Loop functions in thermal QCD

Antonio Vairo

Technische Universität München

Outline

1. Loop functions
2. Polyakov loop
3. Polyakov loop correlator
4. Cyclic Wilson loop

based on

Brambilla Ghiglieri Petreczky Vairo PR D82 (2010) 074019
Berwein Brambilla Ghiglieri Vairo JHEP 1303 (2013) 069
Berwein Brambilla Vairo arXiv:1312.6651

Loop functions

Loop functions

These are gauge invariant quantities measurable by lattice QCD and relevant for the dynamics of static sources in a thermal bath at temperature T.

- `e.g. Petreczky EPJC 43 (2005) 51`

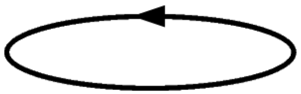
Despite their relevance, not much is known about loop functions in perturbation theory.

Loop functions

- Polyakov loop average in a thermal ensemble at a temperature T

$$P(T)|_R \equiv \frac{1}{d_R}\langle \mathrm{Tr}\ L_R \rangle$$ (R $\equiv$ color representation)

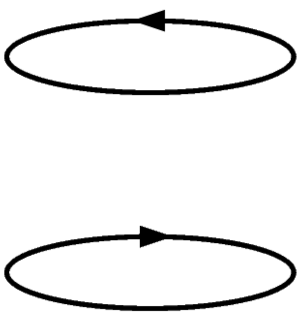
$d_A = N^2 - 1$, $d_F = N$ and $L_R(\mathbf{x}) = \mathrm{P}\exp\left(ig\int_0^{1/T} d\tau A^0(\mathbf{x},\tau)\right)$



- Polyakov loop correlator

$$P_c(r,T) \equiv \frac{1}{N^2}\langle \mathrm{Tr} L_F^\dagger(\mathbf{0}) \mathrm{Tr} L_F(\mathbf{r})\rangle = \frac{1}{N^2}\sum e^{-E_n/T}$$

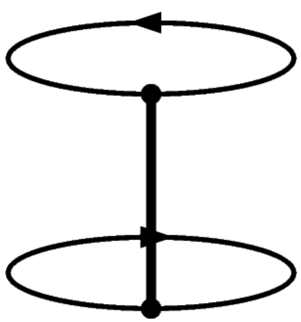
  - `Lüscher Weisz JHEP 0207 (2002) 049`
    `Jahn Philipsen PRD 70 (2004) 074504`



- Cyclic Wilson loop

$$W_c(r,T) \equiv \frac{1}{N}\langle \mathrm{Tr} L_F^\dagger(\mathbf{0}) U^\dagger(1/T) L_F(\mathbf{r}) U(0)\rangle$$

where $U(1/T) = \mathrm{P}\exp\left(ig\int_0^1 ds\,\mathbf{r}\cdot\mathbf{A}(s\mathbf{r},1/T)\right) = U(0)$.



Divergences

- Ultraviolet divergences come from regions where two or more vertices are contracted to one point.
- In the case of internal vertices divergences are removed through renormalization.
- For loop functions one also gets divergences from the contraction of line vertices along the contour.

The superficial degree of divergence

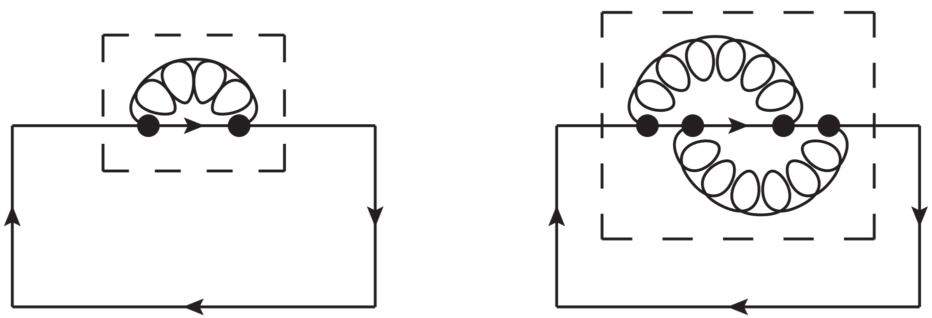
$$\omega = 1 - N_{\rm ex} \quad \text{smooth point}$$
$$\omega = -N_{\rm ex} \quad \text{cusp or intersection}$$

$N_{\rm ex} =$ number of propagators connecting the contraction point to uncontracted vertices.

Divergences

Three possible line vertex divergences.

(1) All vertices are contracted to a smooth point, which leads to a linear divergence;

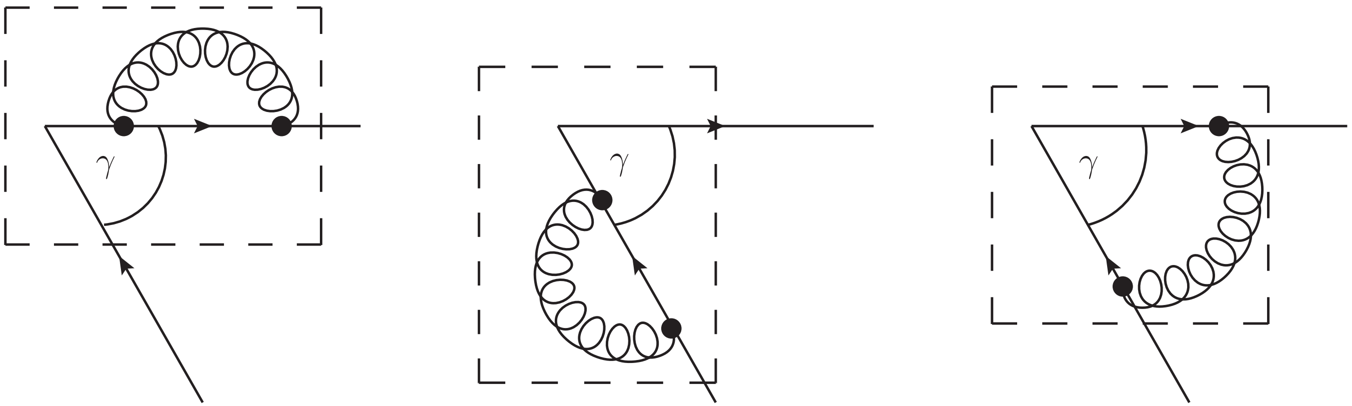


Linear divergences are proportional to the length of the contour and can be removed by a mass term.

(2) The contraction of vertices to a smooth point leaves an external propagator connecting a contracted to an uncontracted vertex: this leads to a logarithmic divergence that can be removed by using renormalized fields and couplings.

(3) All vertices are contracted to a singular point, which gives a logarithmically divergent contribution; these are either cusp or intersection divergences.

Cusps



The renormalization constant for a non-cyclic (time extension smaller than $1/T$) rectangular Wilson loop is determined by four right-angled cusps. In the $\overline{\mathrm{MS}}$-scheme:

$$Z = \exp\left[-2C_F\alpha_{\mathrm{s}}\mu^{-2\varepsilon}/(\pi\bar{\varepsilon})\right]; \qquad \bar{1}/\bar{\varepsilon} \equiv \bar{1}/\varepsilon - \gamma_E + \ln 4\pi$$

Cusp divergences are absent in a cyclic Wilson loop.

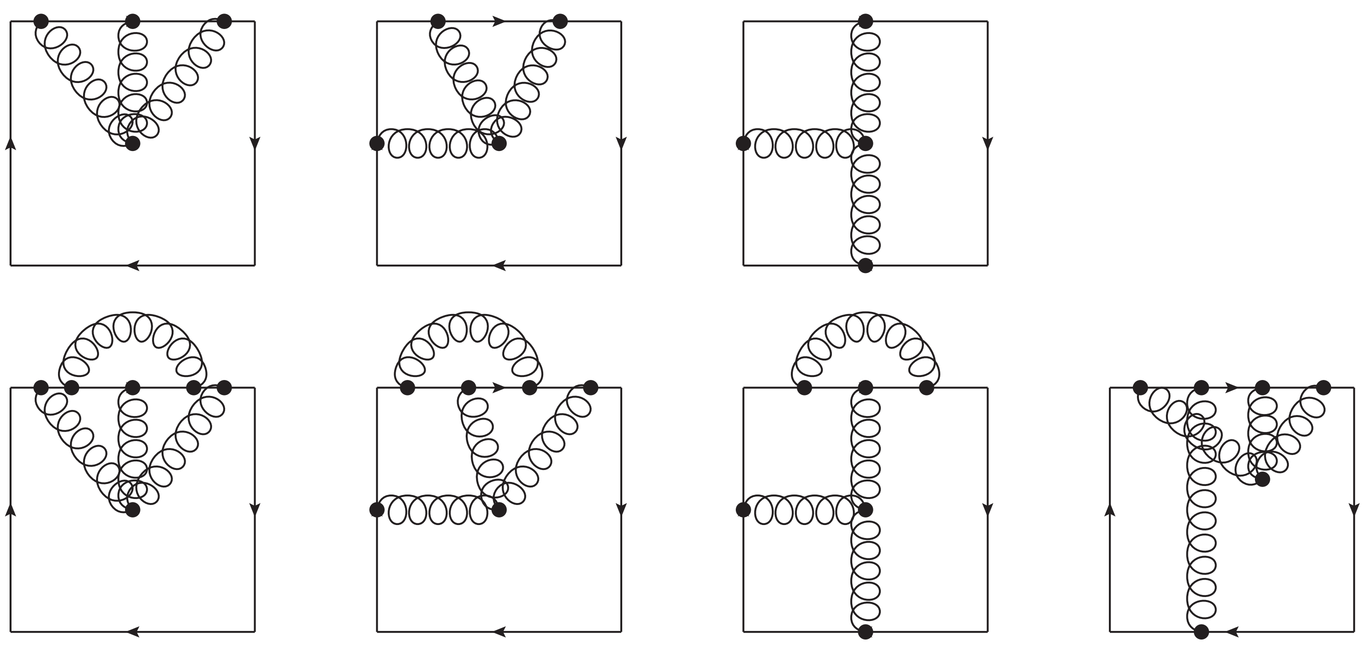
- `Korchemsky Radyushkin NPB 283 (1987) 342`

Intersections

Divergences appear when all vertices are contracted to an intersection point.

- When one vertex is on the string, if every vertex can be contracted to the intersection, then the contribution of the diagram cancels because of cyclicity.
- If all vertices are on a quark line, then the diagram contributes equally to the Polyakov loop, which is finite after charge renormalization.

Hence a connected diagram cannot give rise to an intersection divergence, because either we are in one of the situations above, or it has at least one uncontracted vertex and therefore it is finite.



Polyakov loop

Static and non-static modes

It is convenient to perform the calculation in static gauge $\partial_0 A^0(x) = 0$:

$$L(\mathbf{x}) = \exp\left(\frac{igA^0(\mathbf{x})}{T}\right)$$

Propagators may be split into a static and a non-static component:

$$D_{00}(\omega_n, \mathbf{k}) \;=$$

$$= \frac{\delta_{n0}}{\mathbf{k}^2}$$

$$D_{ij}(\omega_n \neq 0, \mathbf{k}) \;=$$

$$= \frac{1}{\omega_n^2 + \mathbf{k}^2}\left(\delta_{ij} + \frac{k_i k_j}{\omega_n^2}\right)(1 - \delta_{n0})$$

$$D_{ij}(\omega_n = 0, \mathbf{k}) \;=$$

$$= \frac{1}{\mathbf{k}^2}\left(\delta_{ij} - (1-\xi)\frac{k_i k_j}{\mathbf{k}^2}\right)\delta_{n0}$$

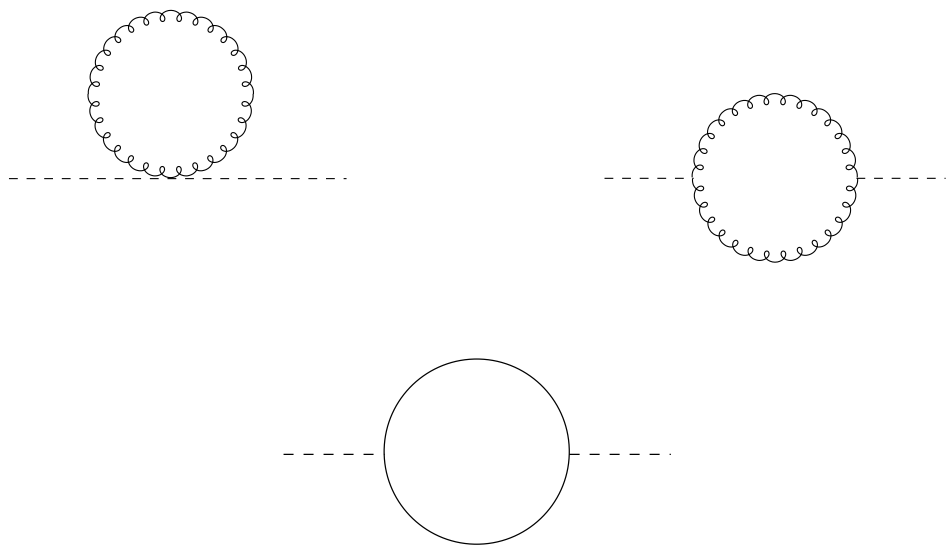
$$D_{\text{ghost}}(\omega_n, \mathbf{k}) \;=$$

$$= \frac{\delta_{n0}}{\mathbf{k}^2}$$

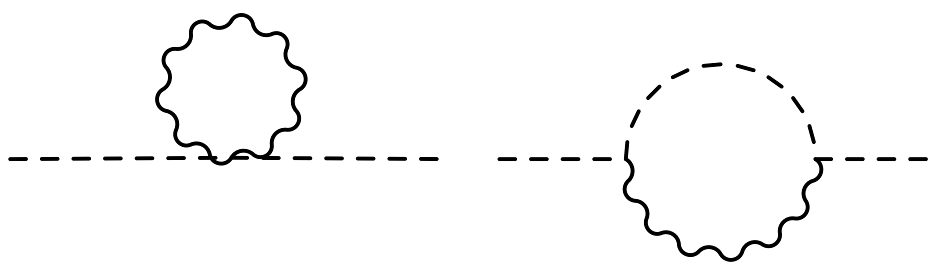
$\omega_n \equiv 2\pi n T$ are the Matsubara frequencies.

Π_{00} at one loop

The temporal component of the gluon self-energy gets non-static



and static contributions



- The calculation is performed in dimensional regularization: $d = 3 - 2\epsilon$.
- $\Pi_{00}(|\mathbf{k}| \ll T) = m_D^2 + \ldots$ where m_D is the Debye mass:

$$m_D^2 \equiv \frac{g^2T^2}{3}\left(N + \frac{n_f}{2}\right).$$

- We keep order ϵ corrections of the type

$$T|\mathbf{k}|^{1-2\epsilon}\epsilon$$

  because the Fourier transform of $|\mathbf{k}|^{1-2\epsilon}/|\mathbf{k}|^4$, coming from a self-energy insertion in a temporal-gluon propagator, is divergent.
- Static loops contribute only through the scale m_D.

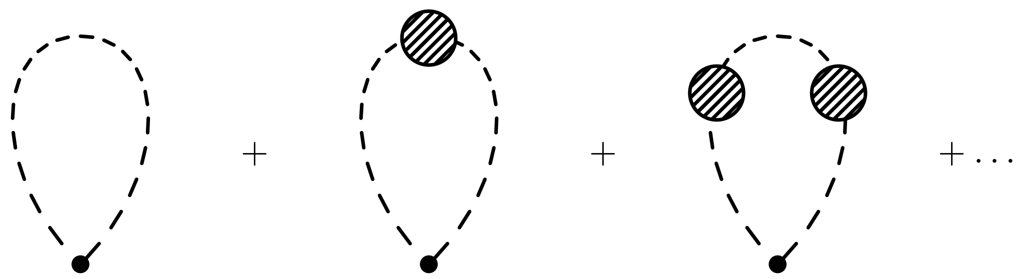
◦ Curci Menotti ZPC 21 (1984) 281
Heinz Kajantie Toimela AP 176 (1987) 218
Rebhan PRD 48 (1993) 3967, NPB 430 (1994) 319

The Polyakov loop at NNLO

We assume the following hierarchy of scales:

$$T \gg m_D$$

Up to NNLO the contributing diagrams are



giving

$$P(T)|_R = 1 + \frac{C_R \alpha_{\mathrm{s}}}{2} \frac{m_D}{T} + \frac{C_R \alpha_{\mathrm{s}}^2}{2} \left[C_A \left(\ln \frac{m_D^2}{T^2} + \frac{1}{2} \right) - n_f \ln 2 \right] + \mathcal{O}(g^5)$$

- At the scale m_D, the gluon-self energies get resummed in the propagator

$$\frac{1}{\mathbf{k}^2 + m_D^2}$$

- The logarithm, $\ln m_D^2/T^2$, signals that an infrared divergence at the scale T has canceled against an ultraviolet divergence at the scale m_D.

Comparison with the literature I

In 1981, Gava and Jengo obtained:

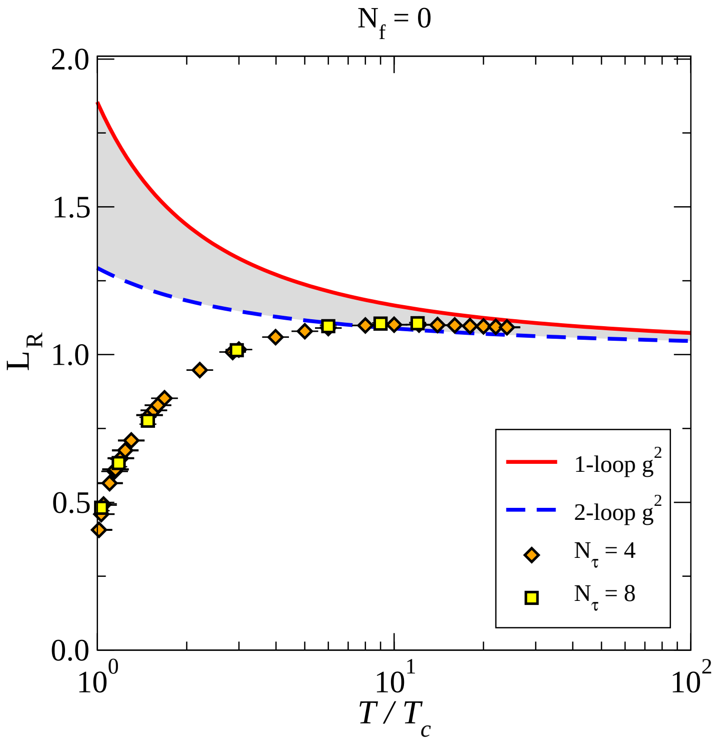
$$P(T)_{\mathrm{GJ}} = 1 + \frac{C_R \alpha_{\mathrm{s}}}{2} \frac{m_D}{T} + \frac{C_R C_A \alpha_{\mathrm{s}}^2}{2} \left(\ln \frac{m_D^2}{T^2} - 2 \ln 2 + \frac{3}{2} \right) + \mathcal{O}(g^5)$$

This result disagrees with ours. The origin of the disagreement has been traced back to not having resummed the Debye mass in the temporal gluons contributing to the static gluon self energy.

○ Gava Jengo PLB 105 (1981) 285

Comparison with the literature II

Our result agrees with the determination of Burnier, Laine and Vepsäläinen, who use a dimensionally reduced EFT framework in a covariant or Coulomb gauge.



- Burnier Laine Vepsäläinen JHEP 1001 (2010) 054
- lattice data from Gupta Hübner Kaczmarek PRD 77 (2008) 034503

The Polyakov loop: some higher order terms

- Non-static modes at the scale m_D:

$$\delta P(T)_{\mathrm{NS},\, m_D} = \frac{3g^4 C_R}{4(4\pi)^3}\frac{m_D}{T}\left[\beta_0 \ln\left(\frac{\mu}{4\pi T}\right)^2 + 2\beta_0\gamma_E + \frac{11}{3}C_A - \frac{2}{3}n_f\,(4\ln 2 - 1)\right]$$

  This contribution fixes the renormalization scale of g^3 in the LO term to $\mu \sim 4\pi T$.

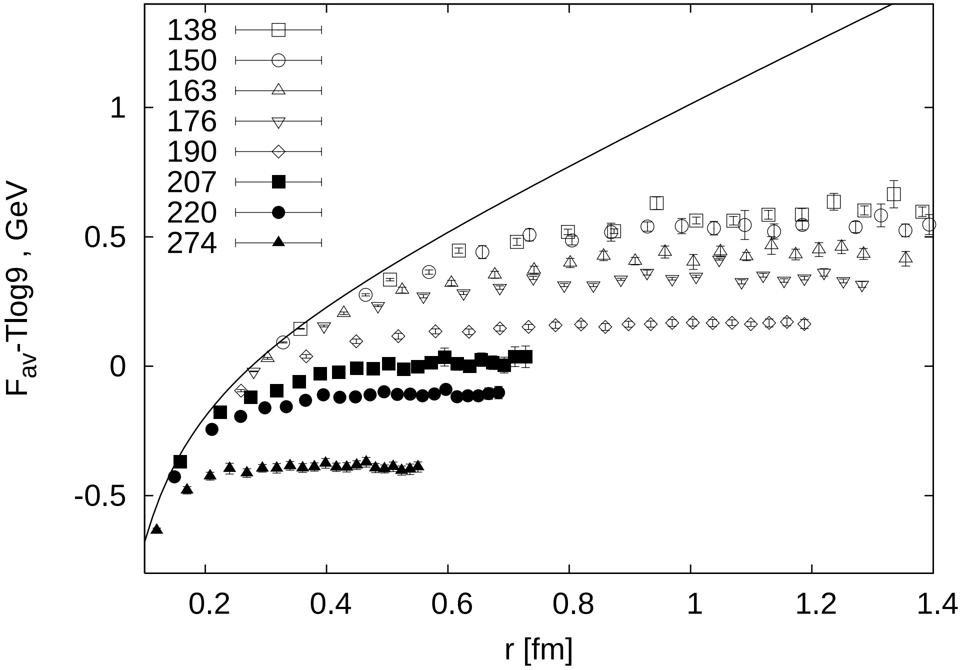
- 

$$\delta P(T) = \left(3C_R^2 - \frac{C_R C_A}{2}\right)\frac{\alpha_{\mathrm{s}}^2}{24}\left(\frac{m_D}{T}\right)^2$$

  This contribution comes from the scale m_D: it is the leading contribution whose color structure is non linear in C_R.

Polyakov loop correlator

The correlator of two Polyakov loops



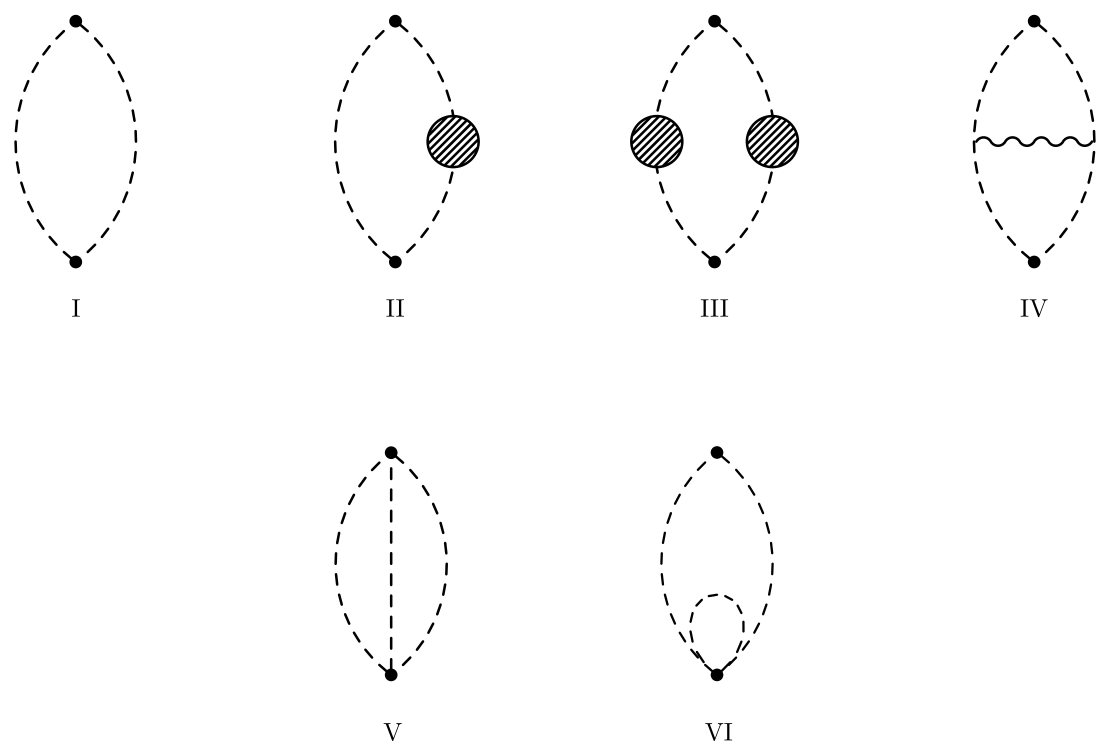
- Petreczky Petrov PRD 70 (2004) 054503

The Polyakov loop correlator at NNLO

We assume the following hierarchy of scales:

$$\frac{1}{r} \gg T \gg m_D \gg \frac{g^2}{r}.$$

We calculate the Polyakov loop correlator up to order $g^6(rT)^0$:



The Polyakov loop correlator at NNLO

We assume the following hierarchy of scales:

$$\frac{1}{r} \gg T \gg m_D \gg \frac{g^2}{r}.$$

We calculate the Polyakov loop correlator up to order $g^6 (rT)^0$:

$$\begin{aligned}
P_c(r,T) \quad = \quad & P(T)^2|_F + \frac{N^2-1}{8N^2}\left\{ \frac{\alpha_s(1/r)^2}{(rT)^2} - 2\frac{\alpha_s^2}{rT}\frac{m_D}{T} \right. \\
& + \frac{\alpha_s^3}{(rT)^3}\frac{N^2-2}{6N} + \frac{1}{2\pi}\frac{\alpha_s^3}{(rT)^2}\left(\frac{31}{9}C_A - \frac{10}{9}n_f + 2\gamma_E\beta_0\right) \\
& + \frac{\alpha_s^3}{rT}\left[C_A\left(-2\ln\frac{m_D^2}{T^2} + 2 - \frac{\pi^2}{4}\right) + 2n_f\ln 2\right] \\
& \left. + \alpha_s^2\frac{m_D^2}{T^2} - \frac{2}{9}\pi\alpha_s^3 C_A \right\} + \mathcal{O}\left(g^6(rT), \frac{g^7}{(rT)^2}\right)
\end{aligned}$$

Comparison with the literature I

In 1986, Nadkarni calculated the Polyakov loop correlator at NNLO assuming the hierarchy:

$$T \gg 1/r \sim m_D$$

Whenever the previous results do not involve the hierarchy $rT \ll 1$, they agree with Nadkarni's ones, expanded for $m_D r \ll 1$.

- `Nadkarni PRD 33 (1986) 3738`

Comparison with the literature II

EFT approaches for the calculation of the correlator of Polyakov loops for the situation $m_D \gtrsim 1/r$ and $T \gg 1/r$ were developed in the past. In that situation, the scale $1/r$ was not integrated out, and the Polyakov-loop correlator was described in terms of dimensionally reduced effective field theories of QCD, while the complexity of the bound-state dynamics remained implicit in the correlator.

Those descriptions are valid for largely separated Polyakov loops when the correlator is either screened by the Debye mass, for $m_D r \sim 1$, or the mass of the lowest-lying glueball, for $m_D r \gg 1$.

- Braaten Nieto PRL 74 (1995) 3530
  Nadkarni PRD 33 (1986) 3738

Comparison with the literature III

In an EFT framework $P_c(r,T)$ can be put in the form

$$P_c(r,T) = \frac{1}{N^2}\left[e^{-f_s(r,T,m_D)/T} + (N^2-1)e^{-f_o(r,T,m_D)/T} + \mathcal{O}\left(\alpha_s^3(rT)^4\right)\right]$$

$f_s = Q\bar{Q}$-color singlet free energy; $f_o = Q\bar{Q}$-color octet free energy.

The color-singlet quark-antiquark potential has been calculated in real-time formalism in the same thermodynamical situation considered here.

- The real part of the real-time potential differs from $f_s(r,T,m_D)$ by

$$\frac{1}{9}\pi N C_F \alpha_s^2 r T^2 - \frac{\pi}{36} N^2 C_F \alpha_s^3 T$$

- The real-time potential has also an imaginary part that is absent in the free energy.

∘ Brambilla Ghiglieri Petreczky Vairo PRD 78 (2008) 014017

Comparison with the literature IV

Jahn and Philipsen have analyzed the gauge structure of the allowed intermediate states in the correlator of Polyakov loops: the quark-antiquark component, φ, of an intermediate state made of a quark located in $\mathbf{x}_1$ and an antiquark located in $\mathbf{x}_2$ should transform as

$$\varphi(\mathbf{x}_1, \mathbf{x}_2) \to g(\mathbf{x}_1)\varphi(\mathbf{x}_1, \mathbf{x}_2)g^\dagger(\mathbf{x}_2)$$

under a gauge transformation g.

- The decomposition of the Polyakov loop correlator in terms of a color singlet and a color octet correlator is in accordance with that result for both a $Q\bar{Q}$ singlet and octet field transform in that way.
- We remark, however, a difference in language: singlet and octet in f_s and f_o refer to the gauge transformation properties of the quark-antiquark fields, while, in Jahn and Philipsen, they refer to the gauge transformation properties of the physical states. In that last sense, of course, octet states cannot exist as intermediate states in the correlator of Polyakov loops.

○ Jahn Philipsen PRD 70 (2004) 074504

Comparison with the literature V

Burnier, Laine and Vepsäläinen have performed a weak-coupling calculation of the untraced Polyakov-loop correlator in Coulomb gauge and of the cyclic Wilson loop up to order g^4.

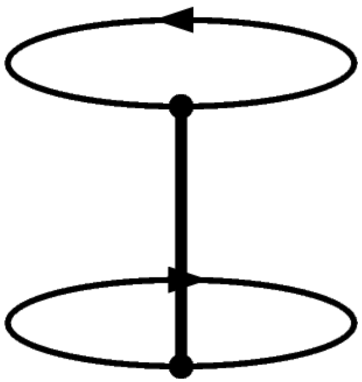
Both these objects may be seen as contributing to the correlator of two Polyakov loops. The first quantity is gauge dependent. We will discuss the relation of the cyclic Wilson loop with the Polyakov-loop correlator.

∘ `Burnier Laine Vepsäläinen JHEP 1001 (2010) 054`

Cyclic Wilson loop

Divergences of the cyclic Wilson loop

Differently from $P(T)$ and $P_c(r,T)$, $W_c(r,T)$ is divergent after charge and field renormalization. This divergence is due to intersection points.



Although it may seem that the cyclic Wilson loop has a continuously infinite number of intersection points, one needs to care only about the two endpoints, for the Wilson loop contour does not lead to divergences in the other ones.

How to renormalize intersection divergences

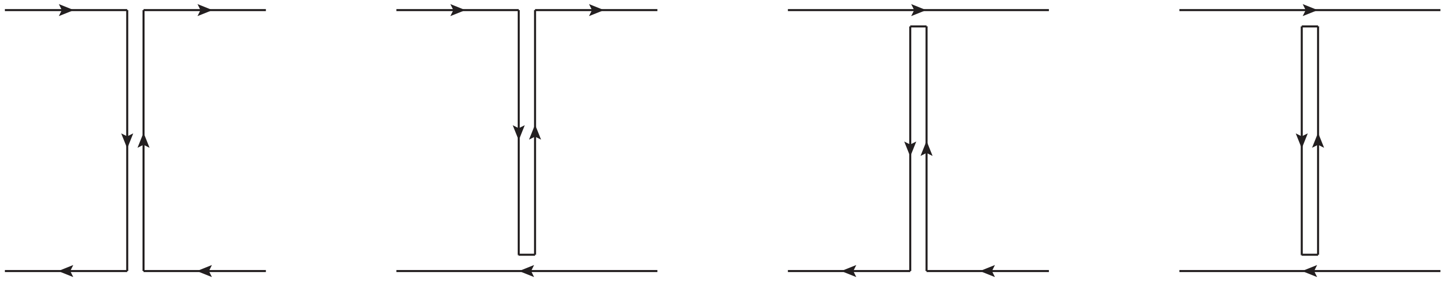
For intersection points connected by 2 Wilson lines (angles θ_k) and cusps (angles φ_l):

$$W^{(R)}_{i_1 i_2 \ldots i_r} = Z_{i_1 j_1}(\theta_1) Z_{i_2 j_2}(\theta_2) \cdots Z_{i_r j_r}(\theta_r) Z(\varphi_1) Z(\varphi_2) \cdots Z(\varphi_s) W_{j_1 j_2 \ldots j_r}$$

- The indices i_k and j_k label the different possible path-ordering prescriptions.
- The loop functions are color-traced and normalized by the number of colours.
- This ensures that all loop functions are gauge invariant.
- The coupling in $W^{(R)}_{i_1 i_2 \ldots i_r}$ is the renormalized coupling.
- The matrices Z are the renormalization matrices.

◦ Brandt Neri Sato PRD 24 (1981) 879

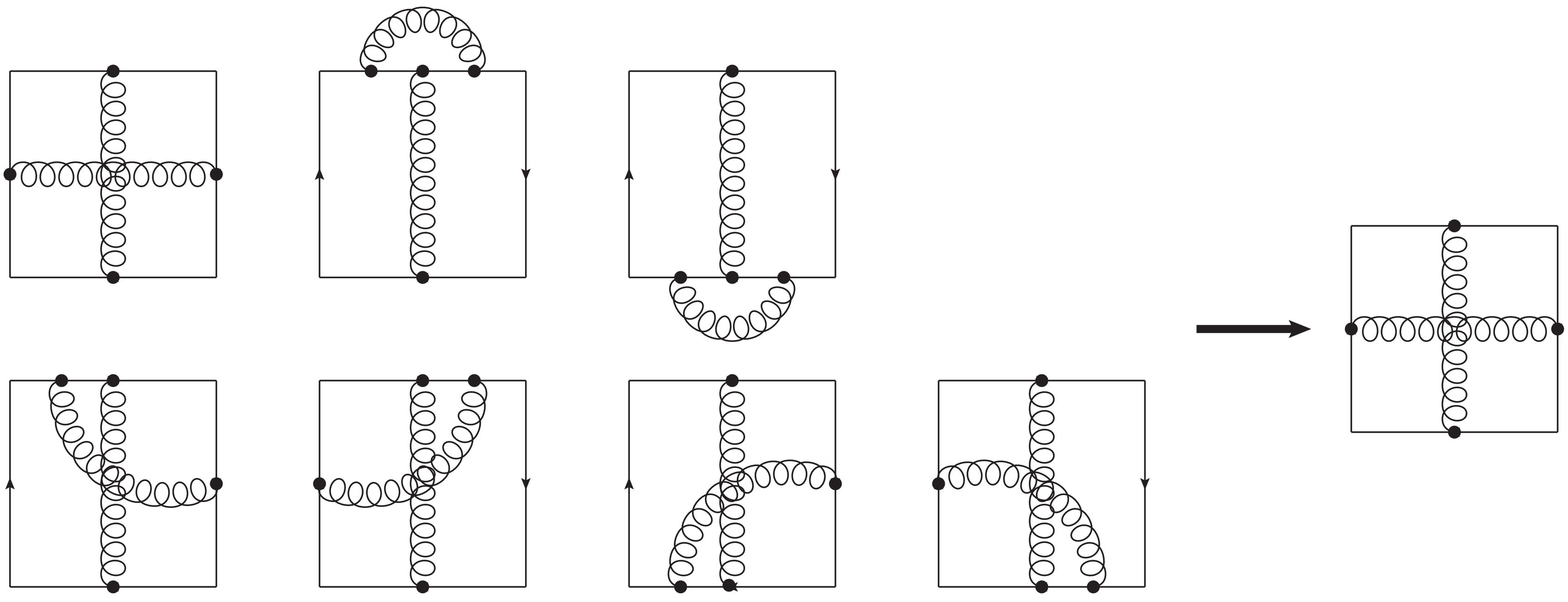
How to renormalize the cyclic Wilson loop



$$\begin{pmatrix} W_c^{(R)} \\ P_c \end{pmatrix} = \begin{pmatrix} Z & 1 - Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W_c \\ P_c \end{pmatrix} .$$

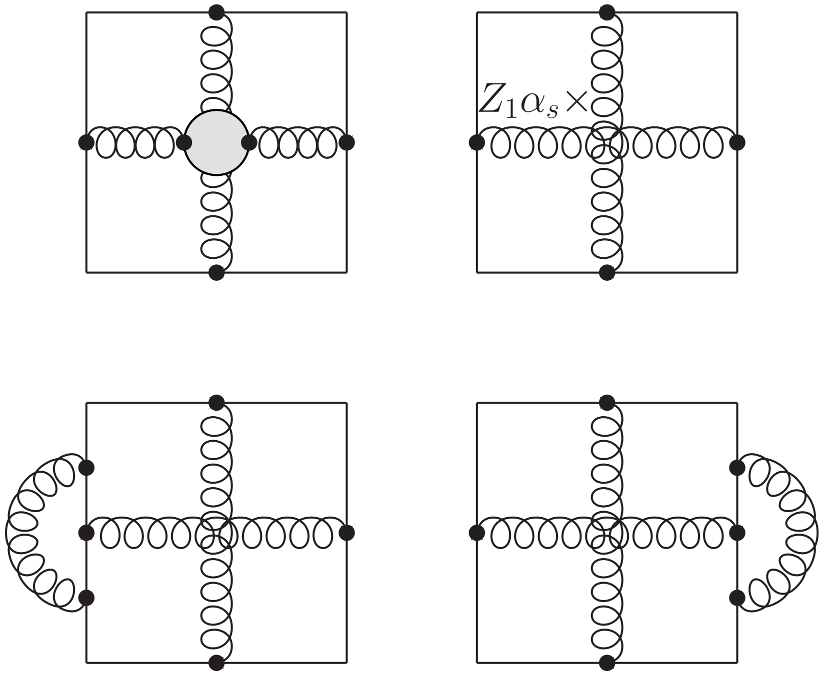
$$Z = 1 + Z_1 \alpha_{\mathrm{s}} \mu^{-2\varepsilon} + Z_2 \left(\alpha_{\mathrm{s}} \mu^{-2\varepsilon}\right)^2 + \mathcal{O}(\alpha_{\mathrm{s}}^3)$$

Z_1



$$Z_1 = -\frac{C_A}{\pi}\frac{1}{\overline{\varepsilon}}$$

Z_2



Z_2 reabsorbs all divergences of the type $\alpha_\mathrm{s}^3/(rT)$.

All other divergences at $\mathcal{O}(\alpha_\mathrm{s}^3)$ are reabsorbed by Z_1 (combined with $P_c(r,T)$ at $\mathcal{O}(\alpha_\mathrm{s}^2)$)!

Renormalization group equation at one loop

$$\begin{cases} \mu \dfrac{\mathrm{d}}{\mathrm{d}\mu} \left(W_c^{(R)} - P_c \right) = \gamma \left(W_c^{(R)} - P_c \right) \\ \mu \dfrac{\mathrm{d}}{\mathrm{d}\mu} \alpha_{\mathrm{s}} = -\dfrac{\alpha_{\mathrm{s}}^2}{2\pi} \beta_0 \end{cases}$$

γ is the anomalous dimension of $W_c^{(R)} - P_c$:

$$\gamma \equiv \frac{1}{Z} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z = 2C_A \frac{\alpha_{\mathrm{s}}}{\pi} + \mathcal{O}(\alpha_{\mathrm{s}}^2)$$

$$\left(W_c^{(R)} - P_c \right)(\mu) = \left(W_c^{(R)} - P_c \right)(1/r) \left(\frac{\alpha_{\mathrm{s}}(\mu)}{\alpha_{\mathrm{s}}(1/r)} \right)^{-4C_A/\beta_0}$$

W_c: final result

In $\overline{\mathrm{MS}}$ at NLO and LL accuracy (i.e. including all terms $\alpha_\mathrm{s}/(rT) \times (\alpha_\mathrm{s} \ln \mu r)^n$), assuming the hierarchy of scales $\frac{1}{r} \gg T \gg m_D \gg \frac{g^2}{r}$, we obtain

$$\begin{aligned}
\ln W_c^{(R)} \quad = \quad & \frac{C_F \alpha_\mathrm{s}(1/r)}{rT} \left\{ 1 + \frac{\alpha_\mathrm{s}}{4\pi} \left[\left(\frac{31}{9} C_A - \frac{10}{9} n_f \right) + 2\,\beta_0 \gamma_E \right] \right. \\
& \left. + \frac{\alpha_\mathrm{s} C_A}{\pi} \left[1 + 2\gamma_E - 2\ln 2 + \sum_{n=1}^{\infty} \frac{2(-1)^n \zeta(2n)}{n(4n^2-1)} (rT)^{2n} \right] \right\} \\
& + \frac{4\pi\alpha_\mathrm{s} C_F}{T} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left(e^{i\mathbf{r}\cdot\mathbf{k}} - 1 \right) \left[\frac{1}{\mathbf{k}^2 + \Pi_{00}^{(T)}(0,\mathbf{k})} - \frac{1}{\mathbf{k}^2} \right] + C_F C_A \alpha_\mathrm{s}^2 \\
& + \frac{C_F \alpha_\mathrm{s}}{rT} \left[\left(\frac{\alpha_\mathrm{s}(\mu)}{\alpha_\mathrm{s}(1/r)} \right)^{-4C_A/\beta_0} - 1 \right] + \mathcal{O}\left(g^5\right)
\end{aligned}$$

$\Pi_{00}^{(T)}(0,\mathbf{k}) =$ (known) thermal part of the gluon self-energy in Coulomb gauge.

Long distance

We have compute W_c for $1/r \gg T \gg m_D \gg \alpha_s/r$, but the renormalization of W_c is general and not bound to this hierarchy. In particular, the renormalization equation must hold also at large distances, $rm_D \sim 1$. There

$$W_c = 1 + \frac{4\pi C_F \alpha_s(\mu)}{T} \frac{e^{-m_D r}}{4\pi r} + \frac{4 C_F C_A \alpha_s^2}{T} \frac{e^{-m_D r}}{4\pi r} \frac{1}{\varepsilon} + \ldots$$

The term $\exp(-m_D r)/(4\pi r)$ comes from the screened temporal gluon propagator, $D_{00}(0, \mathbf{k}) = 1/(\mathbf{k}^2 + m_D^2)$, and the dots stand for finite terms or for h.o. terms.

This expression is renormalized by the same renormalization equation with the same renormalization constant Z as computed at short distances.

This corrects previous analyses finding different UV behaviours at long distances.

∘ Burnier Laine Vepsäläinen JHEP 1001 (2010) 054

Linear divergences

In general, loop functions have power divergences, which factorize and exponentiate to give a factor $\exp[\Lambda L(C)]$, where $L(C)$ is the length of the contour and Λ is some linearly divergent constant. In dimensional regularization such linear divergences are absent, but they would be present in other schemes such as e.g. lattice regularization.

∘ `Polyakov NPB 164 (1980) 171`

An efficient way to calculate the exponent of Wilson loops is the so-called replica trick:

$$\langle W_1 \cdot W_2 \cdots W_N \rangle = 1 + N \ln\langle W \rangle + \mathcal{O}(N^2)$$

$W_i =$ ith copy of W in a replicated theory of QCD not interacting with the others.

∘ `Gardi Laenen Stavenga White JHEP 1011 (2010) 155`
`Gardi Smillie White JHEP 1306 (2013) 088`

$$\exp\left[-2\Lambda_F/T - \Lambda_A r\right] \times Z \times \left(W_c(\mathbf{r}) - P_c(\mathbf{r})\right) \ \text{is finite}$$

Z is in the same renormalization scheme as the linear divergences.

Implication for lattice determinations

The renormalization of W_c allows the proper calculation of this quantity on the lattice.

The right quantity to compute is the multiplicatively renormalizable combination

$$W_c - P_c$$

Alternatively a finite quantity is

$$\frac{(W_c - P_c)(r)}{(W_c - P_c)(r_0)} \times \frac{(W_c - P_c)(2r_0 - r)}{(W_c - P_c)(r_0)}$$

where r_0 is a given fixed distance.