# AN ASYMPTOTIC SOLUTION OF LARGE-N QCD, AND OF N=1 SUSY QCD

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Glueball and meson propagators of any spin in large-N QCD
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and

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Solving QCD (or its SUSY cousin, n=1 SUSY QCD) at large-N is a long standing difficult problem

An easier problem is to solve them only asymptotically in the UV

In a sense we already have an asymptotic solution:
It is standard perturbation theory

But solving the large-N theory, even only asymptotically, is much more interesting:

This solution would replace QCD as a theory of gluons and quarks, that is strongly coupled in the infrared in perturbation theory, with a theory of glueballs and mesons that is weakly coupled at all scales

We have found an asymptotic solution of massless QCD at large-N (and of n=1SUSY QCD) in a sense specified later, by a new purely field-theoretical method, based on fundamental principles, that we call the Asymptotically-Free Bootstrap

It expresses uniquely 2 and 3 point correlators of any spin (explicitly for lower spins) in terms of glueball and meson propagators, in such a way that the result is asymptotic in the UV to RG-improved perturbation theory

It extends to certain primitive r-point correlators and S-matrix amplitudes to all 1/N orders

Why is it interesting ? (Should we really answer this rhetoric question ?)

First and foremost, an asymptotic solution of this kind is a guide to find out an actual solution by other methods, either field theoretical or string theoretical

It an easy way to check forthcoming proposed exact solutions (easy because based only on fundamental principles of field theory)

It has a number of physical applications,
e.g. the pion charged and neutral form factors, light by light scattering amplitudes relevant for QCD corrections to muon anomalous magnetic moment ... and so on, that we will not discuss in this talk

Besides, it provides a quantitative understanding of how much accurate (approximate or would-be exact) solutions proposed in the past years are

In the past years several different proposals have been advanced for the glueball propagators of QCD-like theories based on the the AdS-String /Large-N Gauge Theory Correspondence by the Princetonians (Witten, Klebanov-Strassler, Maldacena-Nunez, Polchinski-Strassler, and many followers . . .)

and more recently on a Topological Field Theory underlying large-N YM (M. B.)

Based on the asymptotic solution we will show that:

None of the proposals for the scalar glueball propagators based on the AdS String/Large-N Gauge Theories correspondence agrees with the universal RG estimate in the UV for any asymptotically free gauge theory

(perhaps as expected, because the AdS models in the supergravity approximation are in fact strongly coupled in the UV)

but the TFT does

But the most fundamental consequence of the asymptotically-free bootstrap is the explicit structure of the asymptotic S-matrix

This puts the strongest constraints on any (string ?) solution for the S-matrix of large-N QCD and of n=1 SUSY QCD

so explicit, and so strong constraints, that we conjecture that they determine uniquely the large-N QCD S-matrix on the string side

as we will see

What makes possible the Asymptotically-Free Bootstrap is a recently-proved Asymptotic Theorem for large-N two-point correlators

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# The large-N limit of SU(N) QCD:

$$Z = \int \delta A \delta\bar{\psi}\delta\psi\, e^{-\frac{N}{2g^2}\int \sum_{\alpha\beta} Tr\left(F^2_{\alpha\beta}\right)+i\sum_f \bar{\psi}_f \gamma^\alpha D_\alpha \psi_f\, d^4x}$$

(G. 't Hooft 1974)

The following remarkable simplifications occur

For example, in the pure glue sector:

$$< \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n) >_{conn} \sim N^{2-n}$$

thus at the leading 1/N order:

$$< \frac{1}{N}\sum_{\alpha\beta} trF^2_{\alpha\beta}(x_1)\ldots\frac{1}{N}\sum_{\alpha\beta} trF^2_{\alpha\beta}(x_k) >=$$

$$< \frac{1}{N}\sum_{\alpha\beta} trF^2_{\alpha\beta}(x_1) > \ldots < \frac{1}{N}\sum_{\alpha\beta} trF^2_{\alpha\beta}(x_k) >$$

At next to leading 1/N order, because of the vanishing of the interaction associated to 3 and multi-point correlators,

two-point correlators are an infinite sum of free fields satisfying the the Kallen-Lehmann representation (A. Migdal, 1977):

$$\int \langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn} e^{-ip\cdot x} d^4x = \sum_{n=1}^{\infty} P^{(s)}\left(\frac{p_\alpha}{m_n^{(s)}}\right) \frac{|<0|\mathcal{O}^{(s)}(0)|p,n,s>'|^2}{p^2+m_n^{(s)2}}$$

$$<0|\mathcal{O}^{(s)}(0)|p,n,s,j>= e_j^{(s)}\left(\frac{p_\alpha}{m}\right) <0|\mathcal{O}^{(s)}(0)|p,n,s>'$$

$$\sum_j e_j^{(s)}\left(\frac{p_\alpha}{m}\right)\overline{e_j^{(s)}\left(\frac{p_\alpha}{m}\right)} = P^{(s)}\left(\frac{p_\alpha}{m}\right)$$

Let me start with the following

simple

but fundamental question

What is the large momentum behavior of two-point correlators of any integer spin s in pure Yang-Mills, in QCD and in n=1 SUSY QCD with massless quarks, or in any confining asymptotically free gauge theory massless in perturbation theory ?

$$\int \langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x = ?$$

For example:

$$\mathcal{O}^{(s)} = Tr(F^2_{\alpha\beta}), \bar{\psi}\gamma^\alpha\psi, T_{\alpha\beta}, \cdots$$

The answer is simple but not completely trivial, as we will see momentarily. We have found it by standard methods:

Perturbation Theory +

Asymptotic Freedom +

Renormalization Group +

Some non-trivial subtlety . . .

$$\int \langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x$$

$$\sim P^{(s)}(\frac{p_\alpha}{p})\, p^{2D-4}\left[\frac{1}{\beta_0 \log(\frac{p^2}{\Lambda^2_{QCD}})}\left(1-\frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{p^2}{\Lambda^2_{QCD}})}{\log(\frac{p^2}{\Lambda^2_{QCD}})}+O(\frac{1}{\log(\frac{p^2}{\Lambda^2_{QCD}})})\right)\right]^{\frac{\gamma_0}{\beta_0}-1}$$

up to a polynomial in momentum, i.e. a contact term, i.e. a distribution supported at x=0 in coordinate space (this is the first subtlety) that must be subtracted;

$P^{(s)}(\frac{p_\alpha}{p})$ is the projector obtained substituting $m^2=-p^2$ in the massive projector of spin s $P^{(s)}(\frac{p_\alpha}{m})$ (this is the second subtlety)

# Definitions:

$$\gamma_{\mathcal{O}^{(s)}}(g) = -\frac{\partial \log Z^{(s)}}{\log \mu} = -\gamma_0 g^2 + \cdots$$

$$\beta(g) = \frac{\partial g}{\partial \log \mu} = -\beta_0 g^3 - \beta_1 g^5 + \cdots$$

Therefore, at the leading large-N order it must hold:

$$\sum_{n=1}^{\infty} P^{(s)}\left(\frac{p_\alpha}{m_n^{(s)}}\right) \frac{|<0|\mathcal{O}^{(s)}(0)|p,n,s>'|^2}{p^2+m_n^{(s)2}}$$

$$\sim P^{(s)}\left(\frac{p_\alpha}{p}\right) p^{2D-4} \left[\frac{1}{\beta_0 \log(\frac{p^2}{\Lambda^2_{QCD}})}\left(1-\frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{p^2}{\Lambda^2_{QCD}})}{\log(\frac{p^2}{\Lambda^2_{QCD}})}+O\left(\frac{1}{\log(\frac{p^2}{\Lambda^2_{QCD}})}\right)\right)\right]^{\frac{\gamma_0}{\beta_0}-1}$$

up to contact terms

Fundamental question:

Which are the constraints on the residues and the poles that follow from this asymptotic equality ?

Oddly, neither Migdal nor other people found any answer (for deep reasons in the case of Migdal, that I will discuss possibly at the end of the talk)

We will answer this question today, after 37 years !

# The answer to the fundamental question is the following Asymptotic Theorem:

$$\int \langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x \sim \sum_{n=1}^{\infty} P^{(s)}\big(\frac{p_\alpha}{m_n^{(s)}}\big)\, \frac{m_n^{(s)2D-4} Z_n^{(s)2} \rho_s^{-1}(m_n^{(s)2})}{p^2+m_n^{(s)2}}$$

$$= P^{(s)}\big(\frac{p_\alpha}{p}\big)\, p^{2D-4} \sum_{n=1}^{\infty} \frac{Z_n^{(s)2} \rho_s^{-1}(m_n^{(s)2})}{p^2+m_n^{(s)2}} + \cdots$$

$$\sim P^{(s)}\big(\frac{p_\alpha}{p}\big)\, p^{2D-4} \int_{m_1^{(s)2}}^{\infty} \frac{Z^{(s)2}(m)}{p^2+m^2} dm^2 + \cdots$$

$$\sim P^{(s)}\big(\frac{p_\alpha}{p}\big)\, p^{2D-4} \Big[\frac{1}{\beta_0 \log(\frac{p^2}{\Lambda^2_{QCD}})}\Big(1-\frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{p^2}{\Lambda^2_{QCD}})}{\log(\frac{p^2}{\Lambda^2_{QCD}})} + O(\frac{1}{\log(\frac{p^2}{\Lambda^2_{QCD}})})\Big)\Big]^{\frac{\gamma_0}{\beta_0}-1}$$

$$\sum_{n=1}^{\infty} f(m_n^{(s)2}) \sim \int_1^{\infty} f(m_n^{(s)2}) dn = \int_{m_1^{(s)2}}^{\infty} f(m^2)\rho_s(m^2) dm^2$$

$$Z_n^{(s)} \equiv Z^{(s)}(m_n^{(s)}) = \exp \int_{g(\mu)}^{g(m_n^{(s)})} \frac{\gamma_{\mathcal{O}^{(s)}}(g)}{\beta(g)} dg$$

$$Z_n^{(s)2} \sim \Big[\frac{1}{\beta_0 \log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}}\Big(1-\frac{\beta_1}{\beta_0^2}\frac{\log\log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}}{\log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}} + O(\frac{1}{\log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}})\Big)\Big]^{\frac{\gamma_0}{\beta_0}}$$

$$\langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn} \sim \sum_{n=1}^{\infty} \frac{1}{(2\pi)^4} \int P^{(s)}\Big(\frac{p_\alpha}{m_n^{(s)}}\Big) \frac{m_n^{(s)2D-4} Z_n^{(s)2} \rho_s^{-1}(m_n^{(s)2})}{p^2 + m_n^{(s)2}} \, e^{ip\cdot x} d^4p$$

## Spin 1:

$$\int \langle \mathcal{O}^{(1)}_{\alpha}(x)\mathcal{O}^{(1)}_{\beta}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x$$

$$\sim \sum_{n=1}^{\infty} (\delta_{\alpha\beta} + \frac{p_\alpha p_\beta}{m_n^{(1)2}}) \frac{m_n^{(1)2D-4} Z_n^{(1)2} \rho_1^{-1}(m_n^{(1)2})}{p^2 + m_n^{(1)2}}$$

$$\sim p^{2D-4}(\delta_{\alpha\beta} - \frac{p_\alpha p_\beta}{p^2}) \sum_{n=1}^{\infty} \frac{Z_n^{(1)2} \rho_1^{-1}(m_n^{(1)2})}{p^2 + m_n^{(1)2}} + \cdots$$

## Spin 2:

$$\int \langle \mathcal{O}^{(2)}_{\alpha\beta}(x)\mathcal{O}^{(2)}_{\gamma\delta}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x$$

$$\sim \sum_{n=1}^{\infty} \left[\frac{1}{2}\eta_{\alpha\gamma}(m_n^{(2)})\eta_{\beta\delta}(m_n^{(2)}) + \frac{1}{2}\eta_{\beta\gamma}(m_n^{(2)})\eta_{\alpha\delta}(m_n^{(2)}) - \frac{1}{3}\eta_{\alpha\beta}(m_n^{(2)})\eta_{\gamma\delta}(m_n^{(2)})\right] \frac{m_n^{(2)2D-4} Z_n^{(2)2} \rho_2^{-1}(m_n^{(2)2})}{p^2 + m_n^{(2)2}}$$

$$\sim p^{2D-4}\left[\frac{1}{2}\eta_{\alpha\gamma}(p)\eta_{\beta\delta}(p) + \frac{1}{2}\eta_{\beta\gamma}(p)\eta_{\alpha\delta}(p) - \frac{1}{3}\eta_{\alpha\beta}(p)\eta_{\gamma\delta}(p)\right] \sum_{n=1}^{\infty} \frac{Z_n^{(2)2} \rho_2^{-1}(m_n^{(2)2})}{p^2 + m_n^{(2)2}} + \cdots$$

$$\eta_{\alpha\beta}(m) = \delta_{\alpha\beta} + \frac{p_\alpha p_\beta}{m^2} \qquad \eta_{\alpha\beta}(p) = \delta_{\alpha\beta} - \frac{p_\alpha p_\beta}{p^2}$$

We now look for a vast generalization of the Asymptotic Theorem to r-point correlators

that we call the Asymptotically-Free Bootstrap

# The Asymptotically-Free Bootstrap (for any spin)

1. Conformal invariance of correlators at lowest order of perturbation theory. For 2 an 3 point correlators structure is fixed uniquely by conformal invariance

2. RG improvement by Callan-Symanzik + asymptyotic freedom ; 1+2 imply that 3 point correlators factorize asymptotically on products of certain coefficients of OPE

3. Kallen-Lehmann representation of coefficients of OPE; This is the new crucial feature, that extends to OPE the aforementioned asymptotic theorem for 2 point correlators

4. 1+2+3 fix uniquely the glueball and meson 3-point correlators asymptotically in the UV

5. primitive r-point correlators follow by iterating the OPE

# The asymptotically-free bootstrap (scalar case, positive charge conjugation)

$$< O(x_1)O(x_2) >_{conn} = G^{(2)}(x_1 - x_2)$$

$$< O(x_1)O(x_2)O(x_3) >_{conn} = G^{(3)}(x_1 - x_2, x_2 - x_3, x_3 - x_1)$$

$$\left(\sum_{i=1}^{i=2} x_i \cdot \frac{\partial}{\partial x_i} + \beta(g)\frac{\partial}{\partial g} + 2(D + \gamma(g))\right) G^{(2)}(x_1 - x_2) = 0$$

$$\left(\sum_{i=1}^{i=3} x_i \cdot \frac{\partial}{\partial x_i} + \beta(g)\frac{\partial}{\partial g} + 3(D + \gamma(g))\right) G^{(3)}(x_1 - x_2, x_2 - x_3, x_3 - x_1) = 0$$

# 1. Conformal invariance of correlators at lowest order of perturbation theory. For 2 an 3 point correlators structure is fixed uniquely by conformal invariance

$$\left(\sum_{i=1}^{i=2} x_i \cdot \frac{\partial}{\partial x_i} + 2D\right) G^{(2)}(x_1 - x_2) = 0$$

$$\left(\sum_{i=1}^{i=3} x_i \cdot \frac{\partial}{\partial x_i} + 3D\right) G^{(3)}(x_1 - x_2, x_2 - x_3, x_3 - x_1) = 0$$

$$G^{(2)}(x_1 - x_2) = C_2 \frac{1}{(x_1 - x_2)^{2D}}$$

$$G^{(3)}(x_1 - x_2, x_2 - x_3, x_3 - x_1) = C_3 \frac{1}{(x_1 - x_2)^D} \frac{1}{(x_2 - x_3)^D} \frac{1}{(x_3 - x_1)^D}$$

# 2. RG improvement by Callan-Symanzik +asymptyotic freedom

$$\left(\sum_{i=1}^{i=2} x_i \cdot \frac{\partial}{\partial x_i} + 2(D - \gamma_0 g^2)\right) G^{(2)}(x_1 - x_2) = 0$$

$$\left(\sum_{i=1}^{i=3} x_i \cdot \frac{\partial}{\partial x_i} + 3(D - \gamma_0 g^2)\right) G^{(3)}(x_1 - x_2, x_2 - x_3, x_3 - x_1) = 0$$

Thus at next-to-leading perturbative order:

$$G^{(2)}(x_1 - x_2) = C_2(1 + O(g^2(\mu)))\frac{1}{(x_1 - x_2)^{2D - \gamma_0 g^2(\mu)}}$$

$$\sim C_2(1 + O(g^2(\mu)))\frac{1}{(x_1 - x_2)^{2D}}(1 + \gamma_0 g^2(\mu)\log(|x_1 - x_2|\mu))$$

$$G^{(3)}(x_1 - x_2, x_2 - x_3, x_3 - x_1) \sim C_3(1 + O(g^2(\mu)))\frac{(1 + \gamma_0 g^2(\mu)\log(|x_1 - x_2|\mu))}{(x_1 - x_2)^D}$$

$$\frac{(1 + \gamma_0 g^2(\mu)\log(|x_2 - x_3|\mu))}{(x_2 - x_3)^D}\frac{(1 + \gamma_0 g^2(\mu)\log(|x_3 - x_1|\mu))}{(x_3 - x_1)^D}$$

# Renormalization-group improvement:

$$1+\gamma_0 g^2(\mu)\log(|x|\mu) \sim (1+\beta_0 g^2(\mu)\log(|x|\mu))^{\frac{\gamma_0}{\beta_0}} \sim (\frac{g(x)}{g(\mu)})^{\frac{\gamma_0}{\beta_0}}$$

$$g(x)^2 \sim \frac{1}{\beta_0 \log(\frac{1}{x^2\Lambda^2_{QCD}})}\left(1-\frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{1}{x^2\Lambda^2_{QCD}})}{\log(\frac{1}{x^2\Lambda^2_{QCD}})}+O(\frac{1}{\log(\frac{1}{x^2\Lambda^2_{QCD}})})\right)$$

$$G^{(2)}(x_1-x_2) \sim C_2(1+O(g^2))\frac{(\frac{g(x_1-x_2)}{g(\mu)})^{\frac{2\gamma_0}{\beta_0}}}{(x_1-x_2)^{2D}}$$

$$G^{(3)}(x_1-x_2,x_2-x_3,x_3-x_1) \sim C_3(1+O(g^2))\frac{(\frac{g(x_1-x_2)}{g(\mu)})^{\frac{\gamma_0}{\beta_0}}}{(x_1-x_2)^{D}}\frac{(\frac{g(x_2-x_3)}{g(\mu)})^{\frac{\gamma_0}{\beta_0}}}{(x_2-x_3)^{D}}\frac{(\frac{g(x_3-x_1)}{g(\mu)})^{\frac{\gamma_0}{\beta_0}}}{(x_3-x_1)^{D}}$$

# 1+2 imply that 3 point correlators factorize asymptotically on products of certain coefficients of OPE

$$O(x)O(0) \sim C(x)O(0) + \cdots$$

$$\left( x \cdot \frac{\partial}{\partial x} + \beta(g)\frac{\partial}{\partial g} + (D + \gamma(g)) \right) C(x) = 0$$

$$C(x) \sim \frac{\left(\frac{g(x)}{g(\mu)}\right)^{\frac{\gamma_0}{\beta_0}}}{x^D}$$

$$< O(x_1)O(x_2)O(x_3) >_{conn} \sim C(x_1 - x_2) < O(x_2)O(x_3) >_{conn}$$

$$\sim C(x_1 - x_2) G^{(2)}(x_2 - x_3)$$

$$\sim C(x_1 - x_2) C^2(x_2 - x_3)$$

$$G^{(3)}(x_1 - x_2, x_2 - x_3, x_3 - x_1) \sim C(x_1 - x_2) C(x_2 - x_3) C(x_3 - x_1)$$

# 3. Kallen-Lehmann (KL) representation of coefficients of OPE; This is the new crucial feature, that extends to OPE the aforementioned asymptotic theorem for 2 point correlators

$$C(x_1 - x_2) \sim \sum_{n=1}^{\infty} \frac{1}{(2\pi)^4} \int \frac{m_n^{D-4} Z_n \rho^{-1}(m_n^2)}{p^2 + m_n^2} e^{ip\cdot(x_1-x_2)} d^4p$$

$$\sim \sum_{n=1}^{\infty} \frac{1}{(2\pi)^4} \int \frac{m_n^{D-4} \left(\frac{g(m_n)}{g(\mu)}\right)^{\frac{\gamma_0}{\beta_0}} \rho^{-1}(m_n^2)}{p^2 + m_n^2} e^{ip\cdot(x_1-x_2)} d^4p$$

$$\sim \frac{\left(\frac{g(x_1-x_2)}{g(\mu)}\right)^{\frac{\gamma_0}{\beta_0}}}{(x_1 - x_2)^D}$$

# 4. 1+2+3 fix uniquely the glueball and meson 3-point vertices asymptotically in the UV

$$< O_{D,\gamma_0}(x_1) O_{D,\gamma_0}(x_2) O_{D,\gamma_0}(x_3) >_{conn} \sim$$

$$\sum_{n_1=1}^{\infty} \frac{1}{(2\pi)^4} \int \frac{m_{n_1}^{D-4} Z_{n_1} \rho^{-1}(m_{n_1}^2)}{p_1^2 + m_{n_1}^2} e^{ip_1 \cdot (x_1 - x_2)} d^4 p_1$$

$$\sum_{n_2=1}^{\infty} \frac{1}{(2\pi)^4} \int \frac{m_{n_2}^{D-4} Z_{n_2} \rho^{-1}(m_{n_2}^2)}{p_2^2 + m_{n_2}^2} e^{ip_2 \cdot (x_2 - x_3)} d^4 p_2$$

$$\sum_{n_3=1}^{\infty} \frac{1}{(2\pi)^4} \int \frac{m_{n_3}^{D-4} Z_{n_3} \rho^{-1}(m_{n_3}^2)}{p_3^2 + m_{n_3}^2} e^{ip_3 \cdot (x_3 - x_1)} d^4 p_3$$

# 5. primitive r-point asymptotic correlators follow by iterating the OPE

$$< O(q_1)O(q_2) >_{conn} \sim \delta(q_1+q_2) \sum_{n=1}^{\infty} \frac{m_n^{2D-4} Z_n^2 \rho^{-1}(m_n^2)}{q_1^2+m_n^2}$$

$$< O_{D,\gamma_0}(q_1)O_{D,\gamma_0}(q_2)O_{D,\gamma_0}(q_3) >_{conn}$$
$$\sim \delta(q_1+q_2+q_3) \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \int \frac{m_{n_1}^{D-4} Z_{n_1}\rho^{-1}(m_{n_1}^2)}{p^2+m_{n_1}^2} \frac{m_{n_2}^{D-4} Z_{n_2}\rho^{-1}(m_{n_2}^2)}{(p+q_2)^2+m_{n_2}^2} \frac{m_{n_3}^{D-4} Z_{n_3}\rho^{-1}(m_{n_3}^2)}{(p+q_2+q_3)^2+m_{n_3}^2} d^4p$$

and so on ...

But this is not the whole story !

We want to find the asymptotic effective action and asymptotic S-matrix

i.e. we want to go from

propagators and correlators

to

kinetic terms and vertices

(this requires some more not-completely-trivial work that we skip, writing only the final answer)

as a result we find some surprises for the S-matrix

# The generating functional of scalar correlation functions in massless large-N QCD and n=1 SUSY QCD asymptotically in the UV

$$
\begin{aligned}
S_{eff} =& \frac{c_2}{2!} \sum_n \int dq_1 dq_2 \delta(q_1+q_2) m_n^{4-2D} Z_n^{-2} \rho(m_n^2) \Phi_n(q_1)(q_1^2+m_n^2)\Phi_n(q_2) \\
&+ \frac{c_3(N)}{3!} \int dq_1 dq_2 dq_3 \delta(q_1+q_2+q_3) \int \sum_{n_1=1}^{\infty} m_{n_1}^2 \frac{m_{n_1}^{-D} Z_{n_1}^{-1} \Phi_{n_1}(q_2)}{p^2+m_{n_1}^2} \\
&\sum_{n_2=1}^{\infty} m_{n_2}^2 \frac{m_{n_2}^{-D} Z_{n_2}^{-1} \Phi_{n_2}(q_3)}{(p+q_2)^2+m_{n_2}^2} \sum_{n_3=1}^{\infty} m_{n_3}^2 \frac{m_{n_3}^{-D} Z_{n_3}^{-1} \Phi_{n_3}(q_1)}{(p+q_2+q_3)^2+m_{n_3}^2} dp \\
&+ \cdots
\end{aligned}
$$

$$c_2 \sim 1$$

$$c_3(N) \sim \frac{1}{N} \text{ for glueballs and gluinoballs}$$

$$c_3(N) \sim \frac{1}{\sqrt{N}} \text{ for mesons and s-mesons}$$

# The generating functional of scalar S-matrix amplitudes in massless large-N QCD and n=1 SUSY QCD asymptotically in the UV

$$S_{can} = \frac{c_2}{2!} \sum_n \int dq_1 dq_2 \delta(q_1 + q_2) \Phi_n(q_1)(q_1^2 + m_n^2) \Phi_n(q_2)$$

$$+ \frac{c_3(N)}{3!} \int dq_1 dq_2 dq_3 \delta(q_1 + q_2 + q_3) \int \sum_{n_1=1}^{\infty} \frac{\rho^{-\frac{1}{2}}(m_{n_1}^2)\Phi_{n_1}(q_2)}{p^2 + m_{n_1}^2}$$

$$\sum_{n_2=1}^{\infty} \frac{\rho^{-\frac{1}{2}}(m_{n_2}^2)\Phi_{n_2}(q_3)}{(p+q_2)^2 + m_{n_2}^2} \sum_{n_3=1}^{\infty} \frac{\rho^{-\frac{1}{2}}(m_{n_3}^2)\Phi_{n_3}(q_1)}{(p+q_2+q_3)^2 + m_{n_3}^2} dp$$

$$+ \cdots$$

The S-matrix depends only on the spectrum but not on the anomalous dimensions ! No conventional string theory has this S-matrix, since vertices are non-local but very much field theoretical (as in super-renormalizable field theories).

# We suggest a possible string candidate that has chances to reproduce these amplitudes at the end of the talk

$$S^{(3)}(n_1,n_2,n_3) = \delta(q_1+q_2+q_3)\int \frac{\rho^{-\frac{1}{2}}(m_{n_1}^2)}{p^2+m_{n_1}^2}\frac{\rho^{-\frac{1}{2}}(m_{n_2}^2)}{(p+q_2)^2+m_{n_2}^2}\frac{\rho^{-\frac{1}{2}}(m_{n_3}^2)}{(p+q_2+q_3)^2+m_{n_3}^2}d^4p$$
$$+\,permutations$$

$$S^{(4)}(n_1,n_2,n_3,n_4) = \delta(q_1+q_2+q_3+q_4)\sum_n \int \frac{\rho^{-\frac{1}{2}}(m_{n_1}^2)}{p^2+m_{n_1}^2}\frac{\rho^{-\frac{1}{2}}(m_{n_2}^2)}{(p+q_2)^2+m_{n_2}^2}\frac{1}{(p-q_1)^2+m_n^2}d^4p$$
$$\frac{\rho^{-1}(m_n^2)}{(q_1+q_2)^2+m_n^2}\int \frac{\rho^{-\frac{1}{2}}(m_{n_3}^2)}{p^2+m_{n_3}^2}\frac{\rho^{-\frac{1}{2}}(m_{n_4}^2)}{(p+q_4)^2+m_{n_4}^2}\frac{1}{(p-q_3)^2+m_n^2}d^4p$$
$$+\,permutations$$

+ the 1PI 4-point amplitude

In fact, in the pure glue sector with positive charge conjugation the generating functional of the S-matrix can be resummed

$$S = \frac{1}{2} Tr \int \Phi(-\Delta + M^2)\Phi \, d^4x + \frac{1}{2} N^2 \log Det_3(-\Delta + M^2 + \frac{c}{N} \rho_0^{-\frac{1}{2}} \Phi)$$

We conjecture that this structure fixes uniquely the string theory that solves QCD, and we suggest a possible candidate at the end of the talk

# Specialize the RG estimate to scalar and pseudoscalar glueball propagators

$$\int \langle \frac{\beta(g)}{gN} tr\big(\sum_{\alpha\beta} F^2_{\alpha\beta}(x)\big) \frac{\beta(g)}{gN} tr\big(\sum_{\alpha\beta} F^2_{\alpha\beta}(0)\big)\rangle_{conn} e^{ip\cdot x} d^4x$$

$$= C_S p^4 \left[ \frac{1}{\beta_0 \log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log \frac{p^2}{\Lambda^2_{\overline{MS}}}}{\log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) + O\left( \frac{1}{\log^2 \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) \right]$$

$$\int \langle \frac{g^2}{N} tr\big(\sum_{\alpha\beta} F_{\alpha\beta}\tilde{F}_{\alpha\beta}(x)\big) \frac{g^2}{N} tr\big(\sum_{\alpha\beta} F_{\alpha\beta}\tilde{F}_{\alpha\beta}(0)\big)\rangle_{conn} e^{ip\cdot x} d^4x$$

$$= C_{PS} p^4 \left[ \frac{1}{\beta_0 \log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log \frac{p^2}{\Lambda^2_{\overline{MS}}}}{\log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) + O\left( \frac{1}{\log^2 \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) \right]$$

$$\int \langle \frac{g^2}{N} tr\big(\sum_{\alpha\beta} F^{-2}_{\alpha\beta}(x)\big) \frac{g^2}{N} tr\big(\sum_{\alpha\beta} F^{-2}_{\alpha\beta}(0)\big)\rangle_{conn} e^{ip\cdot x} d^4x$$

$$= C_{ASD} p^4 \left[ \frac{1}{\beta_0 \log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log \frac{p^2}{\Lambda^2_{\overline{MS}}}}{\log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) + O\left( \frac{1}{\log^2 \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) \right]$$

# Perturbative check: the 3-loop computation by Chetyrkin et al.

$$< trF^2(p) trF^2(-p) >_{conn} \quad = -\frac{N^2-1}{4\pi^2} p^4 \log\frac{p^2}{\mu^2}\Big[1 + g^2(\mu)\Big(f_0 - \beta_0 \log\frac{p^2}{\mu^2}\Big)$$

$$+ g^4(\mu)\Big(f_1 + f_2 \log\frac{p^2}{\mu^2} + f_3 \log^2\frac{p^2}{\mu^2}\Big)\Big]$$

$$\begin{aligned}
f_0 &= \frac{73}{3(4\pi)^2} \\
f_1 - f_3\pi^2 &= \Big(\frac{37631}{54} - \frac{242}{3}\zeta(2) - 110\zeta(3)\Big)\frac{1}{(4\pi)^4} \\
-2\beta_0 &= -2\frac{11}{3(4\pi)^2} \\
2f_2 &= -\frac{313}{(4\pi)^4} \Rightarrow f_2 = -\frac{313}{2(4\pi)^4} \\
3f_3 &= \frac{121}{3(4\pi)^4} \Rightarrow f_3 = \frac{121}{9(4\pi)^4} \Rightarrow f_3 = \beta_0^2 \\
\\
\Rightarrow f_1 &= \Big(\frac{37631}{54} - 110\zeta(3)\Big)\frac{1}{(4\pi)^4}
\end{aligned}$$

# Perturbative check: the 3-loop computation by Chetyrkin et al.

$$< trF\tilde{F}(p)trF\tilde{F}(-p) >_{conn} = -\frac{(N^2-1)}{4\pi^2}p^4\log\frac{p^2}{\mu^2}\Big[1+g^2(\mu)\Big(\tilde{f}_0-\beta_0\log\frac{p^2}{\mu^2}\Big) + g^4(\mu)\Big(\tilde{f}_1+\tilde{f}_2\log\frac{p^2}{\mu^2}+\beta_0^2\log^2\frac{p^2}{\mu^2}\Big)\Big]$$

$$\begin{aligned}
\tilde{f}_0 &= \frac{97}{3(4\pi)^2} \\
\tilde{f}_1 &= \Big(\frac{51959}{54} - 110\zeta(3)\Big)\frac{1}{(4\pi)^4} \\
-2\beta_0 &= -2\frac{11}{3(4\pi)^2} \\
2\tilde{f}_2 &= -\frac{1135}{3(4\pi)^4} \Rightarrow \tilde{f}_2 = -\frac{1135}{6(4\pi)^4}
\end{aligned}$$

# Check of the RG estimate (M.B. and S. Muscinelli, JHEP 08 (2013) 064 [hep-th/1304.6409])

$$\frac{1}{2}\int\langle\frac{g^2}{N}tr\big(\sum_{\alpha\beta}F^{-2}_{\alpha\beta}(x)\big)\frac{g^2}{N}tr\big(\sum_{\alpha\beta}F^{-2}_{\alpha\beta}(0)\big)\rangle_{conn}\,e^{-ip\cdot x}d^4x$$
$$=(1-\frac{1}{N^2})\frac{p^4}{2\pi^2\beta_0}\big(2g^2(p^2)-2g^2(\mu^2)$$
$$+\big(a+\tilde{a}-\frac{\beta_1}{\beta_0}\big)g^4(p^2)-\big(a+\tilde{a}-\frac{\beta_1}{\beta_0}\big)g^4(\mu^2)\big)+O(g^6)$$

$$g^2(p^2)=g^2(\mu^2)\big(1-\beta_0 g^2(\mu^2)\log\frac{p^2}{\mu^2}$$
$$-\beta_1 g^4(\mu)\log\frac{p^2}{\mu^2}+\beta_0^2 g^4(\mu^2)\log^2\frac{p^2}{\mu^2}\big)+\cdots$$

An interesting aside: In the past years several proposals for the glueball propagators have been advanced based on AdS String/ Gauge Theory correspondence, and more recently on a TFT underlying large-N YM

UV test for glueball propagators: AdS String/ Gauge Theory correspondence versus the TFT

TFT (QCD), Polchinski-Strassler or Hard Wall (QCD), Soft Wall (QCD), Klebanov-Strassler background (n=1 cascading SUSY QCD)

# Polchinski-Strassler (Hard Wall)

$$\int < trF^2(x) trF^2(0) >_{conn} e^{-ip\cdot x} d^4x \sim \quad p^4 \left[ 2\frac{K_1(\frac{p}{\mu})}{I_1(\frac{p}{\mu})} - \log p \right] \sim -p^4 \left[ \log p + O(e^{-2\frac{p}{\mu}}) \right]$$

# Soft Wall

$$\int < trF^2(x) trF^2(0) >_{conn} e^{-ip\cdot x} d^4x \sim -p^4 \left[ \log p + O(\frac{\mu^2}{p^2}) \right]$$

# Klebanov-Strassler: n=1 cascading SUSY QCD

$$\frac{\partial g}{\partial \log \Lambda} = -\frac{\frac{3}{(4\pi)^2} g^3}{1 - \frac{2}{(4\pi)^2} g^2}$$

$$\int < trF^2(x) trF^2(0) >_{conn} e^{-ip\cdot x} d^4x \sim p^4 \log^3 \frac{p^2}{\mu^2}$$

All the previous results, disagree with asymptotic freedom and RG by powers of logarithms

It means that the would-be glueball propagators differ from the correct answer in pure YM or in any AF theory for an infinite number of poles and/or residues,

(a fact that raises well motivated doubts on the correctness of the AdS-String spectrum at large-N ... In fact, the AdS-String spectrum disagrees even qualitatively with lattice data)

The main limitation of the asymptotically-free bootstrap is that it does not contain spectral information

but we have constructed a TFT underlying large-N Yang-Mills based on a field theoretical version of Morse-Smale-Floer homology, that contains spectral information

M.B. Yang-Mills mass gap, Floer homology, glueball spectrum and conformal window in large-N QCD
hep-th/1312.1350

By means of the TFT, we have found an answer for the mass gap and the ASD glueball propagator

i.e. the two-point correlator of $O_{ASD} = \sum_{\alpha\beta} Tr(F^-_{\alpha\beta} F^{-\alpha\beta})$

$$F^-_{\alpha\beta} = F^{\alpha\beta} - {}^*F^{\alpha\beta}$$

$${}^*F_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$$ in Euclidean or ultra-hyperbolic signature

$${}^*F_{\alpha\beta} = \frac{i}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$$ in Minkowski

that is compatible

with everything that we know presently about large-N YM, both in the infrared numerically by lattice gauge theory and more importantly in the ultraviolet by first principles (i.e. it agrees with the Asymptotic Theorem)

as we will show momentarily

# The answer

in Euclidean or ultra-hyperbolic signature in large-N YM is:

$$O_S = \sum_{\alpha\beta} Tr F_{\alpha\beta} F^{\alpha\beta}$$

$$O_P = \sum_{\alpha\beta} Tr(F^{\alpha\beta} {}^* F_{\alpha\beta})$$

$$< O_{ASD}(x) O_{ASD}(0) >_{conn} = 4 < O_S(x) O_S(0) >_{conn} +4 < O_P(x) O_P(0) >_{conn}$$

$$\int \langle O_{ASD}(x) O_{ASD}(0) \rangle_{conn} \, e^{-ip\cdot x} d^4x$$

$$= \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{k^2 g_k^4 \Lambda_{\overline{W}}^6}{p^2 + k\Lambda_{\overline{W}}^2} = \frac{2p^4}{\pi^2} \sum_{k=1}^{\infty} \frac{g_k^4 \Lambda_{\overline{W}}^2}{p^2 + k\Lambda_{\overline{W}}^2} + \text{infinite contact terms}$$

$$\sim C_{ADS}^{(0)}(p^2) + 0 < \frac{1}{N} O_{ASD}(0) > + \text{infinite contact terms}$$

$$C_{ASD}^{(0)}(p^2) = \frac{2p^4}{\pi^2 \beta_0} \left[ \frac{1}{\beta_0 \log \frac{p^2}{\Lambda_{\overline{MS}}^2}} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log \frac{p^2}{\Lambda_{\overline{MS}}^2}}{\log \frac{p^2}{\Lambda_{\overline{MS}}^2}} \right) + O\left( \frac{1}{\log^2 \frac{p^2}{\Lambda_{\overline{MS}}^2}} \right) \right]$$

# Conclusion and Conjecture

Construct the Topological String Theory dual to the TFT

Check that the corresponding S-matrix is asymptotic to the answer found by the Asymptotically-Free Bootstrap (likely, because the TFT is asymptotically-free with the correct 2 point correlator)

Then the Topological String Theory would be the string solution of QCD for the spectrum and S-matrix

# Why the Topological String Theory has chances to work ?

Because S-matrix amplitudes for topological strings arise by summing on D-branes, as in Witten Topological Twistor String of n=4 SUSY YM

or by summing on world-sheet instantons as in the Twistorial A-model that is dual to the TFT (M.B.)

and not by summing on Riemann surfaces, as for conventional strings, that in general implies very soft behavior in the UV, more soft than in super-renormalizable field theories

In Witten case the field theoretical MHV amplitudes of n=4 YM are exactly reproduced, i.e. they are hard in the UV

Does the Twistorial A-model dual to the TFT solve QCD
in 't Hooft limit,
perhaps only for the S-matrix ?

We will see ...

The following slides are not part of the talk but contain details useful to answer questions or for further discussion

# The ASD correlator in the TFT needs an exact non-perturbative scheme for the large-N beta function, in such a way that the canonical coupling does not diverge at the infrared Landau pole of the Wilsonian or of the perturbative coupling

M.B. JHEP 05(2009)116

$$\frac{\partial g}{\partial \log \Lambda} = \frac{-\beta_0 g^3 + \frac{1}{(4\pi)^2} g^3 \frac{\partial \log Z}{\partial \log \Lambda}}{1 - \frac{4}{(4\pi)^2} g^2} = -\beta_0 g^3 - \beta_1 g^5 + \cdots$$

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3$$

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{2\gamma_0 g_W^2}{1 + c' g_W^2} = 2\gamma_0 g^2 + \cdots$$

$$\gamma_0 = \frac{1}{(4\pi)^2} \frac{5}{3}$$

$$\begin{aligned}
\frac{\partial g}{\partial \log \Lambda} &= \frac{-\beta_0 g^3 + \frac{2\gamma_0}{(4\pi)^2} g^5}{1 - \frac{4}{(4\pi)^2} g^2} + \cdots \\
&= -\beta_0 g^3 + \frac{2\gamma_0}{(4\pi)^2} g^5 - \frac{4\beta_0}{(4\pi)^2} g^5 + \cdots \\
&= -\beta_0 g^3 - \beta_1 g^5 + \cdots \\
\beta_0 &= \frac{1}{(4\pi)^2} \frac{11}{3} \\
\beta_1 &= \frac{1}{(4\pi)^4} \frac{34}{3}
\end{aligned}$$

Euler-MacLaurin formula, in order to extract the large-momentum asymptotics (Migdal, decades ago ...)

$$\sum_{k=k_1}^{\infty} G_k(p) = \int_{k_1}^{\infty} G_k(p)dk - \sum_{j=1}^{\infty} \frac{B_j}{j!} \left[\partial_k^{j-1} G_k(p)\right]_{k=k_1}$$

## The answer in Minkowski in large-N YM is:

$$< O_{ASD}(x) O_{ASD}(0) >_{conn} = 4 < O_S(x) O_S(0) >_{conn} -4 < O_P(x) O_P(0) >_{conn}$$

+ analytic continuation of momenta (not displayed)

$$\int \langle O_{ASD}(x) O_{ASD}(0) \rangle_{conn} \, e^{-ip\cdot x} d^4x$$

$$= 16\beta_0 < \frac{1}{N} O_{ASD}(0) > \sum_{k=1}^{\infty} \frac{g_k^4 \Lambda_{\overline{W}}^2}{p^2 + k\Lambda_{\overline{W}}^2}$$

$$\sim p^4 0 + C^{(1)}_{ADS}(p^2) < \frac{1}{N} O_{ASD}(0) >$$

$$C^{(1)}_{ADS}(p^2) = 16 \left[ \frac{1}{\beta_0 \log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log \frac{p^2}{\Lambda^2_{\overline{MS}}}}{\log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) + O\left( \frac{1}{\log^2 \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) \right]$$

In n=1 SUSY YM by methods inspired by present work Shifman (2011) has shown in Minkowski:

$$\int \langle O_{ASD}(x) O_{ASD}(0) \rangle_{conn} \, e^{-ip\cdot x} d^4x = 0 + \text{ contact terms}$$

Hence trying to extend to n=1 SUSY YM in Minkowski is pointless !

Since the ASD correlator is the sum of the scalar and pseudoscalar correlators, the prediction of the TFT for the joint scalar and pseudoscalar glueball spectrum of positive C in large-N YM is:

$$m_k^2 = k\Lambda_{QCD}^2;\ k = 1, 2 \cdots$$

Exact linearity, as opposed to asymptotic linearity, is as a strong statement as it sounds very unlikely even at large-N,

but ...

## The prediction of the TFT agrees sharply with

SU(8) lattice YM computation by Meyer-Teper (2004) on the largest lattice (16^3 * 24), presently closest to continuum, i.e. with the smallest value of YM coupling (beta=2N/(g_YM)^2=45.5)

$$r_s = \frac{m_{0^{++*}}}{m_{0^{++}}}$$

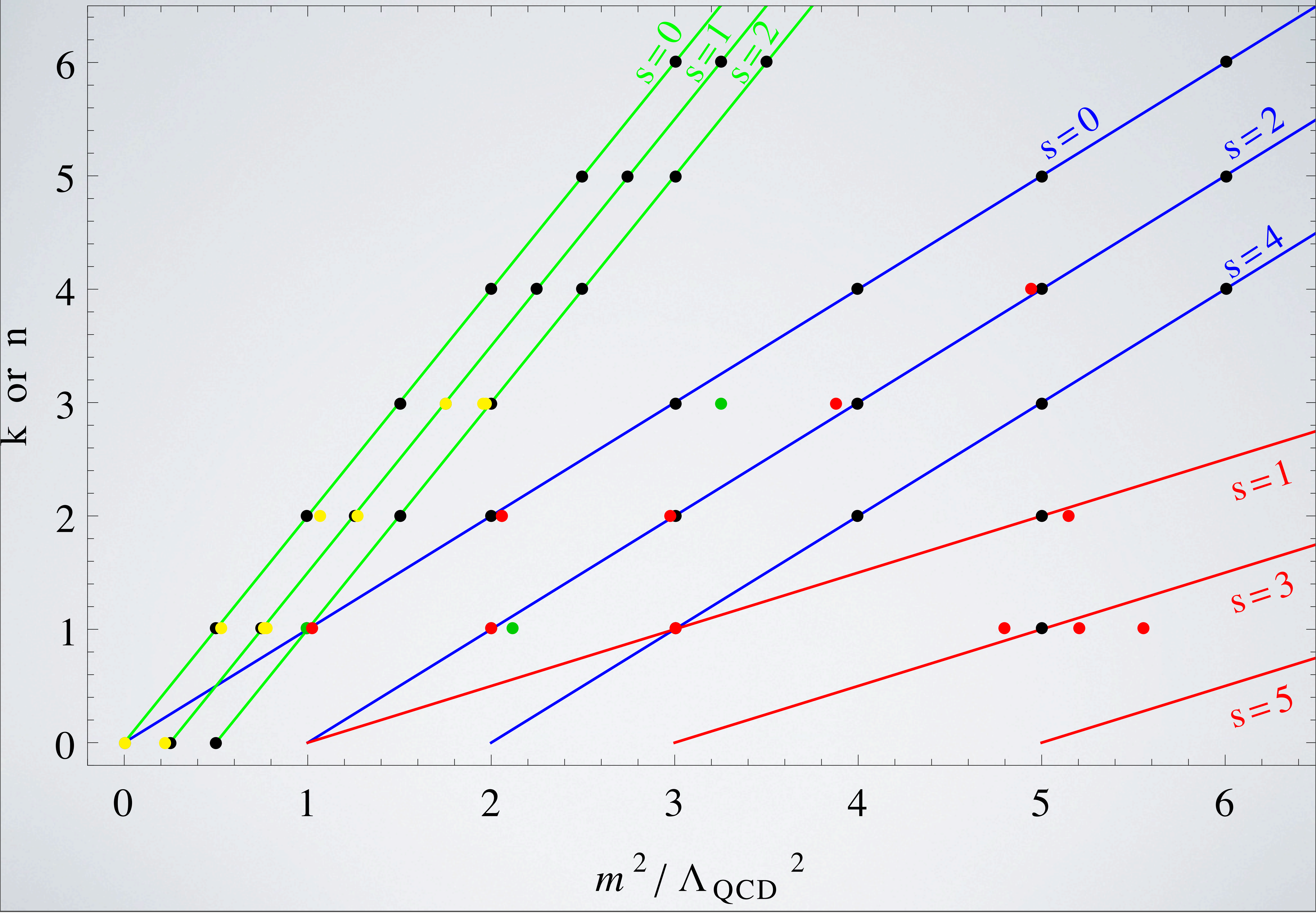
$$r_{ps} = \frac{m_{0^{-+}}}{m_{0^{++}}}$$

$$r_s = r_{ps} = 1.42(11)$$

TFT:

$$r_s = r_{ps} = \sqrt{2} = 1.4142\cdots$$

# Spectrum of large N massless QCD



# Proof of the RG estimate in the coordinate representation using the fact that the operator O is multiplicatively renormalizable in the coordinate representation, because contact terms do not occur for x away from 0

$$\langle \mathcal{O}_D(x)\mathcal{O}_D(0)\rangle_{conn} \sim C_0(x^2)$$

$$\left(x_\alpha \frac{\partial}{\partial x_\alpha} + \beta(g)\frac{\partial}{\partial g} + 2(D + \gamma_{\mathcal{O}_D}(g))\right) C_0(x^2) = 0$$

$$C_0(x^2) = \frac{1}{x^{2D}}\,\mathcal{G}_0(g(x))\,Z^2_{\mathcal{O}_D}(x\mu, g(x))$$

$$\mathcal{G}(g(x)) \sim 1 + O(g^2(x))$$

$$C_0(x^2) \sim \tfrac{1}{x^{2D}}\, g(x)^{\frac{2\gamma_0(\mathcal{O}_D)}{\beta_0}}$$

$$\sim \frac{1}{x^{2D}}\left(\frac{1}{\beta_0 \log(\frac{z_0^2}{x^2\Lambda^2_{QCD}})}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{z_0^2}{x^2\Lambda^2_{QCD}})}{\log(\frac{z_0^2}{x^2\Lambda^2_{QCD}})}\right)\right)^{\frac{\gamma_0}{\beta_0}}$$

$$< O(x)O(0) >_{conn} = \sum_{n=1}^{\infty} \frac{1}{(2\pi)^4} \int \frac{R_n m_n^{2D-4} \rho^{-1}(m_n^2)}{p^2 + m_n^2} e^{ip\cdot x} d^4p$$

$$= \frac{1}{4\pi^2 x^2} \sum_{n=1}^{\infty} R_n m_n^{2D-4} \rho^{-1}(m_n^2) \sqrt{x^2 m_n^2} K_1(\sqrt{x^2 m_n^2})$$

$$\sim \frac{1}{4\pi^2 x^2} \int_1^{\infty} R_n m_n^{2D-4} \rho^{-1}(m_n^2) \sqrt{x^2 m_n^2} K_1(\sqrt{x^2 m_n^2}) dn$$

$$= \frac{1}{4\pi^2 x^2} \int_{m_1^2}^{\infty} R(m) m^{2D-4} \sqrt{x^2 m^2} K_1(\sqrt{x^2 m^2}) dm^2$$

$$= \frac{1}{4\pi^2 x^2} \int_{m_1^2 x^2}^{\infty} R(\frac{z}{x})(\frac{z^2}{x^2})^{D-2} z K_1(z) \frac{dz^2}{x^2}$$

$$= \frac{1}{4\pi^2 (x^2)^D} \int_{m_1^2 x^2}^{\infty} R(\frac{z}{x}) z^{2D-3} K_1(z) dz^2$$

$$z^2 = x^2 m^2$$

# Proof of the Asymoptotic theorem in the coordinate representation

$$\int_{m_1^2 x^2}^{\infty} R(\frac{z}{x}) z^{2D-3} K_1(z) dz^2 \sim \left( \frac{1}{\beta_0 \log(\frac{1}{x^2\Lambda_{QCD}^2})} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log(\frac{1}{x^2\Lambda_{QCD}^2})}{\log(\frac{1}{x^2\Lambda_{QCD}^2})} \right) \right)^{\frac{\gamma_0}{\beta_0}}$$

$$\int_{m_1^2 x^2}^{\infty} R(\frac{z}{x}) z^{2D-3} K_1(z) dz^2$$

$$\sim R(\frac{z_0}{x}) \int_0^{\infty} z^{2D-3} K_1(z) dz^2 \sim \left( \frac{1}{\beta_0 \log(\frac{1}{x^2\Lambda_{QCD}^2})} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log(\frac{1}{x^2\Lambda_{QCD}^2})}{\log(\frac{1}{x^2\Lambda_{QCD}^2})} \right) \right)^{\frac{\gamma_0}{\beta_0}}$$

$$R(\frac{z_0}{x}) \sim \left( \frac{1}{\beta_0 \log(\frac{z_0^2}{x^2\Lambda_{QCD}^2})} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log(\frac{z_0^2}{x^2\Lambda_{QCD}^2})}{\log(\frac{z_0^2}{x^2\Lambda_{QCD}^2})} \right) \right)^{\frac{\gamma_0}{\beta_0}}$$

$$R(\frac{z_0}{x}) \sim Z_{\mathcal{O}}^2(x\mu, g(x))$$

# Proof of the Asymptotic Theorem in momentum representation

$$\int \langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x = \sum_{n=1}^{\infty} \frac{R_n m_n^{2D-4}\rho^{-1}(m_n^2)}{p^2+m_n^2}$$

D even:

$$m_n^{2D-4} = ((m_n^2+p^2)(m_n^2-p^2)+p^4)^{\frac{D}{2}-1}$$

$$\int \langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x = p^{2D-4}\sum_{n=1}^{\infty} \frac{R_n \rho^{-1}(m_n^2)}{p^2+m_n^2} + \cdots$$

D odd:

$$m_n^2 m_n^{2(D-1)-4} = (p^2+m_n^2-p^2)((m_n^2+p^2)(m_n^2-p^2)+p^4)^{\frac{D-1}{2}-1}$$

$$\int \langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x = -p^{2D-4}\sum_{n=1}^{\infty} \frac{R_n \rho^{-1}(m_n^2)}{p^2+m_n^2} + \cdots$$

$$\sum_{n=1}^{\infty} P^{(s)}\big(\frac{p_\alpha}{m_n^{(s)}}\big)\frac{m_n^{(s)2D-4}Z_n^{(s)2}\rho_s^{-1}(m_n^{(s)2})}{p^2+m_n^{(s)2}}$$

$$= P^{(s)}\big(\frac{p_\alpha}{p}\big)\,p^{2D-4}\sum_{n=1}^{\infty}\frac{Z_n^{(s)2}\rho_s^{-1}(m_n^{(s)2})}{p^2+m_n^{(s)2}}+\cdots$$

$$m_n^{(s)2D-4}P^{(s)}\big(\frac{p_\alpha}{m_n^{(s)}}\big)$$

$$m_n^{2d}\to p^{2d};-p^{2d}$$

$$-p^2\to m_n^2$$

$$P^{(s)}\big(\frac{p_\alpha}{m_n}\big)\to P^{(s)}\big(\frac{p_\alpha}{p}\big)$$

# Euler-McLaurin formula:

$$\sum_{k=k_1}^{\infty} G_k(p) = \int_{k_1}^{\infty} G_k(p) dk - \sum_{j=1}^{\infty} \frac{B_j}{j!} \left[\partial_k^{j-1} G_k(p)\right]_{k=k_1}$$

$$\sum_{n=1}^{\infty} \frac{R_n \rho^{-1}(m_n^2)}{p^2 + m_n^2}$$

$$\sim \int_1^{\infty} \frac{R_n \rho^{-1}(m_n^2)}{p^2 + m_n^2} dn$$

$$= \int_{m_1^2}^{\infty} \frac{R(m) \rho^{-1}(m^2)}{p^2 + m^2} \rho(m^2) dm^2$$

$$= \int_{m_1^2}^{\infty} \frac{R(m)}{p^2 + m^2} dm^2$$

$$\nu = \frac{p^2}{\Lambda_{QCD}^2}; k = \frac{m^2}{\Lambda_{QCD}^2}; K = \frac{\Lambda^2}{\Lambda_{QCD}^2}$$

$$\gamma' = \frac{\gamma_0}{\beta_0}$$

$$\beta_0^{-\gamma'} \int_1^\infty \left( \frac{1}{\log(\frac{k}{c})} \left(1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log(\frac{k}{c})}{\log(\frac{k}{c})}\right)\right)^{\gamma'} \frac{dk}{k+\nu}$$

$$\sim \frac{1}{\gamma'-1} \beta_0^{-\gamma'} \left(\log \frac{1+\nu}{c}\right)^{-\gamma'+1} - \frac{\beta_1}{\beta_0^2} \beta_0^{-\gamma'} \left(\log(\frac{1+\nu}{c})\right)^{-\gamma'} \log\log(\frac{1+\nu}{c})$$

$$= \frac{\beta_0^{-\gamma'}}{\gamma'-1} \left(\log \frac{1+\nu}{c}\right)^{-\gamma'+1} \left[1 - \frac{\beta_1(\gamma'-1)}{\beta_0^2} \left(\log(\frac{1+\nu}{c})\right)^{-1} \log\log(\frac{1+\nu}{c})\right]$$

$$\sim \frac{1}{\beta_0(\gamma'-1)} \left(\beta_0 \log \frac{1+\nu}{c}\right)^{-\gamma'+1} \left[1 - \frac{\beta_1}{\beta_0^2} \left(\log(\frac{1+\nu}{c})\right)^{-1} \log\log(\frac{1+\nu}{c})\right]^{\gamma'-1}$$

$$\sim \left( \frac{1}{\beta_0 \log \nu} \left(1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log \nu}{\log \nu}\right)\right)^{\gamma'-1}$$

# It agrees with Naive RG estimate in momentum representation, assuming the operator O to be multiplicatively renormalizable, that is technically false

$$\int \langle \mathcal{O}_D(x)\mathcal{O}_D(0)\rangle_{conn} e^{-ip\cdot x} d^4x \sim C_0(p^2)$$

$$\left(p_\alpha \frac{\partial}{\partial p_\alpha} - \beta(g)\frac{\partial}{\partial g} - 2(D-2+\gamma_{\mathcal{O}_D}(g))\right) C_0(p^2) = 0$$

$$C_0(p^2) = p^{2D-4}\,\mathcal{G}_0(g(p))\,Z^2_{\mathcal{O}_D}(\frac{p}{\mu}, g(p))$$

$$\mathcal{G}(g(p)) \sim \log\frac{p^2}{\Lambda^2_{QCD}} \sim \frac{1}{g^2(p)}$$

$$\int p^{2D-4}\log\frac{p^2}{\mu^2} e^{ipx} d^4p \sim \frac{1}{x^{2D}}$$

$$C_0(p^2) \sim p^{2D-4} g(p)^{\frac{2\gamma_0(\mathcal{O}_D)}{\beta_0}-2}$$

$$\sim p^{2D-4}\left[\frac{1}{\beta_0 \log(\frac{p^2}{\Lambda^2_{QCD}})}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{p^2}{\Lambda^2_{QCD}})}{\log(\frac{p^2}{\Lambda^2_{QCD}})} + O(\frac{1}{\log(\frac{p^2}{\Lambda^2_{QCD}})})\right)\right]^{\frac{\gamma_0}{\beta_0}-1}$$

$$\int_{m_1^2}^{\Lambda^2} \frac{R(m)}{p^2+m^2} dm^2 = Z_{\mathcal{O}}^2(p)\mathcal{G}_0(g(p))$$

$$\nu = \frac{p^2}{\Lambda_{QCD}^2}; k = \frac{m^2}{\Lambda_{QCD}^2}; K = \frac{\Lambda^2}{\Lambda_{QCD}^2}$$

$$\int_{k_1}^{K} \frac{R(\sqrt{k})}{\nu+k} dk = Z_{\mathcal{O}}^2(\sqrt{\nu})\mathcal{G}_0(g(\sqrt{\nu}))$$

$$\int_{k_1}^{K} \frac{R(\sqrt{k})}{\nu+k} dk = \left(\frac{1}{\beta_0 \log \nu}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log\nu}{\log\nu}\right)\right)^{\frac{\gamma_0}{\beta_0}-1}$$

This is an integral equation of Fredholm type, for which a solution exists if and only if it is unique:

$$R(\sqrt{k}) \sim Z^2(\sqrt{k}) \sim \left(\frac{1}{\beta_0 \log \frac{k}{c}}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log\frac{k}{c}}{\log\frac{k}{c}}\right)\right)^{\frac{\gamma_0}{\beta_0}}$$

Given the Kallen-Lehmann representation,

extension of the Asymptotic Theorem to all other coefficients of OPE is straightforward,

taking into account different naive dimensions and anomalous dimensions of each coefficient