Reaction theories – Structure models for light nuclei

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1. General definitions: types of reactions, cross sections, etc.
2. Reaction models (basics)
    1. Single-channel potential/optical model (as simple as possible)
    2. Phase-shift method
    3. Generalizations (Coulomb, numerical calculation, spins, multichannel, absorption)
3. Reaction models (advanced)
    1. The CDCC method (Continuum Discretized Coupled Channel)
    2. Technical aspects: R-matrix, Lagrange meshes
    3. The Eikonal method

4. Structure models for light nuclei
    1. Clustering in nuclei
    2. Non-microscopic models
    3. Microscopic cluster models
5. Recent applications
    1. CDCC (11Be+64Zn, 9Be+208Pb, 7Li+208Pb)
    2. Eikonal (three-body breakup, microscopic eikonal)
6. Nuclear astrophysics: brief overview
7. Conclusion

1. Introduction

General context:

- Two-body systems
- Low energies (E $\lesssim$ Coulomb barrier), few open channels (one)
- Low masses (A $\lesssim$ 15-20)
- Low level densities ($\lesssim$ a few levels/MeV)
- Reactions with neutrons AND charged particles

Main questions to be addressed: determine the choice of the method

A. Type of reactions
   - Elastic, inelastic, transfer, etc.

B. Energy range
   - Partial wave expansion
   - Number of open channels $\rightarrow$ influences the absorption

C. Level densities

1. Introduction

A. Different types of reactions

1. Elastic collision : entrance channel=exit channel

A+B $\rightarrow$ A+B: Q=0

2. Inelastic collision (Q≠0)

A+B $\rightarrow$ A*+B (A*=excited state)
A+B $\rightarrow$ A+B*
etc..

3. Transfer reactions

A+B $\rightarrow$ C+D
A+B $\rightarrow$ C+D+E
etc…

4. Radiative capture reactions

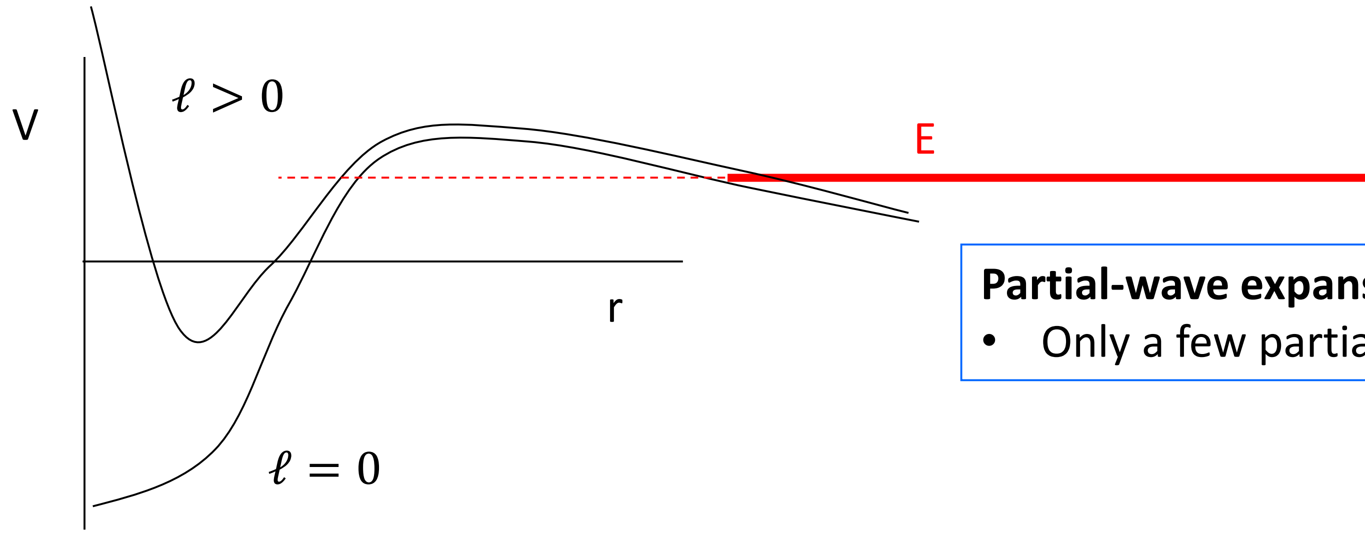
A+B$\rightarrow$ C + γ

5. Breakup: main tool to investigate exotic nuclei

1. Introduction

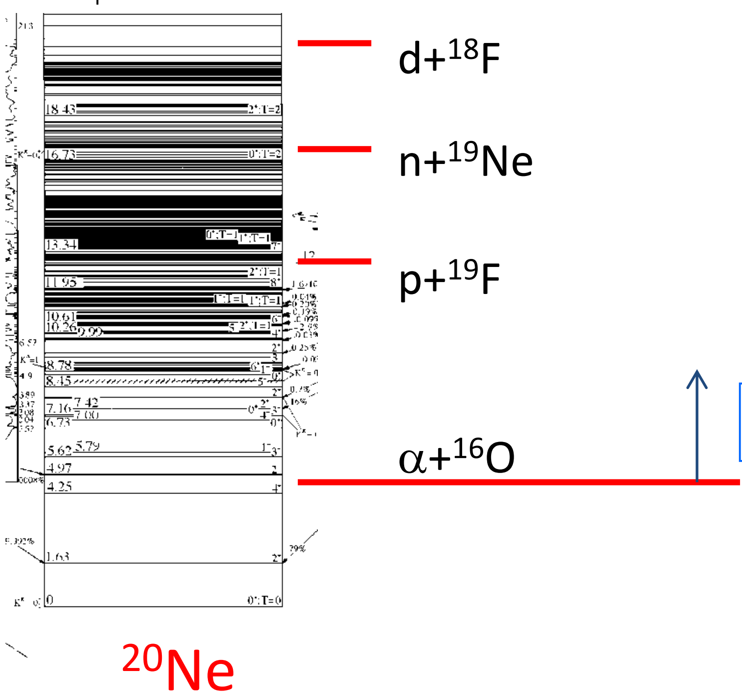
B. Energy

Low energies (~ Coulomb barrier)



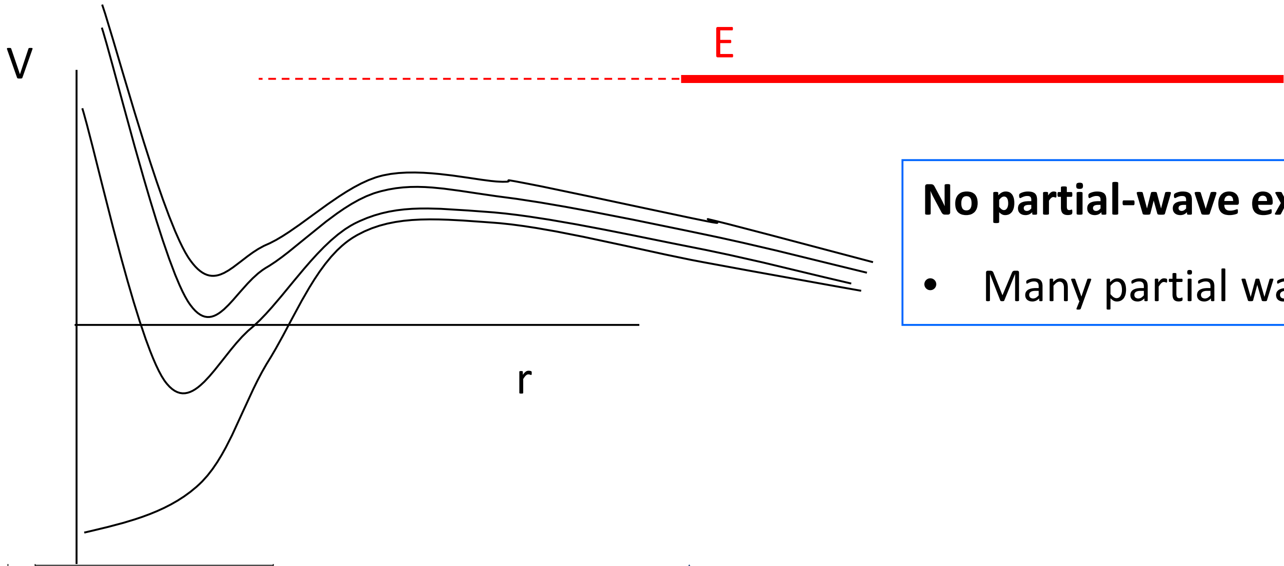
Partial-wave expansion

- Only a few partial waves contribute



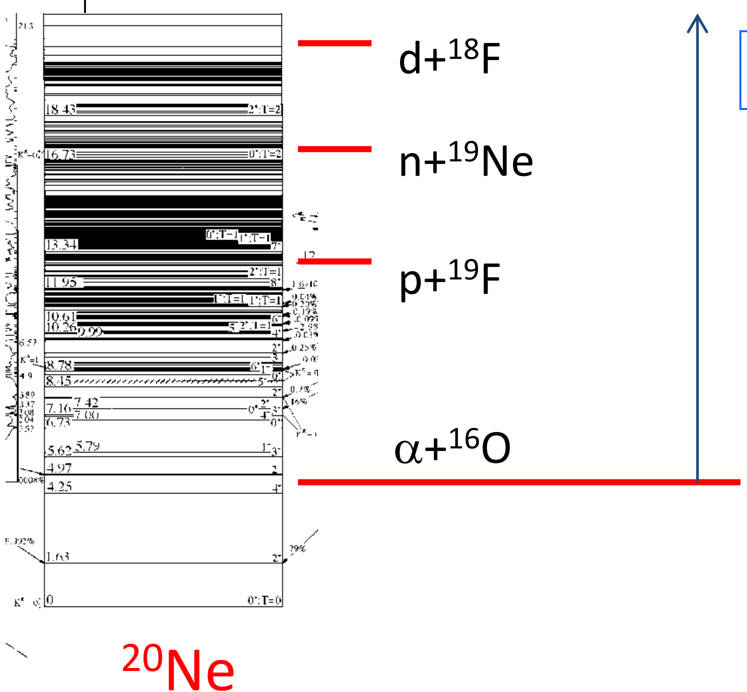
Single-channel calculation (no absorption)

High energies



No partial-wave expansion (high E)

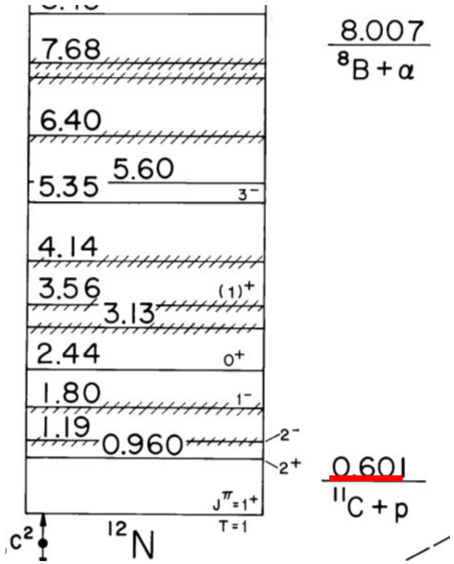
- Many partial waves



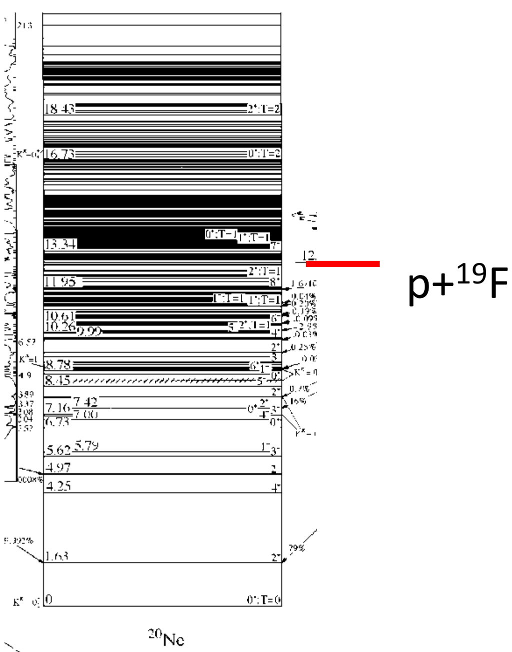
Many open channels →absorption important

1. Introduction

C. Level density



^{11}C+p: **Low** level density (typical of exotic nuclei)



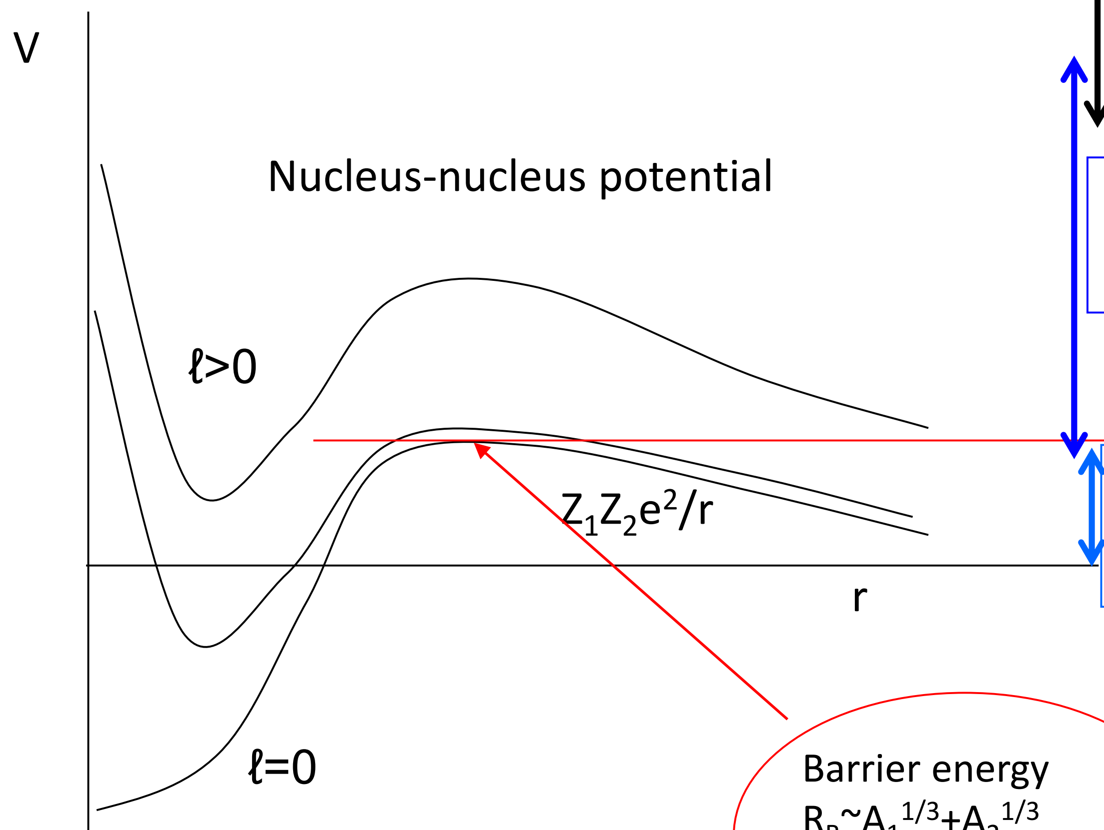
^{19}F+p: **High** level density (typical of stable nuclei)

➔ different models

1. Introduction

Various models

Far above the barrier
(Too) many ℓ values
→ no partial wave expansion
ex: Eikonal, semi-classic



Near the barrier
Limited ℓ values→ partial wave expansion
ex: Optical model, CDCC, DWBA, etc.

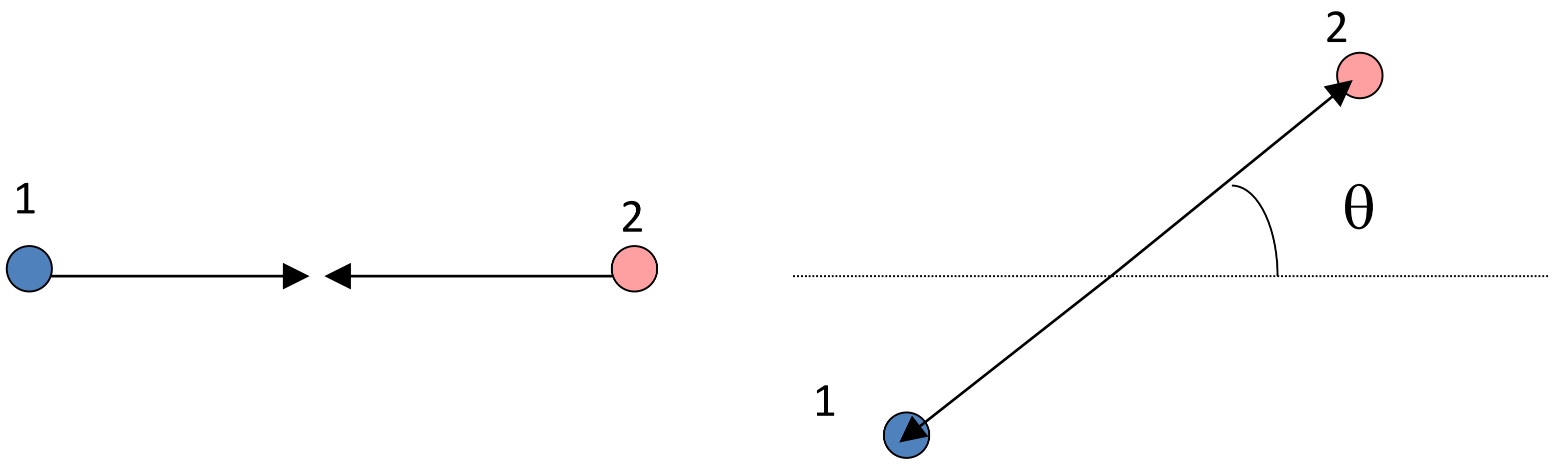
Below the barrier
Very few ℓ values
R matrix, GCM

Nuclear Astrophysics

Barrier energy
$R_B \sim A_1^{1/3}+A_2^{1/3}$
$V_B \sim Z_1Z_2e^2/R_B$

2. Single-channel potential/optical model

Scheme of the collision (elastic scattering)



Before collision

After collision

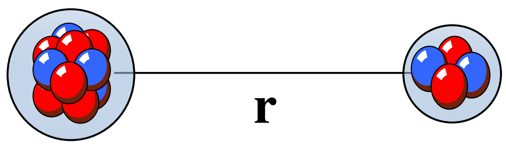
Center-of-mass system

2. Single-channel optical model

A. Definitions

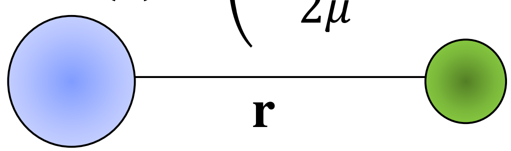
Schrödinger equation: $H\Psi(\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots \boldsymbol{r_A}) = E\Psi(\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots \boldsymbol{r_A})$ with $E > 0$: scattering states

- A-body equation (microscopic models) $H = \sum_i T_i + \frac{1}{2}\sum_{i,j} V_{ij}(\boldsymbol{r_i} - \boldsymbol{r_j})$



- Optical model: internal structure of the nuclei is neglected
  the particles interact by a potential
  absorption simulated by the imaginary part = optical potential

$$H\Psi(\boldsymbol{r}) = \left(-\frac{\hbar^2}{2\mu}\Delta + V(\boldsymbol{r})\right)\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r})$$



- Additional assumptions: elastic scattering
  no Coulomb interaction
  spins zero

2. Single-channel optical model

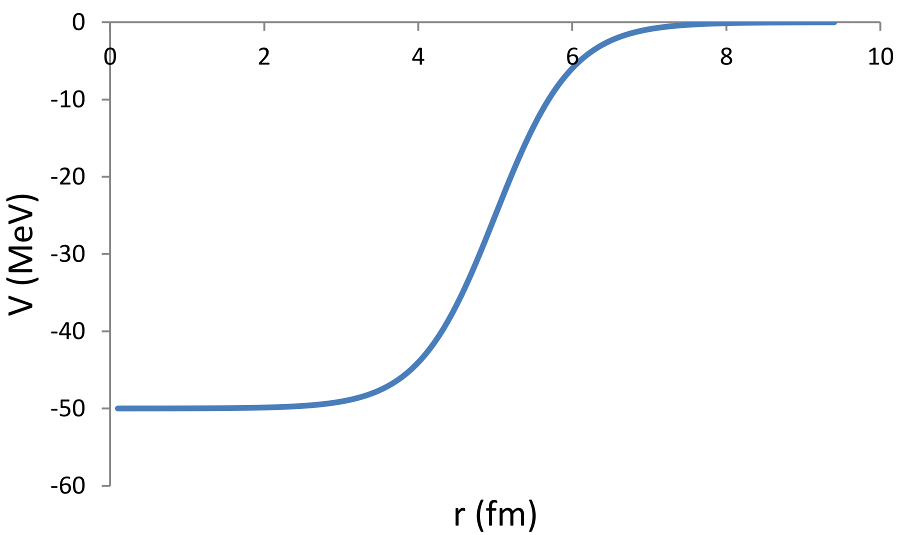
Two contributions to the optical potential: nuclear $V_N(r)$ and Coulomb $V_C(r)$

Typical nuclear potential: $V_N(r)$ (short range, attractive)

- examples: Gaussian $V_N(r) = -V_0 \exp(-(r/r_0)^2)$

  Woods-Saxon: $V_N(r) = -\frac{V_0}{1+\exp\left(\frac{r-r_0}{a}\right)}$

- parameters are fitted to experiment
- no analytical solution of the Schrödinger equation



Woods-Saxon potential
r_0 =range (~sum of the radii)
a= diffuseness (~0.5 fm)

Figure: V_0=50 MeV, r_0=5 fm, $a = 0.5$ fm

2. Single-channel optical model

Coulomb potential: long range, repulsive

- « point-point » potential : $V_C(r) = \frac{Z_1 Z_2 e^2}{r}$

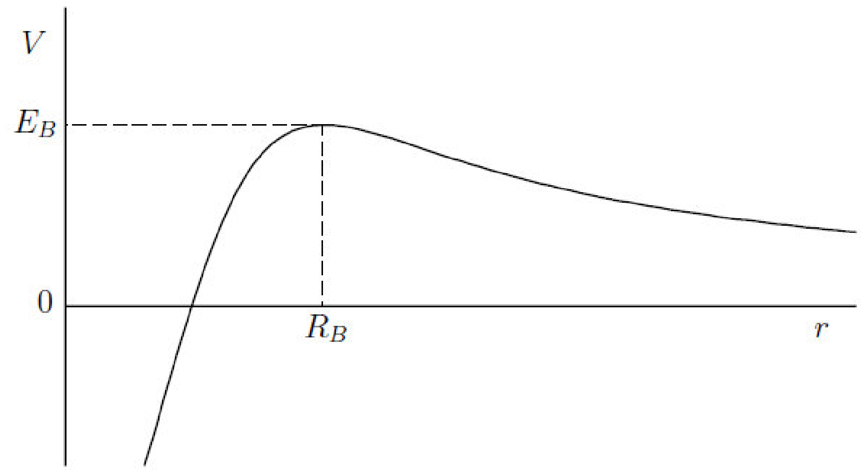
- « point-sphere » potential : (radius R_C)

$$V_C(r) = \frac{Z_1 Z_2 e^2}{r} \text{ for } r \geq R_C$$

$$V_C(r) = \frac{Z_1 Z_2 e^2}{2R_C}\left(3 - \left(\frac{r}{R_C}\right)^2\right) \text{ for } r \leq R_C$$

Total potential : $V(r) = V_N(r) + V_C(r)$: presents a maxium at the Coulomb barrier

- radius $r = R_B$
- height $V(R_B) = E_B$



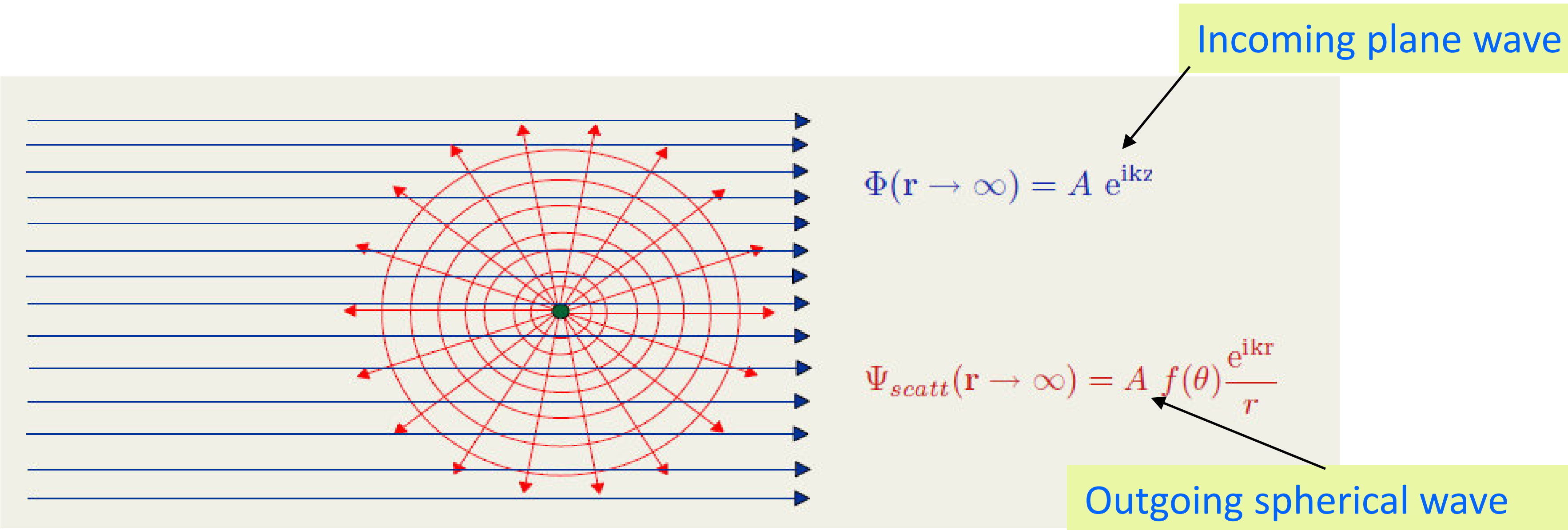
B. General solution

$$H\Psi(\boldsymbol{r}) = \left(-\frac{\hbar^2}{2\mu}\Delta + V(\boldsymbol{r})\right)\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r})$$

with $\Psi(\boldsymbol{r}) = \Phi(r) + \Psi_{scatt}(r)$ ($\Phi(\boldsymbol{r})$ corresponds to $V(r)$=0)

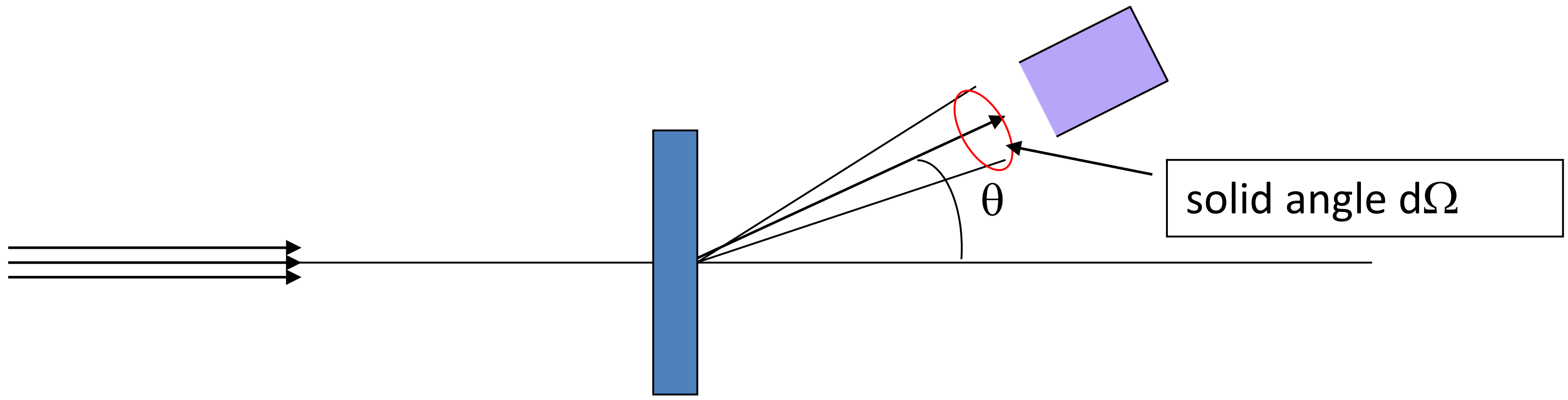
At large distances : $\Psi(\boldsymbol{r}) \to A\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(\theta)\frac{e^{ikr}}{r}\right)$ (with *z* along the beam axis)

where: k=wave number: $k^2 = 2mE/\hbar^2$
A =amplitude (scattering wave function is not normalized)
$f(\theta)$ =scattering amplitude (length)



2. Single-channel optical model

C. Cross sections



Cross section: $$\frac{d\sigma}{d\Omega} = |f(\theta)|^2, \sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

- Cross section obtained from the asymptotic part of the wave function
  General problem for scattering states: the wave function must be known up to large distances
- "Direct" problem: determine σ from the potential
- "Inverse" problem : determine the potential V from σ
- Angular distribution: E fixed, θ variable
- Excitation function: θ variable, E fixed,

2. Single-channel optical model

Main issue: determining the scattering amplitude $f(\theta)$ (and wave function $\Psi(\boldsymbol{r})$)

- *Method 1: partial wave expansion*: $\Psi(\boldsymbol{r}) = \sum_{lm} \Psi_l(r) Y_l^m(\theta,\phi)$
  - Must be determined for each partial wave l → **phase-shit method**
  - At low energies, few partial waves
  - $f(\theta)$ determined by the long-range part of $\Psi(\boldsymbol{r})$

- *Method 2: Formal theory- Lippman-Schwinger equation*

$$f(\theta) = -\frac{2\mu}{4\pi\hbar^2}\int \exp(-ikr'\cos\theta)V(\boldsymbol{r}')\Psi(\boldsymbol{r}')d\boldsymbol{r}'$$

- equivalent to the Schrödinger equation
- $V(\boldsymbol{r})$ has a short range → $\Psi(\boldsymbol{r})$ is not necessary at large distances
- approximations: valid if $V(r)$ is small or E is large
  - Born approximation : $\Psi(\boldsymbol{r}) = \exp(i\boldsymbol{k}\cdot\boldsymbol{r})$
  - Eikonal approximation $\Psi(\boldsymbol{r}) = \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\widehat{\Psi}(\boldsymbol{r})$

3. Phase-shift method: potential model

3. Phase-shift method: potential model

- Goal: solving the Schrodinger equation

$$\left(-\frac{\hbar^2}{2\mu}\Delta + V(\boldsymbol{r})\right)\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r})$$

  with a partial-wave expansion

$$\Psi(\boldsymbol{r}) = \sum_{\ell,m} \frac{u_\ell(r)}{r} Y_\ell^m(\Omega_r)\, Y_\ell^{m*}(\Omega_k)$$

- Simplifying assumtions
  - neutral systems (no Coulomb interaction)
  - spins zero
  - single-channel calculations → elastic scattering

- Generalizations briefly illustrated in the next section

3. Phase-shift method: Definition, cross section

- The wave function is expanded as

$$\Psi(\boldsymbol{r}) = \sum_{\ell,m} \frac{u_\ell(r)}{r} Y_\ell^m(\Omega_r)\, Y_\ell^{m*}(\Omega_k)$$

- This provides the Schrödinger equation for each partial wave ($\Omega_k = 0 \ \rightarrow m = 0$)

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right) u_\ell + V(r) u_\ell = E u_\ell$$

- Large distances : $r \rightarrow \infty, V(r) \rightarrow 0$

$$u_\ell'' - \frac{\ell(\ell+1)}{r^2} u_\ell + k^2 u_\ell = 0 \ \text{ Bessel equation } \ \rightarrow u_\ell(r) = r j_\ell(kr), r n_\ell(kr)$$

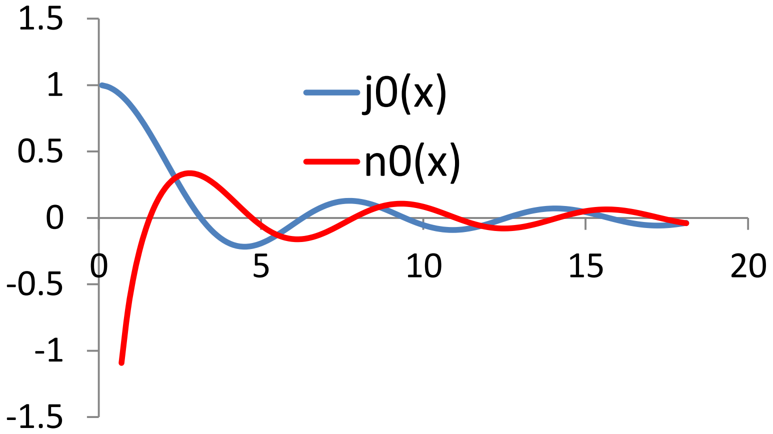
- Remarks
  - must be solved for all l values
  - at low energies: few partial waves in the expansion
  - at small r: $u_\ell(r) \rightarrow r^{\ell+1}$

3. Phase-shift method: Definition, cross section

For small x: $j_l(x) \to \frac{x^l}{(2l+1)!!}$
$n_l(x) \to -\frac{(2l-1)!!}{x^{l+1}}$

For large x: $j_l(x) \to \frac{1}{x}\sin(x - l\pi/2)$
$n_l(x) \to -\frac{1}{x}\cos(x - l\pi/2)$

Examples: $j_0(x) = \frac{\sin x}{x}$, $n_0(x) = -\frac{\cos\ x}{x}$



At large distances: $u_\ell(r)$ is a linear combination of $rj_\ell(kr)$ and $rn_\ell(kr)$

$$u_\ell(r) \to C_l\, r\left(j_\ell(kr)\ - \tan \delta_\ell \times n_\ell(kr)\right)$$

With δ_ℓ = phase shift (information about the potential):

If V=0 → $\delta_\ell = 0$

3. Phase-shift method: Definition, cross section

Derivation of the elastic cross section

- Identify the asymptotic behaviours

$$\Psi(\boldsymbol{r}) \to A\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(\theta)\frac{e^{ikr}}{r}\right)$$

$$\Psi(\boldsymbol{r}) \to \sum_\ell C_\ell\big(j_\ell(kr) \; - \tan \delta_\ell \times n_\ell(kr)\big)Y_\ell^0(\Omega_r)\sqrt{\frac{2\ell+1}{4\pi}}$$

- Provides coefficients C_ℓ and scattering amplitude $f(\theta)$

$$f(\theta,E) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\delta_\ell(E)) - 1)\, P_\ell(\cos\theta)$$

$$\frac{d\sigma(\theta,E)}{d\Omega} = |f(\theta,E)|^2$$

- Integrated cross section (neutral systems only)

$$\sigma = \frac{\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

- In practice, the summation over ℓ is limited to some ℓ_{max}

3. Phase-shift method: Definition, cross section

$$\frac{d\sigma(\theta,E)}{d\Omega} = |f(\theta,E)|^2 \text{ with } f(\theta,E) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\delta_\ell(E))-1)\, P_\ell(\cos\theta)$$

➔ factorization of the dependences in E and θ

low energies: small number of ℓ values ($\delta_\ell \to 0$ when ℓ increases)

high energies: large number (→ alternative methods)

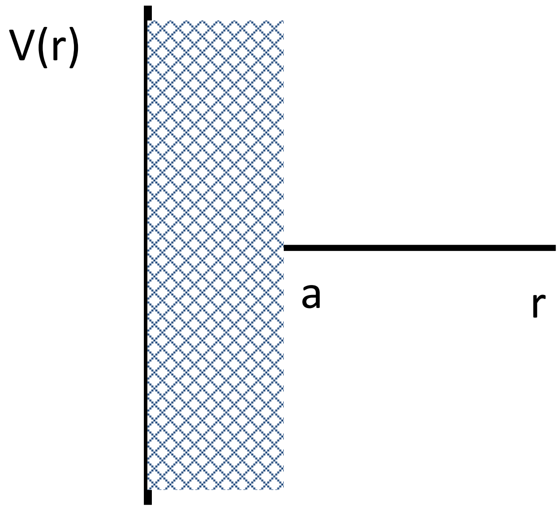
General properties of the phase shifts

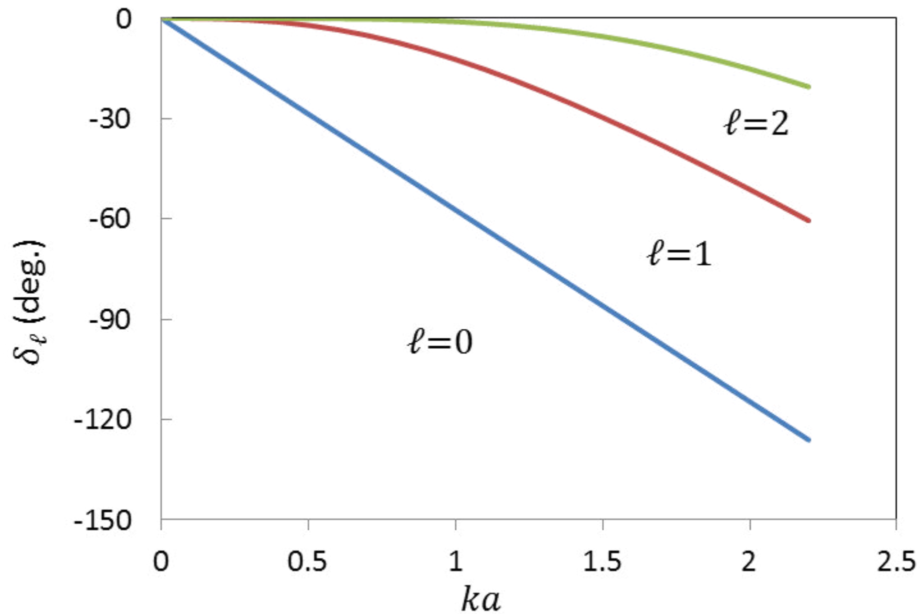
1. The phase shift (and all derivatives) are continuous functions of E
2. The phase shift is known within nπ: $\exp 2i\delta = \exp(2i(\delta + n\pi))$
3. Levinson theorem
    - $\delta_\ell(E=0)$ is arbitrary
    - $\delta_\ell(0) - \delta_\ell(\infty) = \mathrm{N}\pi$, where N is the number of bound states in partial wave ℓ
    - Example: p+n, $\ell = 0$: $\delta_0(0) - \delta_0(\infty) = \pi$ (bound deuteron)
      $\ell = 1$: $\delta_1(0) - \delta_1(\infty) = 0$ (no bound state for $\ell = 1$)

3. Phase-shift method: example

- Example: hard sphere (radius a)
- continuity at $r = a \rightarrow j_\ell(ka) - \tan\delta_\ell \times n_\ell(ka) = 0$ $\quad \rightarrow \tan\delta_\ell = \frac{j_\ell(ka)}{n_\ell(ka)}$

$\rightarrow \delta_0 = -ka$





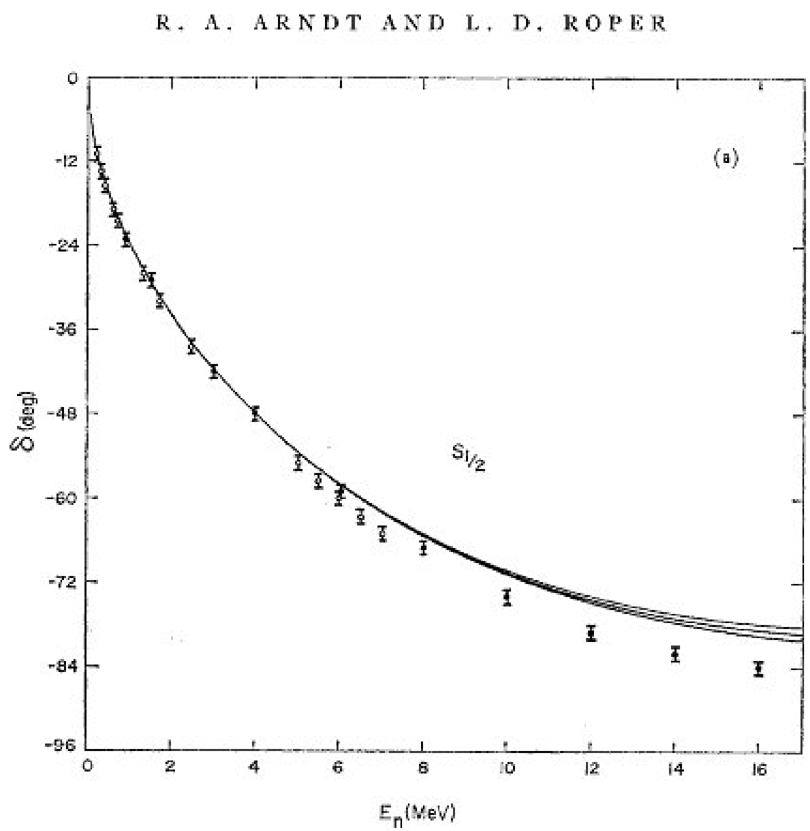
At low energies: $\delta_\ell(E) \rightarrow -\frac{(ka)^{2\ell+1}}{(2\ell+1)!!(2l-1)!!}$, in general: $\delta_\ell(E) \sim k^{2\ell+1}$

➔ Strong difference between $\ell = 0$ (no barrier) et $\ell \neq 0$ (centrifugal barrier)

3. Phase-shift method: example

example : α+n phase shift $\ell = 0$

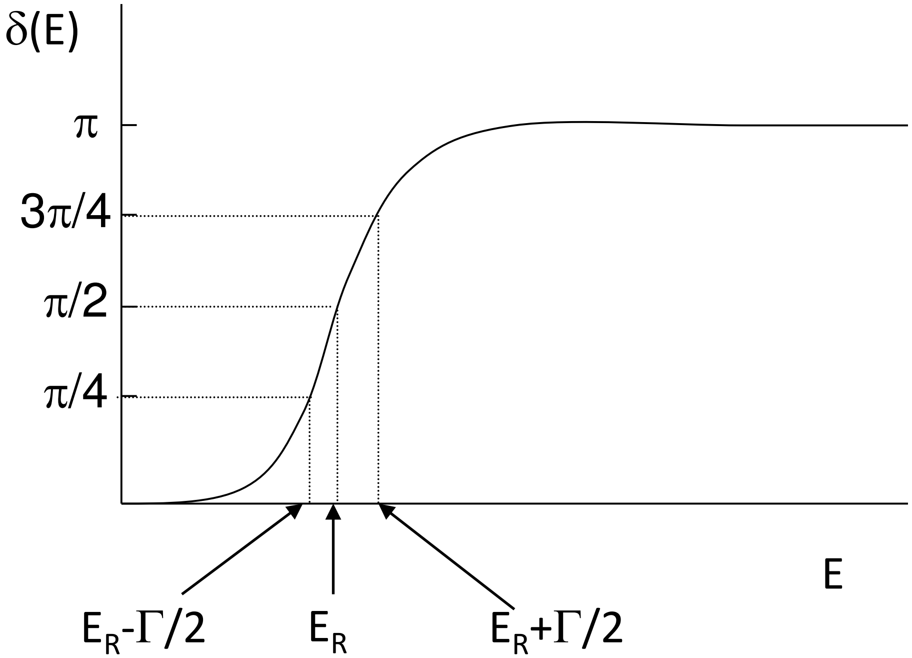
consistent with the hard sphere ($a \sim 2.2$ fm)



3. Phase-shift method: resonances

Resonances: $\delta_R(E) \approx \text{atan}\,\frac{\Gamma}{2(E_R - E)}$ = Breit-Wigner approximation

E_R=resonance energy
Γ=resonance width



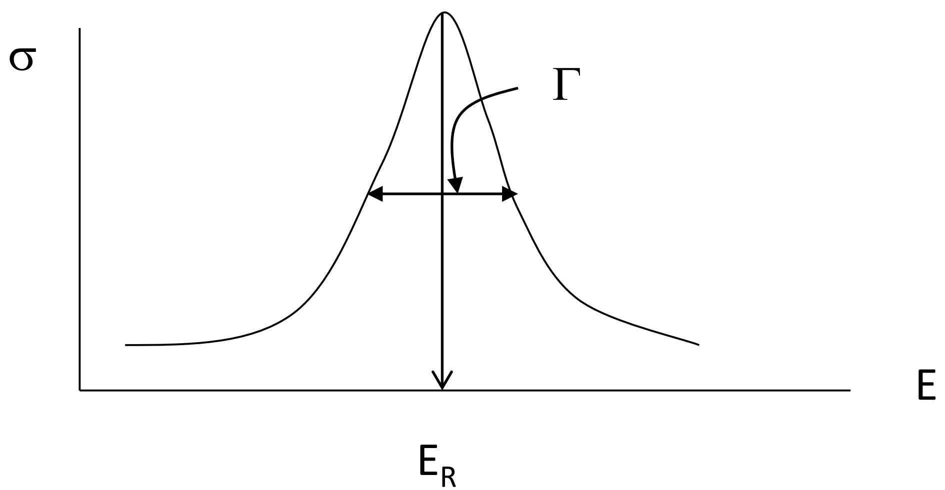
- Narrow resonance: Γ small
- Broad resonance: Γ large
- Bound states: Γ =0 ($E_R < 0$)

3. Phase-shift method: resonances

Cross section

$\sigma(E) = \frac{\pi}{k^2}\sum_\ell (2\ell+1)|\exp(2i\delta_\ell)-1|^2$ maximum for $\delta = \frac{\pi}{2}$

Near the resonance: $\sigma(E) \approx \frac{4\pi}{k^2}(2\ell_R+1)\frac{\Gamma^2/4}{(E_R-E)^2+\Gamma^2/4}$, where ℓ_R=resonant partial wave

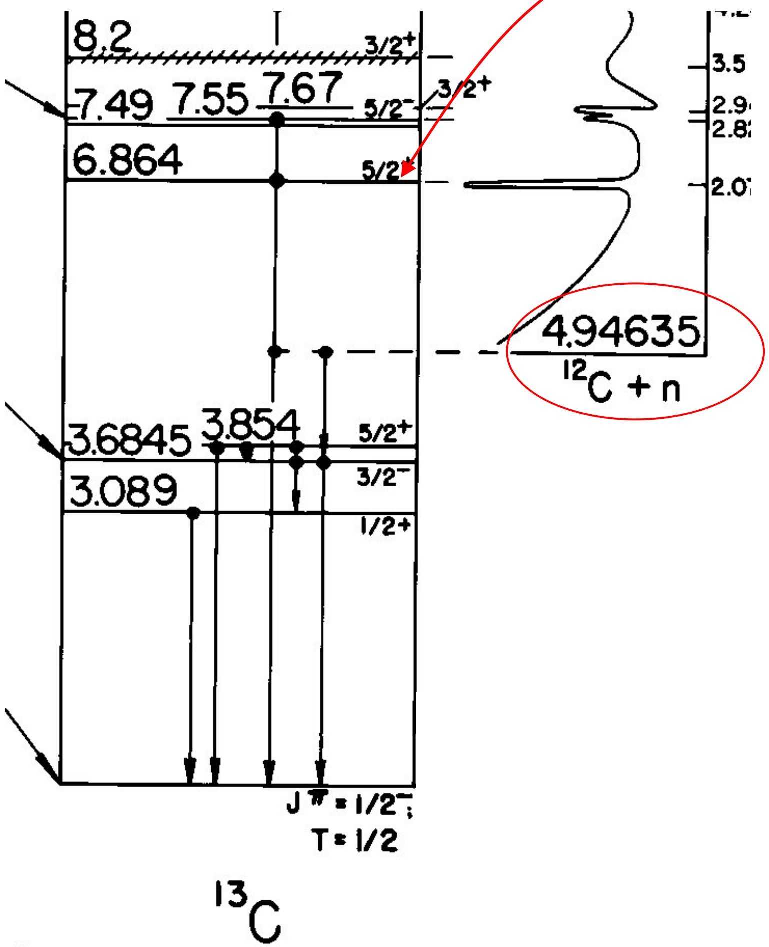


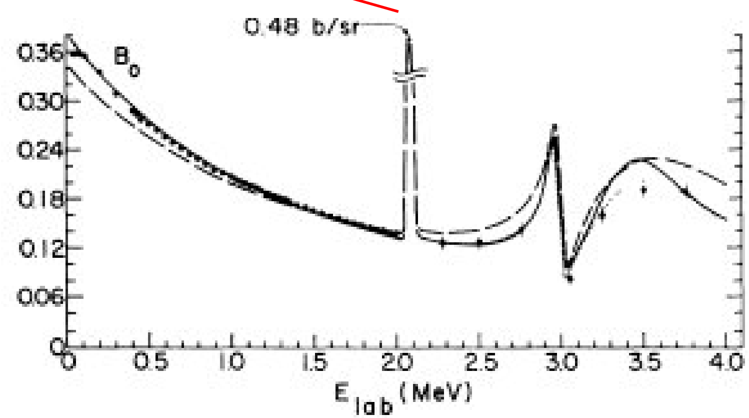
In practice:

- Peak not symmetric (Γ depends on E)
- « Background » neglected (other ℓ values)
- Differences with respect to Breit-Wigner

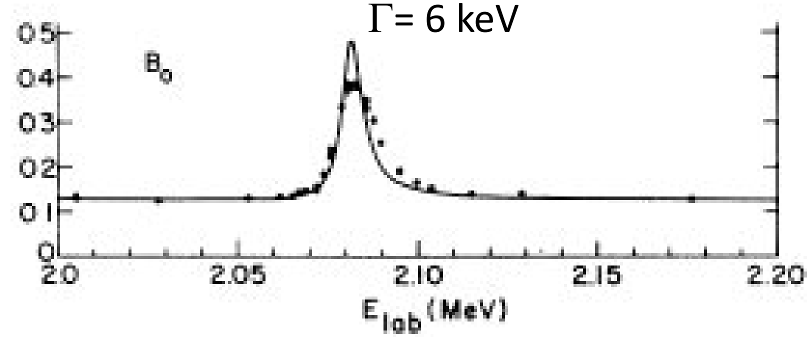
3. Phase-shift method: resonances

Example: n+^{12}C





E_{cm}=1.92 MeV
Γ= 6 keV



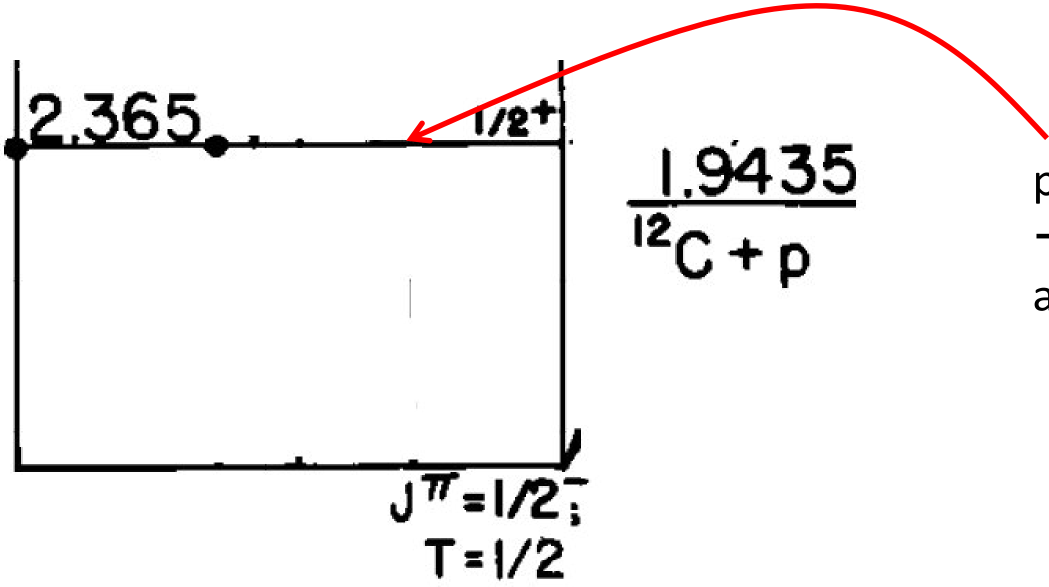
Comparison of 2 typical times:

a. Lifetime of the resonance: $\tau_R = \hbar/\Gamma \approx \frac{197}{3.10^{23}\times 6.10^{-3}} \approx 1.1\times 10^{-19} s$

b. Interaction time without resonance: $\tau_{NR} = d/v \approx 5.2\times 10^{-22} s$ ➔ $\tau_{NR} << \tau_R$

Narrow resonances

- Small particle width
- long lifetime
- can be approximetly treated by *neglecting the asymptotic behaviour of the wave function*



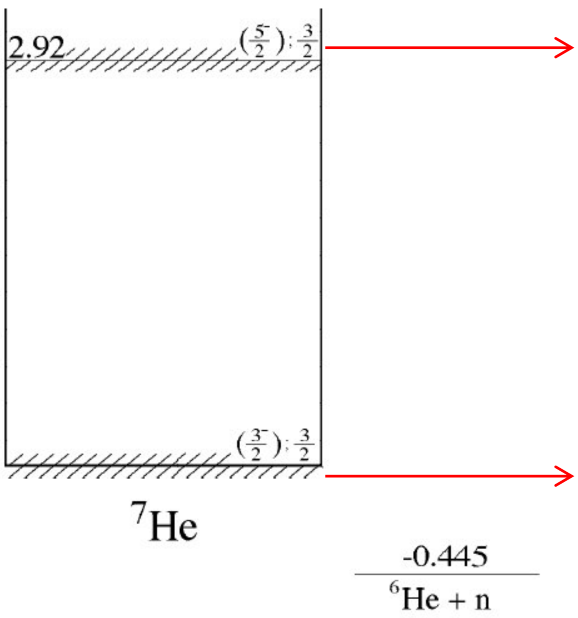
proton width=32 keV

→ can be described in a bound-state approximation

3. Phase-shift method: resonances

Broad resonances

- Large particle width
- Short lifetime
- *asymptotic behaviour of the wave function is important*
  → rigorous scattering theory
  → bound-state model complemented by other tools (complex scaling, etc.)



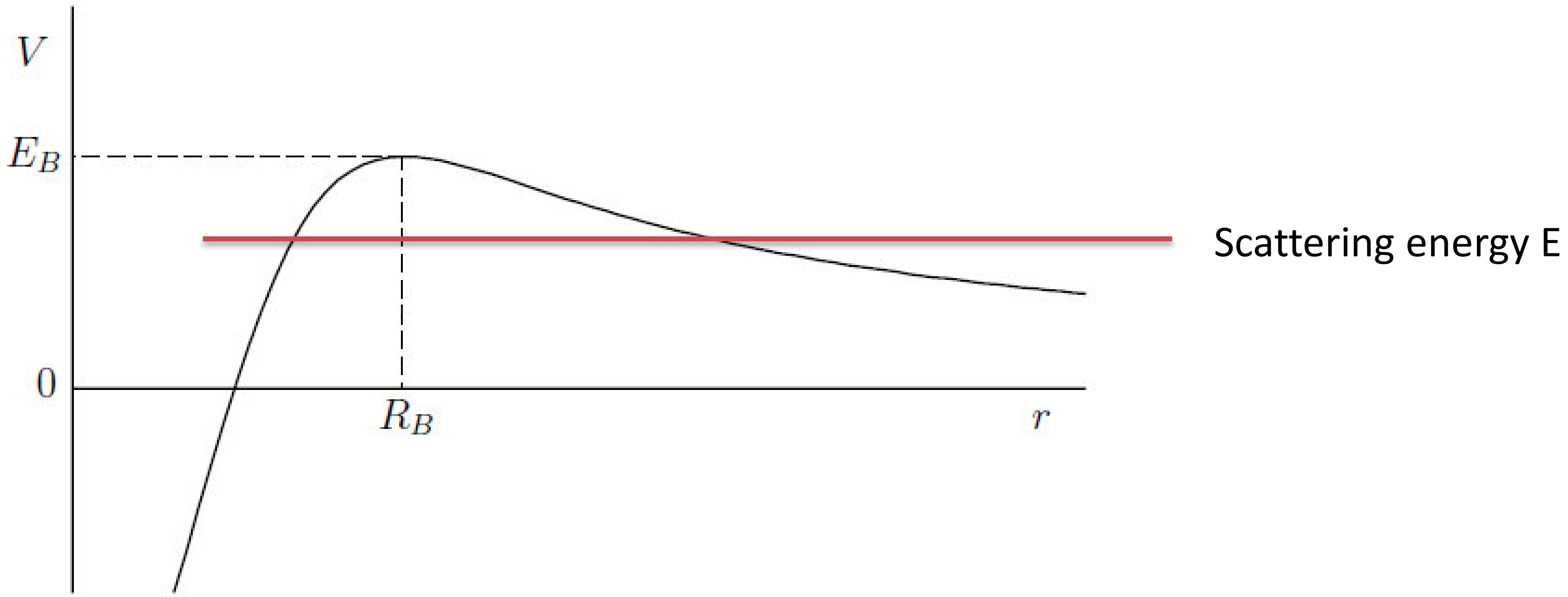
very broad resonance: Γ=1990 keV

ground state unstable: Γ=120 keV

4. Generalizations

- Extension to charged systems
- Numerical calculation
- Optical model (high energies → absorption)
- Extension to multichannel problems

Generalization 1: charged systems



$E \gg E_B$: weak coulomb effects (V negligible with respect to E)
$E < E_B$: strong coulomb effects (ex: nuclear astrophysics)

4. Generalizations

A. Asymptotic behaviour

Neutral systems

$$\left(-\frac{\hbar^2}{2\mu}\Delta + V_N(r) - E\right)\Psi(\boldsymbol{r}) = 0$$

$$\Psi(\boldsymbol{r}) \to \exp(i\boldsymbol{k}\cdot\boldsymbol{r}) + f(\theta)\frac{\exp(ikr)}{r}$$

Charged systems

$$\left(-\frac{\hbar^2}{2\mu}\Delta + V_N(r) + \frac{Z_1 Z_2 e^2}{r} - E\right)\Psi(\boldsymbol{r}) = 0$$

$$\Psi(\boldsymbol{r}) \to \exp(i\boldsymbol{k}\cdot\boldsymbol{r} + i\eta\ln(\boldsymbol{k}\cdot\boldsymbol{r} - kr)) + f(\theta)\frac{\exp(i(kr - \eta\ln 2kr))}{r}$$

$$\eta = \frac{Z_1 Z_2 e^2}{\hbar v}$$

- Sommerfeld parameter
- « measurement » of coulomb effects
- Increases at low energies
- Decreases at high energies

4. Generalizations

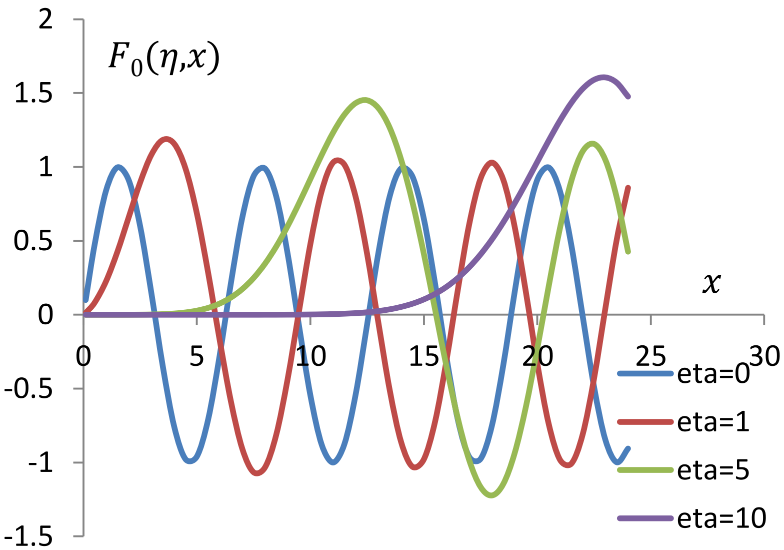
B. Phase shifts with the coulomb potential

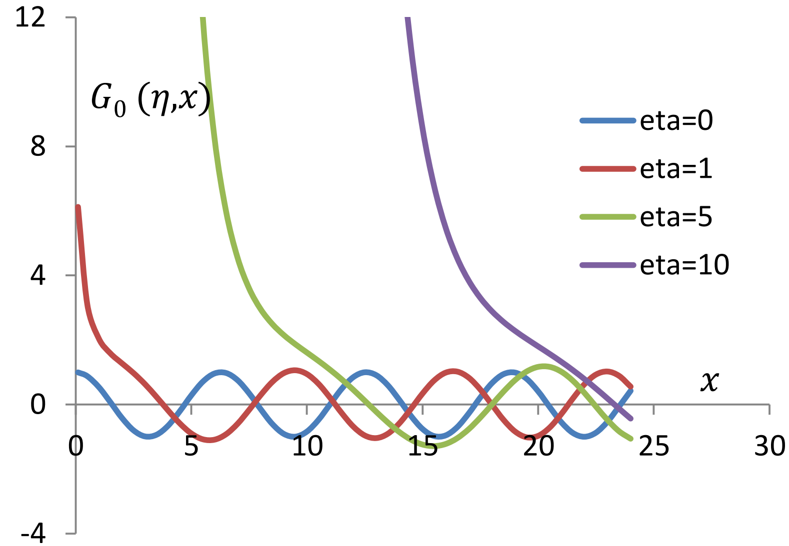
Neutral system: $$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + k^2\right) R_\ell = 0$$

Bessel equation : solutions $j_\ell(kr), n_\ell(kr)$

Charged system: $$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2\,\frac{\eta k}{r} + k^2\right) R_\ell = 0:$$

Coulomb equation: solutions $F_\ell(\eta, kr), G_\ell(\eta, kr)$





4. Generalizations

- Incoming and outgoing functions (complex)

$$I_\ell(\eta,x) = G_\ell(\eta,x) - iF_\ell(\eta,x) \to e^{-i\left(x-\frac{\ell\pi}{2}-\eta\ln 2x+\sigma_\ell\right)}\text{: incoming wave}$$

$$O_\ell(\eta,x) = G_\ell(\eta,x) + iF_\ell(\eta,x) \to e^{i\left(x-\frac{\ell\pi}{2}-\eta\ln 2x+\sigma_\ell\right)}\text{: outgoing wave}$$

- Phase-shift definition
  - neutral systems : $R_\ell(r) \to rA\big(j_\ell(kr) - \tan\delta_\ell\, n_\ell(kr)\big)$
  - charged systems: $R_\ell(r) \to A\big(F_\ell(\eta,kr) + \tan\delta_\ell\, G_\ell(\eta,kr)\big)$

$$\to B\left(\cos\delta_\ell\, F_\ell(\eta,kr) + \sin\delta_\ell\, G_\ell(\eta,kr)\right)$$
$$\to C\big(I_\ell(\eta,kr) - U_\ell O_\ell(\eta,kr)\big)$$

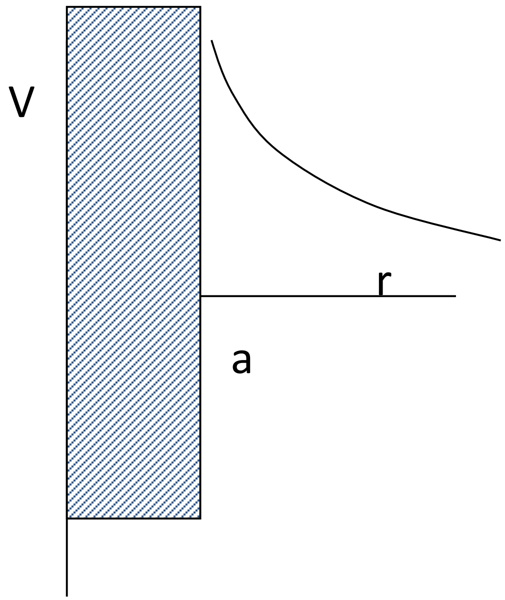
3 equivalent definitions (amplitude is different)

Collision matrix (=scattering matrix)

$$U_\ell = e^{2i\delta_\ell}\text{ : module }|U_\ell| = 1$$

4. Generalizations

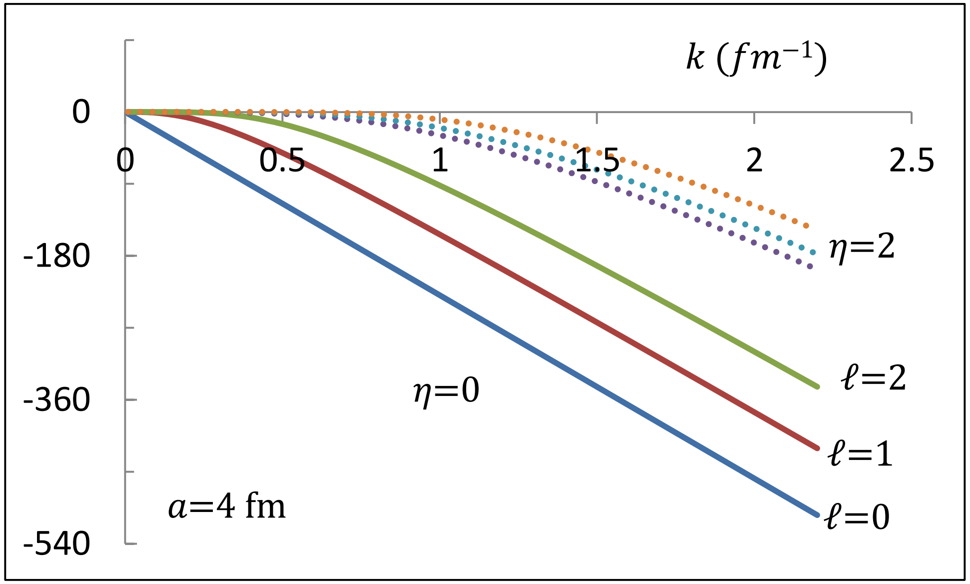
Example: hard-sphere potential



$$V(r) = \frac{Z_1 Z_2 e^2}{r} for\ r > a$$

$$\infty\ for\ r < a$$

phase shift: $\tan\delta_\ell = -\frac{F_\ell(\eta,ka)}{G_\ell(\eta,ka)}$



4. Generalizations

C. Rutherford cross section

For a Coulomb potential ($V_N = 0$):

- scattering amplitude : $f_c(\theta) = -\frac{\eta}{2k \sin^2 \theta/2} e^{2i(\sigma_0 - \eta \ln \sin \theta/2)}$
- Coulomb phase shift for $\ell = 0$: $\sigma_0 = \arg \Gamma(1 + i\eta)$

We get the Rutherford cross section:

$$\frac{d\sigma_C}{d\Omega} = |f_c(\theta)|^2 = \left(\frac{Z_1 Z_2 e^2}{4E \sin^2 \theta/2}\right)^2$$

- Increases at low energies
- Diverges at $\theta = 0$ → no integrated cross section

4. Generalizations

D. Cross sections with nuclear and Coulomb potentials

- The general defintions

$$f(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\delta_\ell)-1)\,P_\ell(\cos\theta)$$
$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

are still valid

- $Problem$: very slow convergence with ℓ
  → separation of the nuclear and coulomb phase shifts

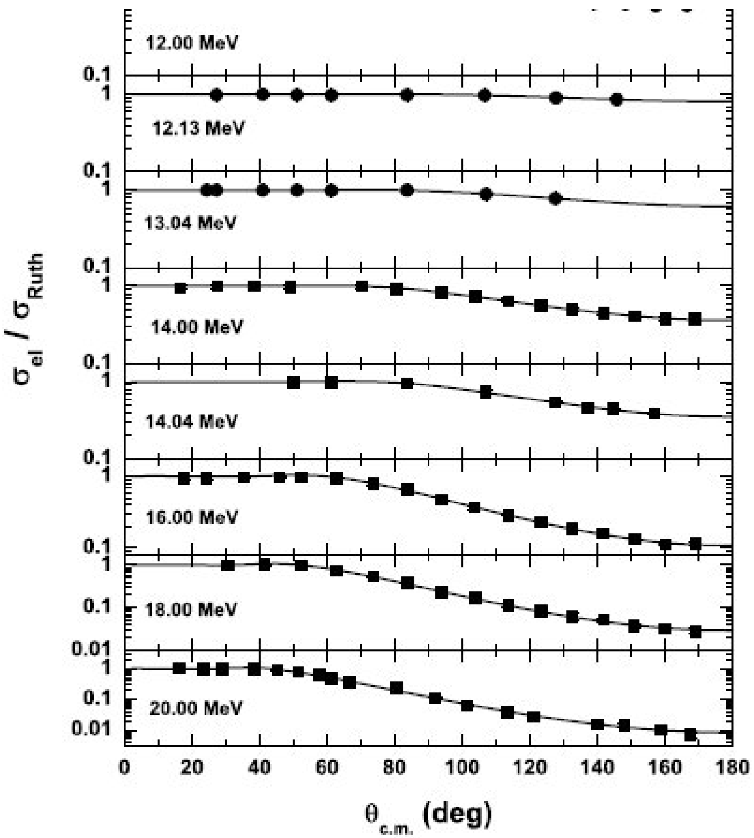
$$\delta_\ell = \delta_\ell^N + \sigma_\ell$$
$$\sigma_\ell = \arg\Gamma(1+\ell+i\eta)$$

- Scattering amplitude $f(\theta)$ written as $f(\theta) = f^C(\theta) + f^N(\theta)$
  - $f^C(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\sigma_\ell)-1)\,P_\ell(\cos\theta) = -\frac{\eta}{2k\sin^2\theta/2}e^{2i(\sigma_0-\eta\ln\sin\theta/2)}$
    → analytical
  - $f^N(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\exp(2i\sigma_\ell)\ \left(\exp(2i\delta_\ell^N)-1\right)P_\ell(\cos\theta)$
    → converges rapidly

4. Generalizations

Total cross section: $\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = |f^C(\theta) + f^N(\theta)|^2$

- Nuclear term dominant at 180°
- Coulomb term coulombien dominant at small angles → used to normalize experiments
- Coulomb amplitude strongly depends on the angle → $\frac{d\sigma/d\Omega}{d\sigma_C/d\Omega}$
- Integrated cross section $\int \frac{d\sigma}{d\Omega} d\Omega$ is not defined



System $^6Li+^{58}Ni$

- $E_{cm} = \frac{58}{64} E_{lab}$
- Coulomb barrier
$$E_B \sim \frac{3 * 28 * 1.44}{7} \sim 17 \text{ MeV}$$
- Below the barrier: $\sigma \sim \sigma_C$
- Above E_B : σ is different from σ_C

Generalization 2: numerical calculation

For some potentials: analytic solution of the Schrödinger equation

In general: no analytical solution → numerical approach

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}u_\ell(r) + (V(r) - E)u_\ell(r) = 0$$

with: $$V(r) = V_N(r) + \frac{Z_1 Z_2 e^2}{r} + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}$$

$$u_\ell(r) \to F_\ell(kr,\eta)\cos\delta_\ell + G_\ell(kr,\eta)\sin\delta_\ell$$

Numerical solution : discretization N points, with mesh size h

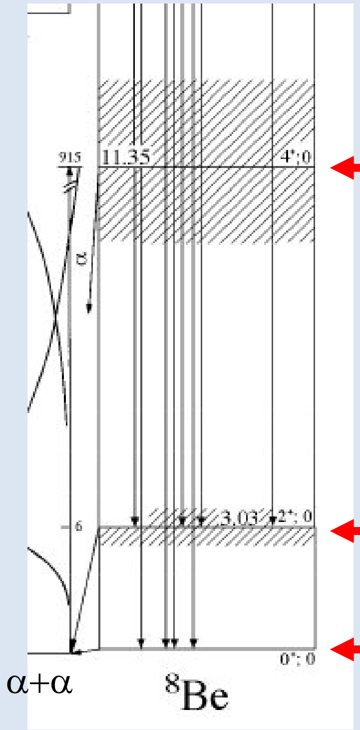
- $u_l(0) = 0$
- $u_l(h) = 1$ (or any constant)
- $u_l(2h)$ is determined numerically from $u_l(0)$ and $u_l(h)$ (Numerov algorithm)
- $u_l(3h), \ldots u_l(Nh)$
- for large r: matching to the asymptotic behaviour → phase shift

Bound states: same idea (but energy is unknown)

4. Generalizations

Example: α+α

Experimental spectrum of ^{8}Be

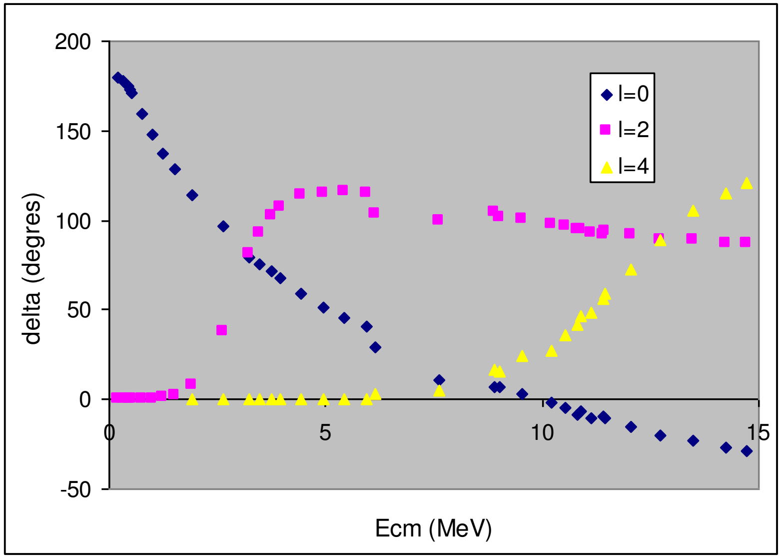


4^+
E~11 MeV
Γ~3.5 MeV

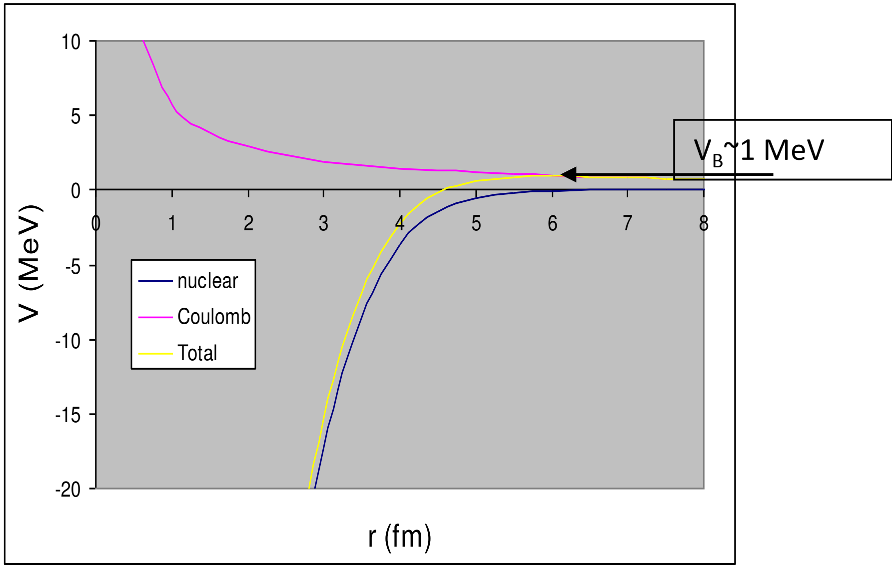
2^+
E~3 MeV
Γ~1.5 MeV

0^+
E=0.09 MeV
Γ=6 eV

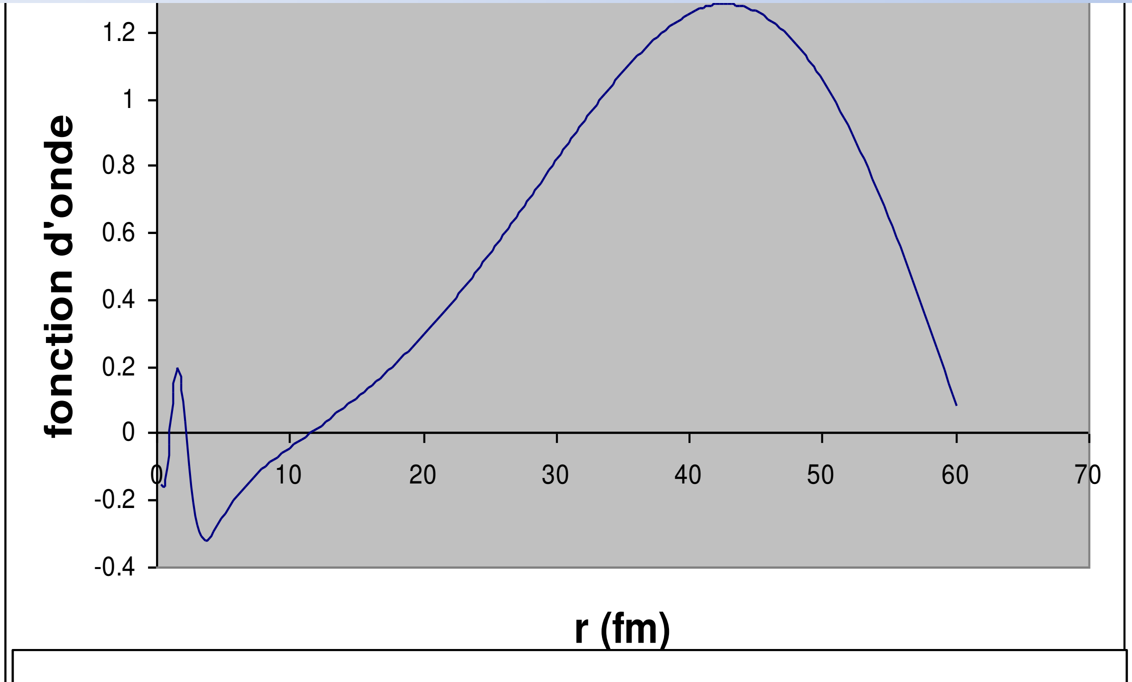
Experimental phase shifts

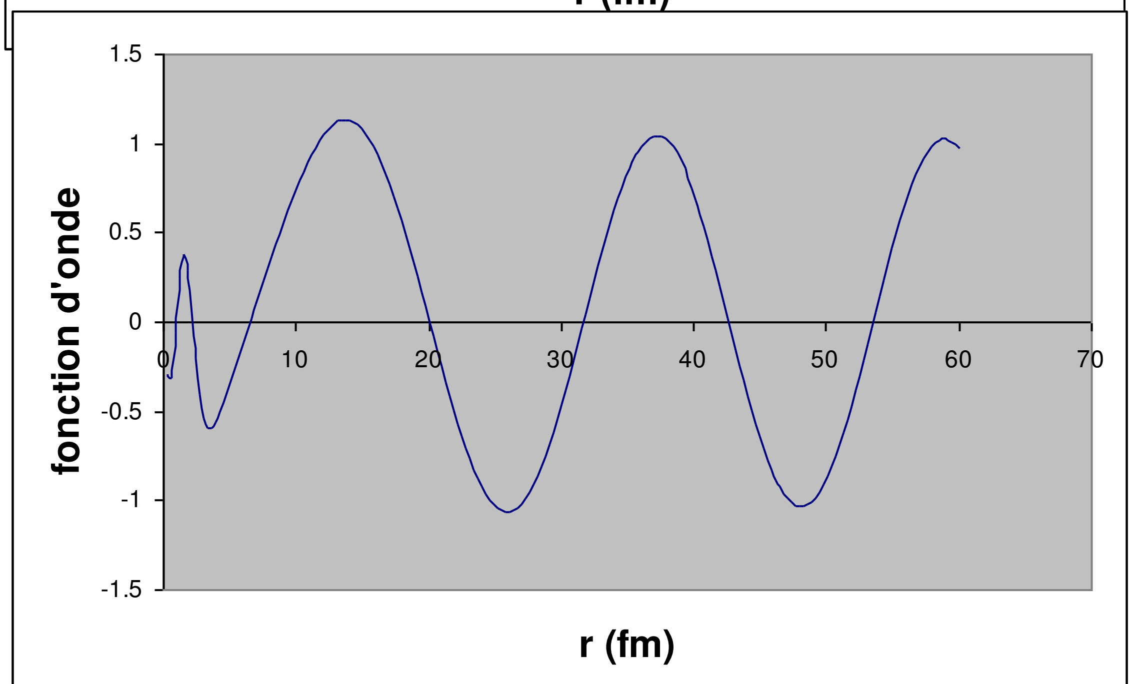


Potential: $V_N(r)=-122.3*\exp(-(r/2.13)^2)$



4. Generalizations





$\alpha+\alpha$ wave function for $\ell = 0$

E=0.2 MeV

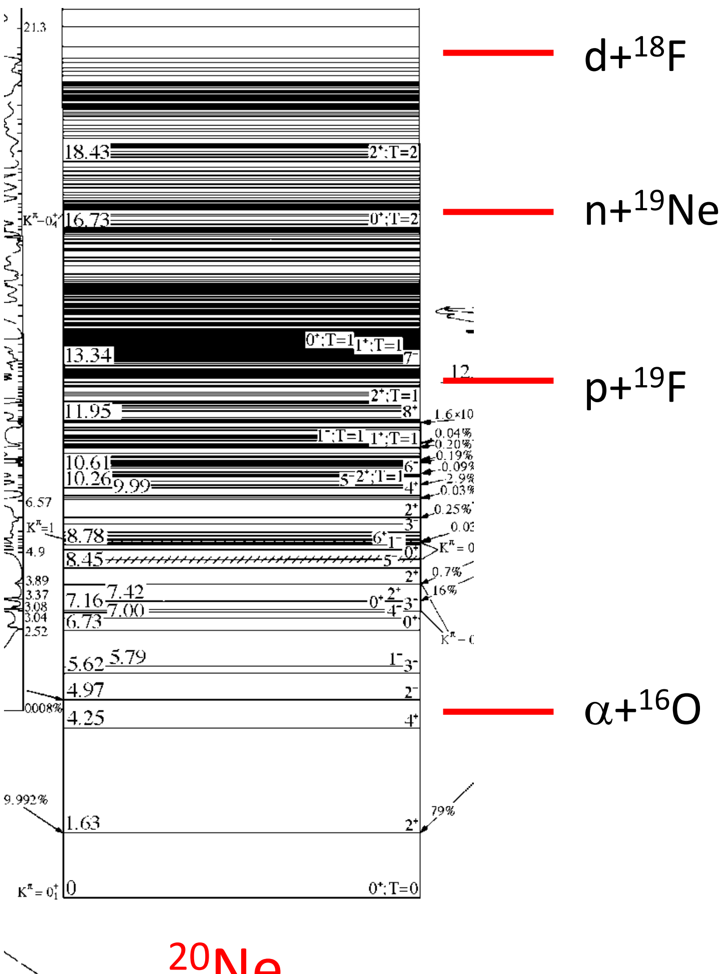
- $E < E_B$
- Small amplitude for r small

E=1 MeV

- $E \approx E_B$

Generalization 3: complex potentials

Goal: to simulate absorption channels



^{20}Ne

High energies:

- many open channels
- strong absorption
- potential model extended to complex potentials (« optical »)

Phase shift is complex: $\delta = \delta_R + i\delta_I$

collision matrix: $U = \exp(2i\delta) = \eta \exp(2i\delta_R)$

where $\eta = \exp(-2\delta_I) < 1$

Elastic cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{4k^2}\left|\sum_{\ell}(2\ell+1)(\eta_\ell \exp(2i\delta_\ell)-1)P_\ell(\cos\theta)\right|^2$$

Reaction cross section:

$$\sigma = \frac{\pi}{k^2}\sum_{\ell}(2\ell+1)(1-\eta_\ell^2)$$

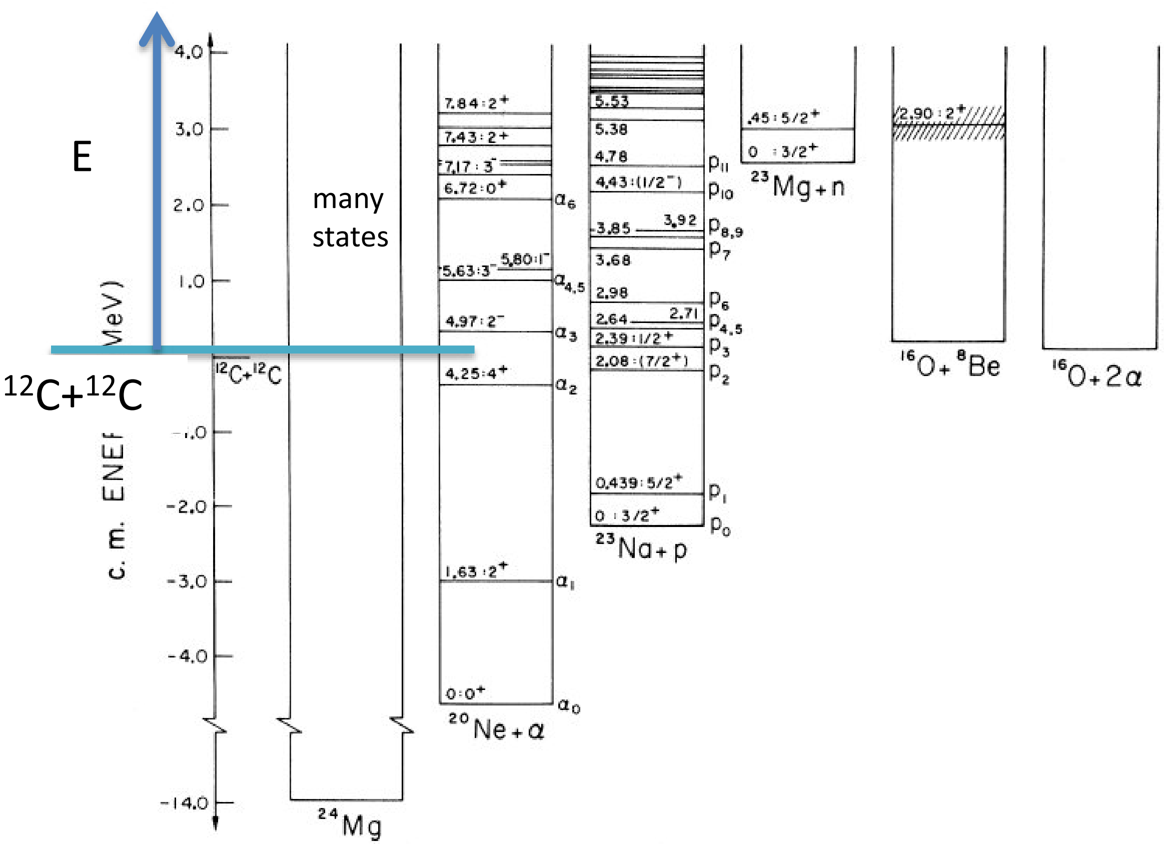
4. Generalizations

In astrophysics, optical potentials are used to compute fusion cross sections

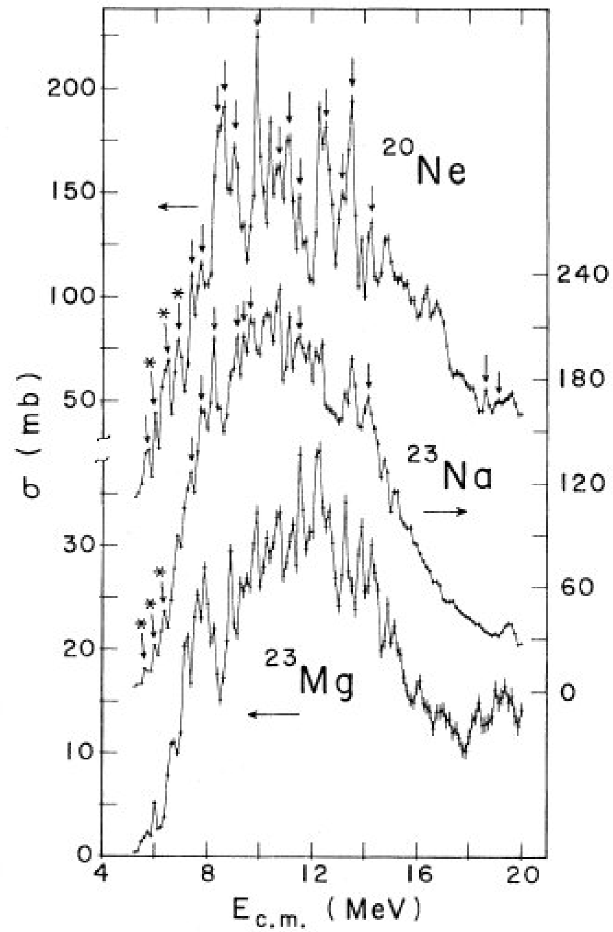
Fusion cross section: includes many channels

Example: $^{12}C+^{12}C$: Essentially $^{20}Ne+\alpha$, $^{23}Na+p$, $^{23}Mg+n$ channels

→ absorption simulated by a complex potential $V = V_R + iW$



experimental cross section
Satkowiak et al. PRC 26 (1982) 2027



4. Generalizations

Origin of the complex potential

- Time-dependent Schrödinger equation: $i\hbar \frac{\partial \Psi}{\partial t} = H\Psi = (T+V)\Psi$

  is equivalent to (real potential): $\frac{\partial \rho}{\partial t} + \mathrm{div}\,\boldsymbol{J} = 0$, with $\boldsymbol{J} = \frac{1}{\mu} Re\,(\Psi^* \boldsymbol{p} \Psi)$

  → constant current J

  for a complex potential: $V = V_R + iV_I$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\boldsymbol{J} = \frac{2}{\hbar} \mathrm{V_I} \rho$$

→ $V_I < 0$ simulates absorption (inelastic, transfer, etc) not explicitly included

Simple interpretation

- Let us assume a constant potential $V = -V_0$

  → wave function=plane wave $\Psi \sim \exp(ik_0 r) \sim \exp(i\sqrt{\frac{2\mu(E+V_0)}{\hbar^2}}r)$ → $|\Psi|^2 = 1$

- For a complex potential $V = -V_0 - iW_0$ (W_0 small)
  → wave function $\Psi \sim \exp(ik_0 r)\exp(-k_I r)$: → $|\Psi|^2 \sim \exp(-2k_I r)$
  → incoming particles « disappear » (=absorption)

4. Generalizations

Generalization 4 : system with spins (multichannel)

- Allows to deal with inelatic, transfer, etc..
- Phase shift (single-channel) → collision (scattering) matrix

A. Quantum numbers

- Good quantum numbers: total angular momentum J and parity π
- Additional indices
  - Channel α defined by 2 nuclei with spins I_1, I_2 et parités π_1, π_2
  - Channel spin $I = I_1 + I_2$
  - Relative angular momentum ℓ

  with $$J = I + \ell$$ $$\pi = \pi_1 \pi_2 (-1)^{\ell}$$

Examples:

1) α+n : I_1=0, I_2=1/2 → I=1/2, $\ell = |J - \frac{1}{2}|$ or $J + \frac{1}{2}$: channel number =1

2) p+n : I_1=I_2=1/2 → $I = 0$ or 1: channel number depends on J

4. Generalizations

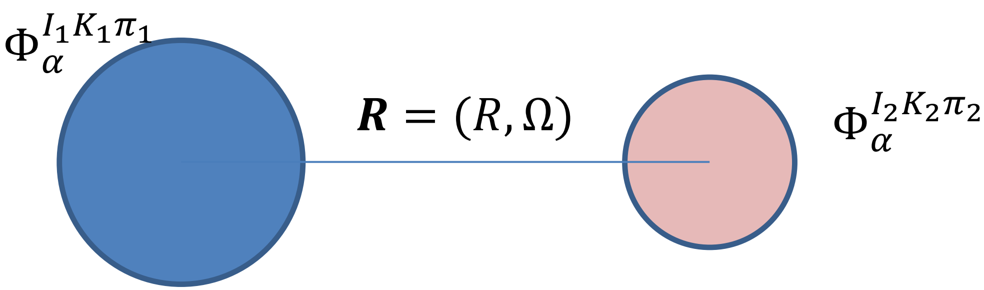
3) Reaction $^6\mathrm{Li} + p \to {}^3\mathrm{He} + \alpha$

- channel 1: $^6\mathrm{Li} + p$, spin(^{6}Li) I_1=1$^+$, spin(p) I_2=1/2$^+$
- channel 2: $^3\mathrm{He} + \alpha$, spin (^{3}He)=1/2$^+$, spin(α)=0$^+$

$J\pi$	**channel** $\alpha = 1$	**channel** $\alpha = 2$	$\alpha I \ell$
$1/2^+$	$I = 1/2, \ell = 0$ $I = 3/2, \ell = 2$	$I = 1/2, \ell = 0$	3 values
$1/2^-$	$I = 1/2, \ell = 1$ $I = 3/2, \ell = 1$	$I = 1/2, \ell = 1$	3 values
$3/2^+$	$I = 1/2, \ell = 2$ $I = 3/2, \ell = 0,2$	$I = 1/2, \ell = 2$	4 values
$3/2^-$	$I = 1/2, \ell = 1$ $I = 3/2, \ell = 1,3$	$I = 1/2, \ell = 1$	4 values

→ Size of the collision matrix is: 3x3 or 4x4

B. Coupled-channel wave functions



1. Internal wave functions of nuclei 1 and 2 : $\Phi_\alpha^{I_1K_1\pi_1}$ and $\Phi_\alpha^{I_2K_2\pi_2}$

2. Coupling of the projectile+target spins: $I = I_1 \oplus I_2$

$$\Phi_\alpha^{IK\pi_1\pi_2} = \sum_{K_1K_2} < I_1K_1I_2K_2|IK > \Phi_\alpha^{I_1K_1\pi_1}\Phi_\alpha^{I_2K_2\pi_2} = \left[\Phi_\alpha^{I_1\pi_1} \otimes \Phi_\alpha^{I_2\pi_2}\right]^{IK}$$

3. Channel function is defined by ($J = \ell \oplus I$)

$$\varphi_{\alpha I\ell}^{JM\pi}(\Omega) = \left[\Phi_\alpha^{I\pi_1\pi_2} \otimes Y_\ell(\Omega)\right]^{JM}$$

4. Total wave function for given J and π :

$$\Psi^{JM\pi} = \sum_{\alpha I\ell} u_{\alpha I\ell}^{J\pi}(r)\, \varphi_{\alpha I\ell}^{JM\pi}(\Omega)$$

4. Generalizations

4. Total wave function for given J and π :$\Psi^{JM\pi} = \sum_{\alpha I \ell} u^{J\pi}_{\alpha I \ell}(r)\, \varphi^{JM\pi}_{\alpha I \ell}(\Omega)$

5. Radial functions $u^{J\pi}_{\alpha I \ell}(r)$ are obtained from a set of coupled equations

- Single-channel: $-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right) u_\ell + V(r) u_\ell = E u_\ell$

  With $u_l(r) \to I_\ell(\eta, kr) - U_\ell O_\ell(\eta, kr)$

  $U_\ell = \exp(2i\delta_\ell)$ = « matrix » 1x1

- Multichannel

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right) u^{J\pi}_{\alpha I \ell} + \sum_{\alpha' I' \ell'} V^{J\pi}_{\alpha I \ell, \alpha' I' \ell'}(r) u^{J\pi}_{\alpha' I' \ell'} = E u^{J\pi}_{\alpha I \ell}$$

with $u^{J\pi}_{\alpha I \ell}(r) \to I_\ell(r)\delta_{\alpha\omega} - U^{J\pi}_{\alpha I \ell, \alpha' I' \ell'} O_{\ell'}(r)$

Collision matrix provides cross sections (several J values are necessary)

4. Generalizations

Comments on the multichannel system

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}\right)u^{J\pi}_{\alpha I\ell}+\sum_{\alpha I'\ell'}V^{J\pi}_{\alpha I\ell,\alpha' I'\ell'}(r)u^{J\pi}_{\alpha' I'\ell'}=Eu^{J\pi}_{\alpha I\ell}$$

- Standard form of many scattering theories (CDCC, folding, microscopic, 3-body, etc.)
- Theories differ by the calculation of the potentials
- Diagonal and non-diagonal potentials $V^{J\pi}_{\alpha I\ell,\alpha' I'\ell'}(r)$
  non-diagonal $V^{J\pi}_{\alpha I\ell,\alpha' I'\ell'}(r)\rightarrow 0$ for large r
  diagonal $V^{J\pi}_{\alpha I\ell,\alpha Il}(r)\rightarrow \frac{Z_p Z_t e^2}{r}$ for large r
- Main problems
  Sometimes: more than 100 channels are included
  Long range of the potential → numerical difficulties
- Numerical resolution can be time consuming
  2 main methods: Numerov (+ improvements)
  R-matrix method

4. Generalizations

C. Cross sections in a multichannel formalism

One channel: $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$

$$f(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\delta_\ell) - 1)\,P_\ell(\cos\theta)$$

Multi channel: $\frac{d\sigma}{d\Omega}(\alpha \to \alpha') = \sum_{K_1K_2K_1'K_2'} \left|f_{K_1K_2,K_1'K_2'}(\theta)\right|^2$

$$f_{K_1K_2,K_1'K_2'}(\theta) = \sum_{J\pi}\sum_{lI,l'I'} \ldots U^{J\pi}_{\alpha I\ell,\alpha' I'\ell'} Y_{l'}(\theta,0)$$

Collision matrix

- generalization of d: $U_{ij}=\eta_{ij}\exp(2i\delta_{ij})$
- determines the cross section

With: K_1K_2=spin orientations in the entrance channel

$K_1'K_2'$=spin orientations in the exit channel