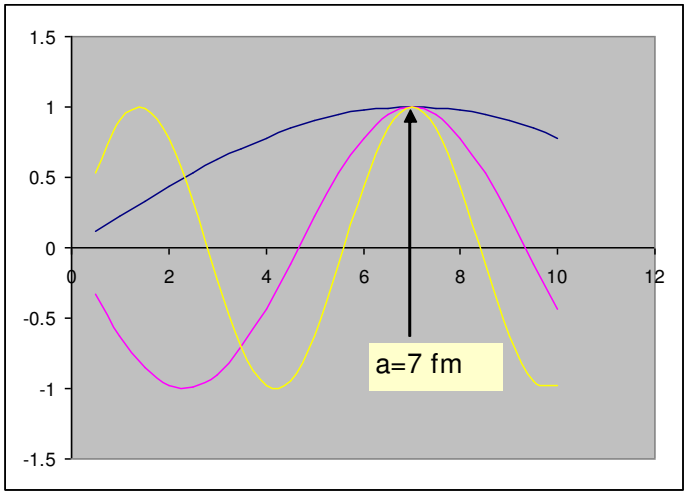
In general: a too small → not in the R-matrix conditions (V_N negligible)

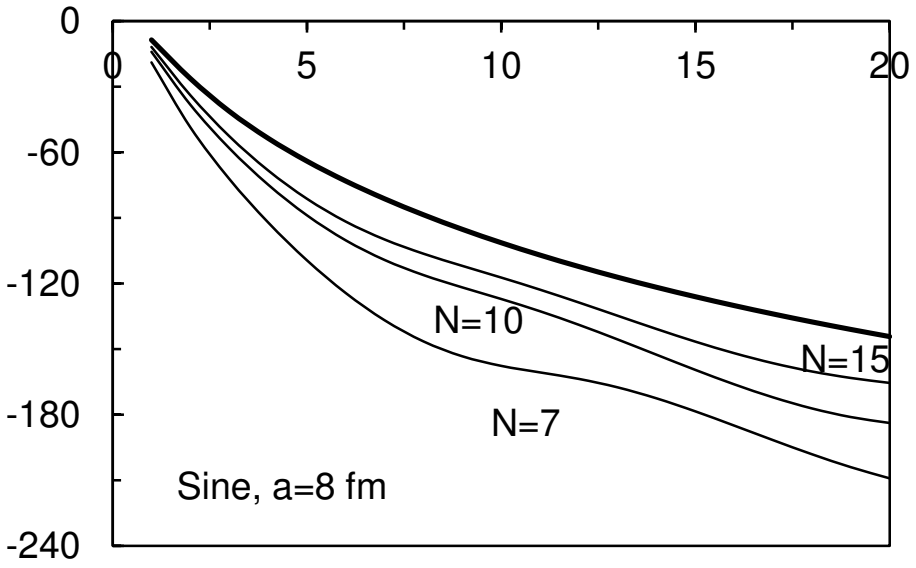
a too large: needs many basis functions

➔ compromize

4. Technical aspects associated with CDCC: The Lagrange-mesh method

sine functions: $u_i'(a) = 0$
→ not flexibility

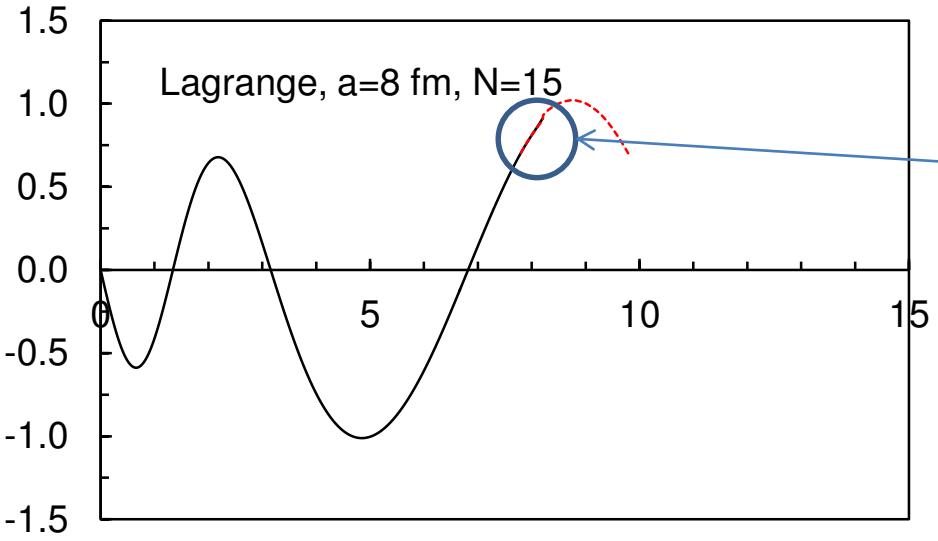




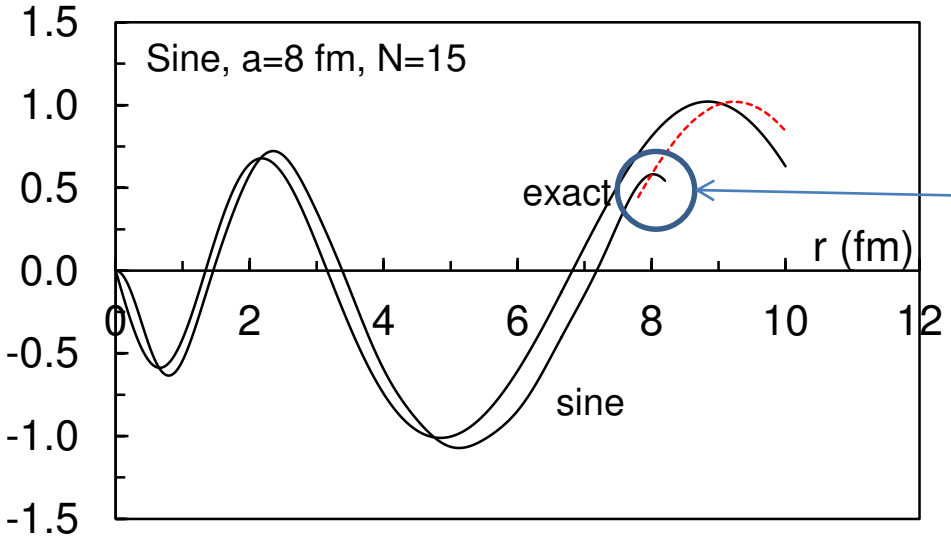
very slow convergence
of the phase shift

4. Technical aspects associated with CDCC: The Lagrange-mesh method

Analysis of the wave functions (E=8 MeV, a=8fm)



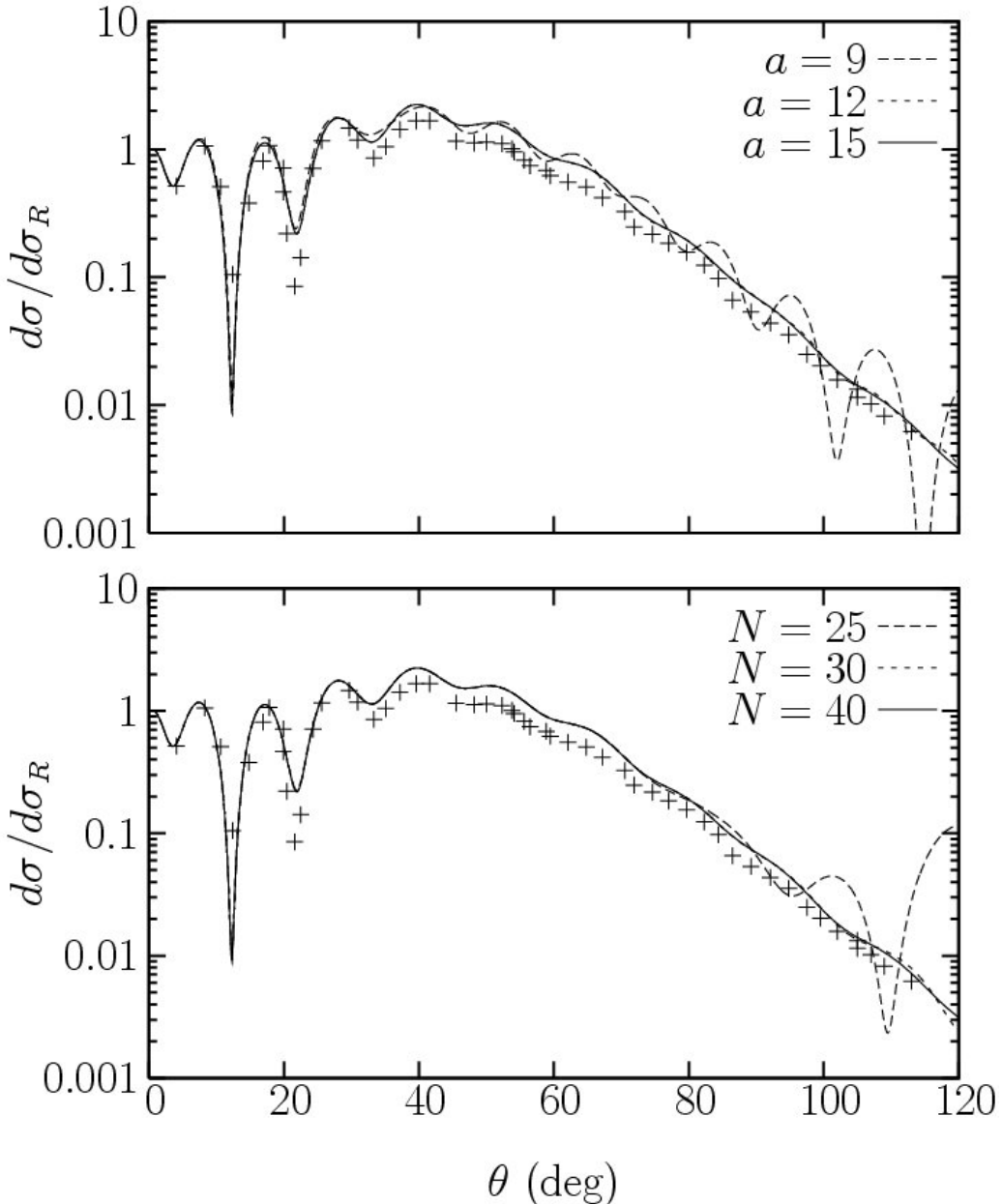
excellent matching at r=a
→ accurate phase shift



poor matching at r=a
→ inaccurate phase shift

4. Technical aspects associated with CDCC: The Lagrange-mesh method

Application of the R-matrix to CDCC: Test case: d+^{58}Ni at E_{lab}=80 MeV



channel radius *a (N=30)*

Number of basis (Lagrange) functions *N (a=15* fm)

Computer time: 2 main parts

- Matrix elements: very fast with Lagrange functions
- Inversion of (complex) matrix C → R-matrix (long times for large matrices)

For reactions involving halo nuclei:

- Long range of the potentials (Coulomb)

$$\frac{Z_1 Z_t e^2}{\boldsymbol{R} + \frac{A_2}{A_p}\boldsymbol{r}} + \frac{Z_2 Z_t e^2}{\boldsymbol{R} - \frac{A_1}{A_p}\boldsymbol{r}} = \sum_\lambda V_\lambda(r,R) P_\lambda(\cos\theta_{Rr})$$

$$V_{cc'}(R) \approx \frac{Z_p Z_t e^2}{R} + \frac{Z_t Q_p}{R^3} + \ldots$$

Can be large (large quadrupole moments of PS)

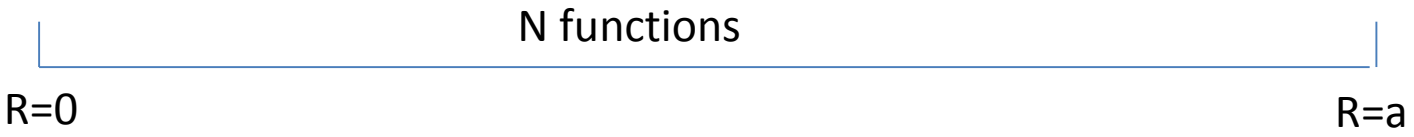
- Radius *a* must be large
- Many basis functions (*N* large)

→

- Distorted Coulomb functions (FRESCO)
- Propagation techniques in the R-matrix (well known in atomic physics)

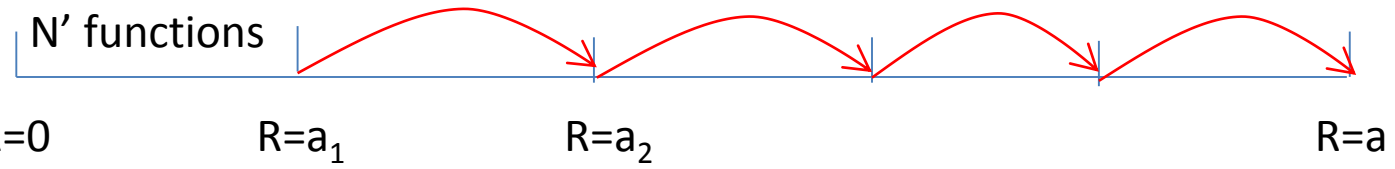
  Ref.: Baluja et al. Comp. Phys. Comm. 27 (1982) 299

Without propagation



- Matrix elements integrated over [0,a]
- Inversion of a matrix of dimension N x N_c

With propagation



- The interval [0,a] is split in N_S subintervals
- In each subinterval N'~N/N_S: $\langle \varphi_n \mid (T_c + \mathcal{L}_c + E_c - E)\delta_{cc'} + V_{cc'}^{J\pi} \mid \varphi_{n'} \rangle_{a_i - a_{i+1}}$
  - Interval 1: determine R(a_1)
  - Interval 2 : R(a_2) from R(a_1)
  - Interval N_S: R(a) from R(a_{NS-1})

  } (inversion of a matrix with size N' x N_c)

➔ N_S smaller calculations:

Lagrange functions well adapted to matrix elements over [a_i,a_{i+1}]

5. The eikonal method

5. The eikonal method

Ref: *R.J. Glauber, Lect. Notes in Physics, Vol. 1 (1959)*
Y. Suzuki et al., "Structure and reactions with light exotic nuclei" (2003)
M. Yahiro et al. Prog. Theor. Phys. 126 (2011) 167

- The wave function is factorized as $\Psi(r) = \exp(ikz)\,\hat{\Psi}(r)$

- The main energy dependence (high E) is in $\exp(ikz)$→ factorization

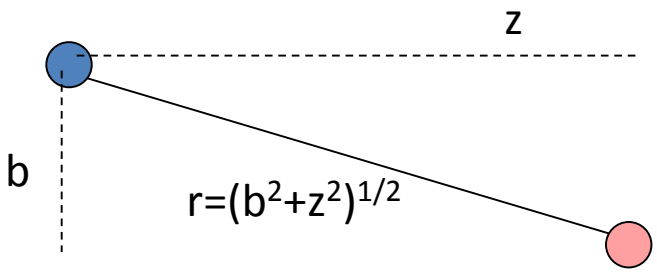
- Then $-\frac{\hbar^2}{2\mu}\Delta\hat{\psi}(r) - i\hbar v\frac{d}{dz}\hat{\psi}(r) + V(r)\hat{\psi}(r) = 0$

= Eikonal approximation (first-order equation) : valid at high energy

1. One-body projectile: very simple (elastic only)
2. Two and three-body projectile:
   more complicated
   but: elastic, inelastic, breakup, etc.

5. The eikonal method

One-body projectile: Only elastic scattering



$$-i\hbar v\frac{d}{dz}\hat{\psi}(r)+V(r)\hat{\psi}(r)=0$$

$$\hat{\psi}(r)=\exp(-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,z')dz)$$

$$f(\theta)\approx ik\int_0^{\infty}J_0(qb)(1-e^{i\chi(b)})b\,db$$

$$\chi(b)=-\frac{1}{\hbar v}\int_{-\infty}^{\infty}V(b,z)dz=\text{eikonal phase}$$

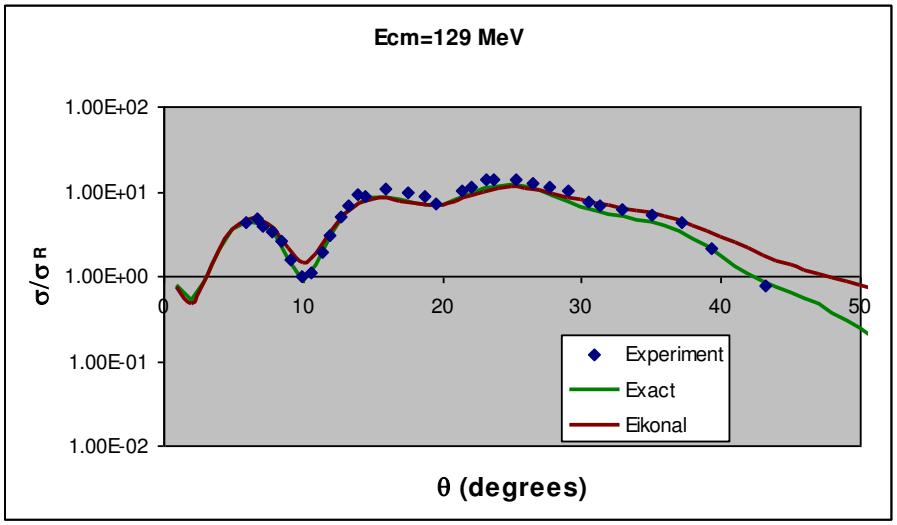
With b=transverse part of r: r=(b,0,z)=impact parameter

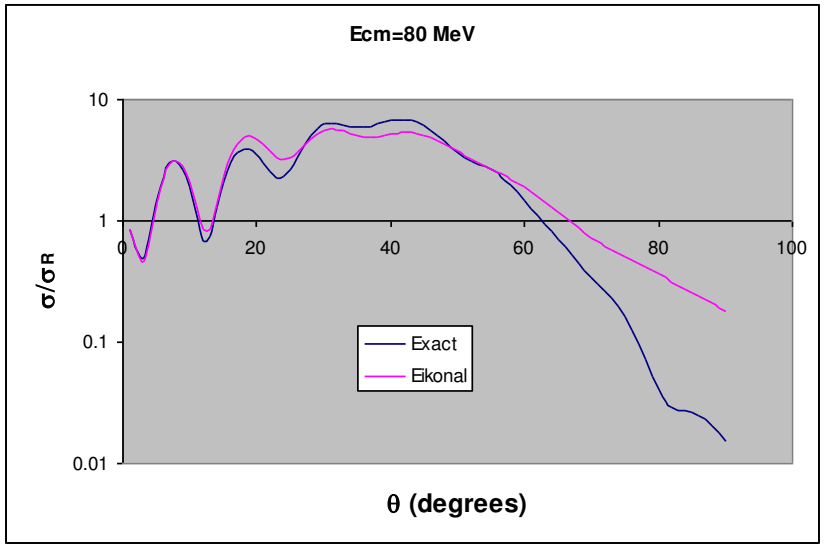
➔ very simple: integral over z, and over b

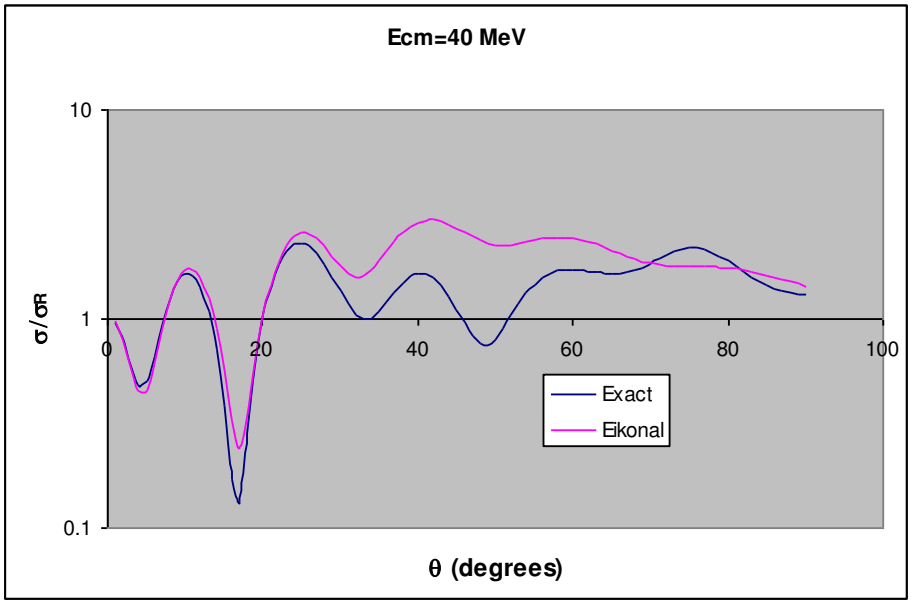
➔ Only valid at high energies (and small angles)

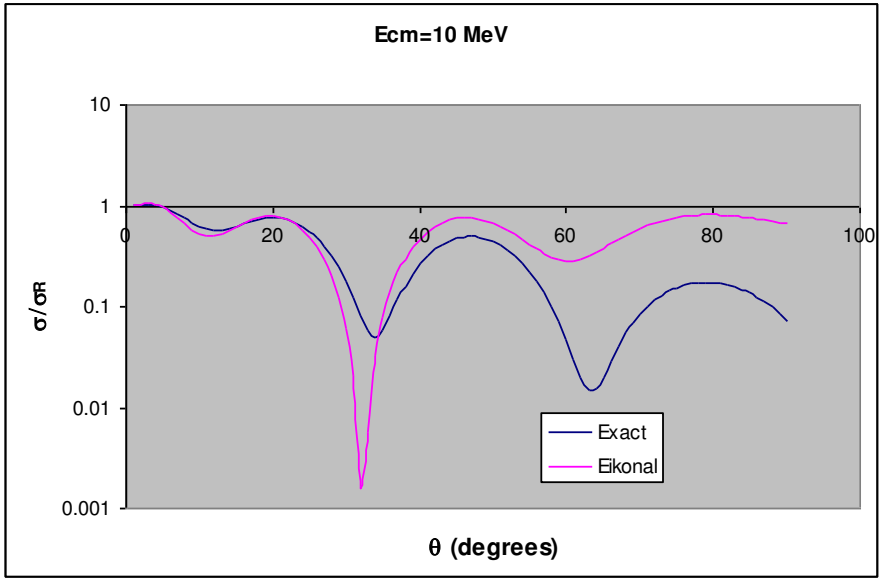
5. The eikonal method

Coulomb barrier ~ 2 MeV







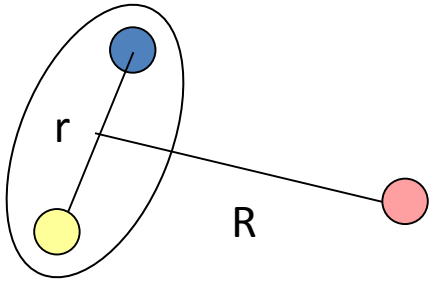


➔ accurate at high energies, poor approximation at low energies

5. The eikonal method: two-body projectile

Two-body projectile:
2 coordinates: R,r

→ elastic, breakup

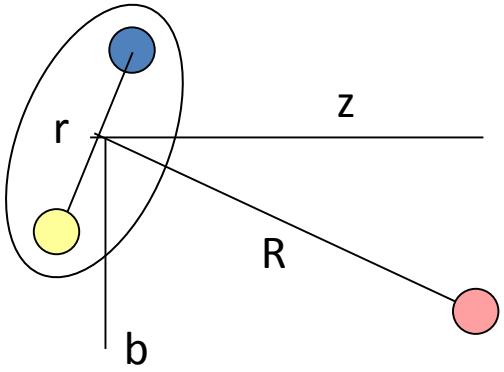


Ex: ^{11}Be+^{208}Pb

With ^{11}Be=^{10}Be+n

potential: $V(\boldsymbol{R},\boldsymbol{r}) = V_{t1}\left(\boldsymbol{R} + \frac{A_2}{A_p}\boldsymbol{r}\right) + V_{t2}\left(\boldsymbol{R} - \frac{A_1}{A_p}\boldsymbol{r}\right)$

eikonal phase: $\chi(b,\boldsymbol{r}) = \chi_1(b,\boldsymbol{r}) + \chi_2(b,\boldsymbol{r})$



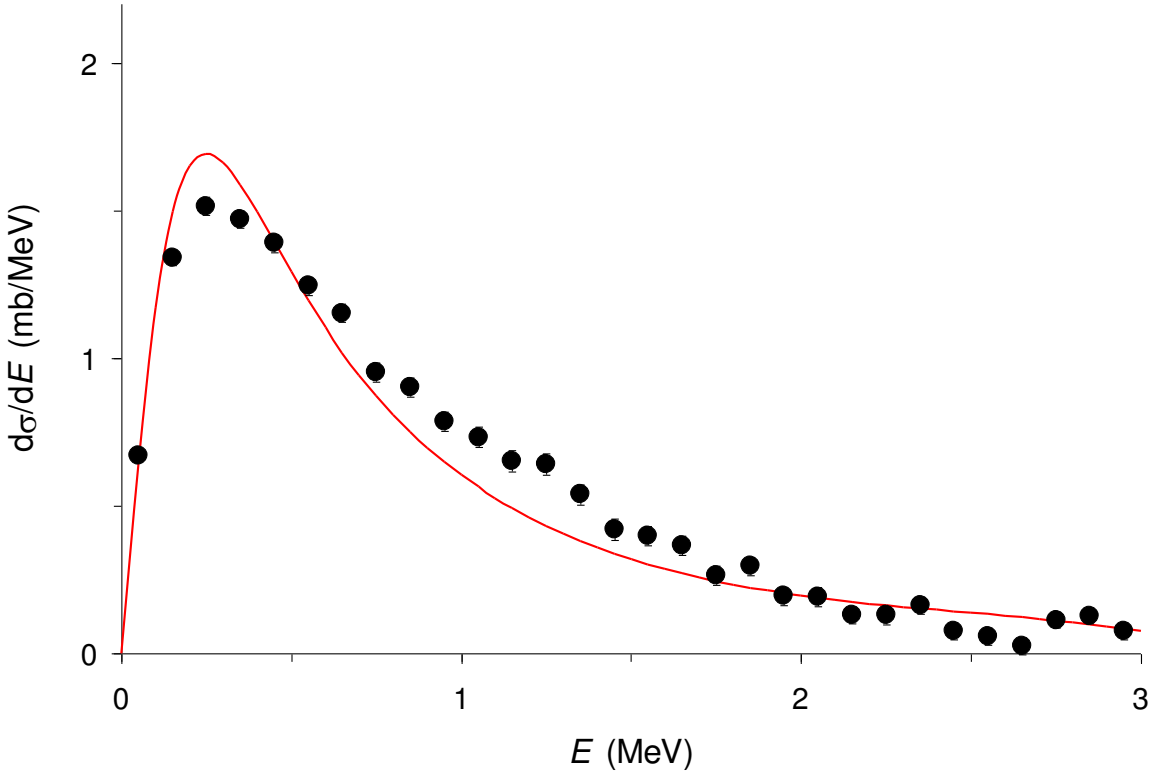
- Wave function of the projectile $\Phi_0(r)$
- Assume gs wave function (adiabatic approx.)
- Elastic cross sections: $S(b) = < \Phi_0(\boldsymbol{r}) \left| e^{i\chi(b,\boldsymbol{r})} \right| \Phi_0(\boldsymbol{r}) >$
- Breakup $S(k,b) = < \Psi_0(k,\boldsymbol{r}) \left| e^{i\chi(b,\boldsymbol{r})} \right| \Phi_0(\boldsymbol{r}) >$
- In practice: $e^{i\chi(b,\boldsymbol{r})}$ is expanded in multipoles
- Ref: D. Baye, EJP, Spec. Topics 156 (2008)

5. The eikonal method: two-body projectile

Elastic $S(b) = < \phi_0(\mathbf{r}) | e^{i\chi(b,\mathbf{r})} | \phi_0(\mathbf{r}) >$

breakup $S(b) = < \psi_0(\mathbf{r},k) | e^{i\chi(b,\mathbf{r})} | \phi_0(\mathbf{r}) >, \ \psi_0(\mathbf{r},k) =$ continuum state

Example: ^{11}Be breakup on ^{208}Pb at 68 MeV/u (RIKEN data)
D. Baye, EPJ ST156 (2008) 93



5. The eikonal method: extensions

- *Three-body projectiles*
  ^{6}He: D. Baye, P. Capel, P. Descouvemont, and Y. Suzuki, *Phys. Rev. C 79 (2009) 024607*
  ^{11}Li*:*E. C. Pinilla, P. Descouvemont, D. Baye, *Phys. Rev. C 85 (2012) 054610*

- *Dynamic eikonal approximation (no adiabatic approximation)*
  D. Baye, P. Capel, G. Goldstein, *Phys. Rev. Lett. 95 (2005) 082502*

- *Eikonal CDCC*: the projectile is described over a basis → multichannel problems
  K. Ogata et al., Phys. Rev. C 68, 064609 (2003)

- *Extensions to relativistic energies*
  K. Ogata and C. Bertulani, PTP 123 (2010) 701)

- *Neutron removal reactions*
  M. Yahiro, K. Ogata, K. Minomo, Prog. Theor. Phys. 126 (2011) 167

- *Microscopic eikonal:* projectile described by a microscopic model
  ^{7}Li*:*E. C. Pinilla, P. Descouvemont, in progress…