Conformal anomaly actions for dilaton interactions

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Chiral and conformal anomalies

Chiral anomaly

$$\begin{aligned} \partial_\mu \langle j_5^\mu \rangle_s &= \frac{Q^2}{16\,\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\,F_{\mu\nu}\,F_{\alpha\beta} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \quad j_5^\mu \equiv \bar\psi\gamma^\mu\gamma^5\psi \end{aligned}$$

Conformal anomaly

$$\begin{aligned} g_{\mu\nu}\langle T^{\mu\nu}\rangle_s &= \sum_{I=s,f,V} n_I\left[\beta_a(I)\,F + \beta_b(I)\,E_4 + \beta_c(I)\,\Box R\right] - \frac{\kappa}{4}\,n_V\,F^{\mu\nu}\,F_{\mu\nu} \\ F &\equiv C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2\,R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2 \\ E_4 &\equiv R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4\,R^{\alpha\beta}R_{\alpha\beta} + R^2 \end{aligned}$$

Anomalies in the perturbative picture

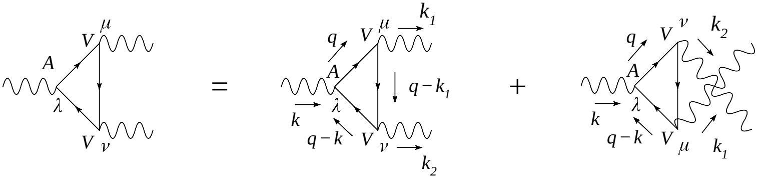
Anomalies provide **a perturbative window on nonperturbative physics**, as they show up in perturbation theory but are otherwise independent of the energy scale, thus **affecting both UV and IR physics**:

- 't Hooft anomaly matching conditions (1979)
- Komargodski-Schwimmer's weak proof of the weak a-theorem (2011)
- Massless scalar degrees of freedom in IR gravity (Mottola, 2008)

The chiral anomaly (Adler, 1968; Bell, Jackiw, 1969) has solved the phenomenological puzzle of the decay $\pi^0 \to \gamma\gamma$ through modified PCAC.

In the 1-particle-irreducible effective action, the perturbative signature of both chiral and conformal anomalies are **anomaly poles**, especially featured by 3-point correlation functions (Dolgov and Zakharov in 1970 for the *AVV*, Mottola and Giannotti in 2008 for the *TVV*...).

The AVV and the chiral anomaly pole



The on-shell matrix element in the massless limit

$$\Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{i\,Q^2}{2\,\pi^2}\,\frac{k^\lambda}{k^2}\,\epsilon^{\mu\nu\alpha\beta}\,k_{1\,\alpha}\,k_{2\,\beta}\,\tilde{A}_\mu(k_1)\,\tilde{A}_\nu(k_2)$$

where the anomaly pole is manifest and is coupled (non-zero residue) in the massless limit.

$\Rightarrow$ **The pole is the perturbative signature of the pion !**

Sum rules approach shades new light on the issue of anomaly poles: listen to Luigi delle Rose on Wednesday.

Wess-Zumino anomaly actions

The 1PI effective action is non-local but the appearance of anomaly poles implies **the possibility of extra local degrees of freedom**.

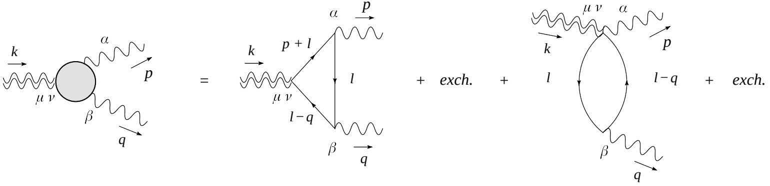
Local solution to the anomaly equations are provided by
Wess-Zumino anomaly actions
$\Rightarrow$ all anomalous interactions encoded.

For the case $SU_L(3) \times SU_R(3)$ it was done by Wess and Zumino (1970).

Local degrees of freedom = pion fields.
5-pion vertex:

$$\frac{1}{6\,\pi^2\,F_\pi^5}\,\epsilon^{\mu\nu\alpha\beta}\,tr\,(\Pi\partial_\mu\Pi\partial_\nu\Pi\partial_\alpha\Pi\partial_\beta\Pi)\ ,\qquad \Pi = \frac{1}{2}\,\lambda_i\,\Pi_i$$

The anomaly pole for the TVV



(a) (b) (c)

The conformal anomaly shows **strikingly similar features**: on shell correlator in the massless limit of QED

$$
\Gamma^{\mu\nu\alpha\beta}(k^2,0,0) = -\frac{e^2}{48\pi^2}\Bigg\{\frac{1}{k^2}\left[\left(2\,p^\beta\,q^\alpha - k^2\,g^{\alpha\beta}\right)\left(2\,p^\mu\,p^\nu + 2\,q^\mu\,q^\nu - k^2\,g^{\mu\nu}\right)\right]
+\frac{1}{3}\left(12\,\log\left(\frac{k^2}{\mu^2}\right) - 35\right)\Big[(p^\mu q^\nu + p^\nu q^\mu)\eta^{\alpha\beta} + \frac{k^2}{2}(\eta^{\alpha\nu}\eta^{\beta\mu} + \eta^{\alpha\mu}\eta^{\beta\nu})
-\eta^{\mu\nu}(\frac{k^2}{2}\eta^{\alpha\beta} - q^\alpha p^\beta) - (\eta^{\beta\nu}p^\mu + \eta^{\beta\mu}p^\nu)q^\alpha - (\eta^{\alpha\nu}q^\mu + \eta^{\alpha\mu}q^\nu)p^\beta\Big]\Bigg\}\tilde{A}_\alpha(p)\tilde{A}_\beta(q)
$$

C. Corianò, L. Delle Rose, A. Quintavalle, M.S., JHEP 1306 (2013) 077

It is quite natural to identify the pole as a signature of the (pseudo-) Goldstone boson of scale symmetry, **the dilaton** (τ).

Under scale transformations

$$x^\mu \to e^\sigma x^\mu \Leftrightarrow \tau \to \tau + \Lambda\,\sigma$$

The 1PI effective action in the gauge sector (F^2) of the conformal anomaly is easily obtained from the diagrammatic computation:

$$\Gamma[A_\mu, \tau] = \int d^4x\, \frac{\tau}{\Lambda}\, F^{\mu\nu} F_{\mu\nu} + \ldots (\text{mass terms})\,, \quad \Lambda = \text{conformal scale}$$

This implies **anomalous enhancements in the 2-photon and 2-gluon channels !**

But what about the rest...?

There is no dilaton self-interaction in here...anything else ?

Classical Weyl gauging

Question : how to write an anomalous effective (low energy) action for the dilaton, **encoding all the anomalous interactions** ?

Answer : thoroughly exploit Weyl symmery and its relation with conformal symmetry (Zumino, 1970): a theory which is Weyl invariant in curved space is conformal invariant in the flat limit.

General strategy for making a classical field theory Weyl invariant: **Weyl-gauging** (Iorio, O'Raifertaigh, Sachs, Wiesendanger, 1996): for a scale-invariant theory, embed it in curved space and make the replacements

$$\begin{aligned} \Phi &\rightarrow \Phi\, e^{d_\Phi \tau/\Lambda} \\ \nabla_\mu &\rightarrow \nabla_\mu + (-d_\Phi\, \delta^\nu{}_\mu + 2\,\Sigma^\nu{}_\mu)\; W_\nu \end{aligned}$$

W_μ is an abelian gauge vector field

$$\begin{aligned} g_{\mu\nu} &\rightarrow g_{\mu\nu} e^{2\,\sigma(x)}\, g_{\mu\nu}\,, \quad \text{Weyl transf.} \\ W_\mu &\rightarrow W_\mu + \partial_\mu \sigma \end{aligned}$$

Just like electrodynamics...

If you want to include dimensionful parameters, no way: need a dilaton (τ) !
Just a compensator field at this stage...

$$\mu \to \mu e^{-d_\mu \tau/\Lambda}$$

Under Weyl symmetry

$$g_{\mu\nu} \to e^{2\,\sigma(x)}\, g_{\mu\nu} \Leftrightarrow \tau \to \tau + \Lambda\,\sigma(x)$$

One can make **a minimal choice** and introduce **only one new d.o.f.**

$$W_\mu \equiv \frac{\partial_\mu \tau}{\Lambda}$$

Then Weyl gauging is simply given by

$$\begin{aligned}
m \quad &\to \quad m\, e^{-\tau/\Lambda} \\
g_{\mu\nu} \quad &\to \quad \hat{g}_{\mu\nu} \equiv g_{\mu\nu}\, e^{-2\tau/\Lambda} \\
\Phi \quad &\to \quad \hat{\Phi} \equiv \Phi\, e^{d_\Phi \tau/\Lambda} \\
\nabla_\mu \quad &\to \quad \nabla_\mu + (-d_\Phi\, \delta^\nu{}_\mu + 2\, \Sigma^\nu{}_\mu)\, \frac{\partial_\nu \tau}{\Lambda}
\end{aligned}$$

Dynamics from Weyl gauging

Example: the free scalar field

$$\mathcal{S}_\phi = \frac{1}{2}\int d^d x \sqrt{g}\left(g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + m^2\,\phi^2\right) \to$$

$$\to \mathcal{S}_{\phi,\tau} = \frac{1}{2}\int d^d x\,\sqrt{g}\left\{(\partial\phi)^2 + m^2\,\phi^2\,e^{-2\tau/\Lambda} + \frac{d-2}{2}\,\phi^2\,\frac{\Box\tau}{\Lambda} + \left(\frac{d-2}{2}\right)^2\,\phi^2\,\frac{(\partial\tau)^2}{\Lambda^2}\right\}$$

Complete Weyl variation:

$$\delta_W \mathcal{S}_{\phi,\tau} \equiv \frac{\delta\mathcal{S}_{\phi,\tau}}{\delta g_{\mu\nu}}(\delta_W g_{\mu\nu}) + \frac{\delta\mathcal{S}_{\phi,\tau}}{\delta\tau}(\delta_W\tau) = 0$$

Hint: we have got interactions $\sim \phi^2\,\frac{\Box\tau}{\Lambda}$ and $\sim \phi^2\,\frac{(\partial\tau)^2}{\Lambda^2}$ **authomatically...**

Quantum Weyl gauging

There can be two kinds of contributions to the dilaton effective action:

1. Diff- $\times$ Weyl-invariant terms, carrying non information on anomalous interactions;
2. Diff- but not Weyl-invariant terms, encoding anomalous interactions.

Easy to classify all the **diff** $\times$ **Weyl invariant** contributions,

$$\mathcal{J}_n \sim \frac{1}{\Lambda^{2(n-2)}} \int d^4x \sqrt{\hat{g}}\,\hat{R}^n$$

and get what the **non anomalous part of the dilaton effective action** looks like

$$\Gamma_0[g,\tau] \sim \sum_n \mathcal{J}_n[\hat{g}] \sim \int d^4x \left[e^{-\frac{4\,\tau}{\Lambda}}\,\alpha + \frac{1}{2}\, e^{-\frac{2\,\tau}{\Lambda}}\,(\partial\tau)^2 + 36\,\gamma \left(\frac{\Box\tau}{\Lambda} - \frac{(\partial\tau)^2}{\Lambda^2} \right) \right] + \dots$$

We focus on classically Weyl-invariant theories !

It is most interesting to investigate the effect of Weyl gauging on the Weyl-non-invariant part of the effective action, which is best done working in **dimensional regularization**, because in such a scheme

Anomaly $\Leftrightarrow$ 1-loop counterterms

$$
\begin{aligned}
\Gamma[g] &= \Gamma_0[g] + \Gamma_{\mathrm{Ct}}[g] \\
\Gamma_{\mathrm{Ct}}[g] &= -\frac{\mu^{-\epsilon}}{\epsilon}\int d^d x\,\sqrt{g}\left(\beta_a F + \beta_b E_4\right), \quad \epsilon = 4-d \\
g_{\mu\nu}\frac{\delta\Gamma_0[g]}{\delta g_{\mu\nu}}\bigg|_{d\to4} &= 0 \\
\frac{2}{\sqrt{g}}g_{\mu\nu}\frac{\delta\Gamma_{\mathrm{Ct}}[g]}{\delta g_{\mu\nu}}\bigg|_{d\to4} &= \beta_a\left(F - \frac{2}{3}\Box R\right) + \beta_b\, E_4
\end{aligned}
$$

What does the Weyl gauged effective action look like ?

$$
\hat{\Gamma}_0[g] = 0
$$
$$
\hat{\Gamma}_{\mathrm{Ct}}[g] = ?
$$

Weyl gauging of the counterterms

Start with a very general expansion of the gauged counterterms (everything computed in d dimensions)...

$$-\frac{1}{\epsilon}\int d^dx\,\sqrt{\hat{g}}\,\hat{F}\,(\hat{E}_4) = -\frac{1}{\epsilon}\int d^dx\sum_{i,j=0}^{\infty}\frac{1}{i!j!}\,\epsilon^i\,\frac{1}{\Lambda^j}\,\frac{\partial^{i+j}\left[\sqrt{\hat{g}}\,\hat{F}\,(\hat{E}_4)\right]}{\partial\epsilon^i\,\partial(1/\Lambda)^j}$$

but only the $O(\epsilon)$ contributions are significant !!!

$$\Gamma_{WZ}[g,\tau] = \Gamma_{\rm ren}[g,\tau] - \hat{\Gamma}_{\rm ren}[g,\tau] = \int d^4x\,\sqrt{g}\,\Bigg\{$$
$$\beta_a\left[\frac{\tau}{\Lambda}\left(F - \frac{2}{3}\Box R\right) + \frac{2}{\Lambda^2}\left(\frac{R}{3}\,(\partial\tau)^2 + (\Box\tau)^2\right) - \frac{4}{\Lambda^3}\,(\partial\tau)^2\,\Box\tau + \frac{2}{\Lambda^4}\,(\partial\tau)^4\right]$$
$$+\beta_b\left[\frac{\tau}{\Lambda}\,E_4 - \frac{4}{\Lambda^2}\left(R^{\alpha\beta} - \frac{R}{2}\,g^{\alpha\beta}\right)\partial_\alpha\tau\,\partial_\beta\tau - \frac{4}{\Lambda^3}\,(\partial\tau)^2\,\Box\tau + \frac{2}{\Lambda^4}\,(\partial\tau)^4\right]\Bigg\}$$

$$\begin{aligned} \hat{\Gamma}_{\rm ren}[g,\tau] &= \Gamma_{\rm ren}[g,\tau] - \Gamma_{WZ}[g,\tau] \\ \delta_W \hat{\Gamma}_{\rm ren}[g,\tau] &= 0 \end{aligned}$$

We have got a Weyl-invariant quantum effective action at the price of introducing the dilaton.

1. If there is a dilaton, then it should describe the IR limit of some theory to be unvealed at the (presently unknown) scale Λ; the analogy with the pion can be pushed quite far: composite dilaton, new conformal sector... (Grinstein B., Goldberger, W. D., Skiba W., 2007; Grinstein B., Uttayarat P. 2011).
   It predicts all the anomalous self-interactions. By far the **most interesting scenario**...
2. If no dilaton is there, then we are just using a mathematical trick to isolate the anomalous contritubion to the effective action, but this is not for nothing, we can draw some interesting conclusion anyhow...

$$\Gamma_{WZ}[\delta,\tau] = \int d^4x \left[\frac{2\,\beta_a}{\Lambda^2}\,(\Box\tau)^2 + (\beta_a+\beta_b)\left(-\frac{4}{\Lambda^3}\,(\partial\tau)^2\,\Box\tau + \frac{2}{\Lambda^4}\,(\partial\tau)^4\right)\right]$$

After coupling to gravity, we get the most general effective action for **residual anomalous self-interactions in the flat limit as well...**(computed also in 6 dimensions)

C. Corianò, L. Delle Rose, C. Marzo, M.S., Phys.Lett. B726 (2013) 4-5, 896-905

C. Corianò, L. Delle Rose, C. Marzo, M.S., Class.Quant.Grav. 31 (2014) 105009

Easy to extract the dilaton vertices $\mathcal{I}_n(x_1,\ldots,x_n)$:

$$\mathcal{I}_n(x_1,\ldots,x_n) = \frac{\delta^n\left(\hat{\Gamma}_{\mathrm{ren}}[\delta,\tau]-\Gamma_{\mathrm{ren}}[\delta,\tau]\right)}{\delta\tau(x_1)\ldots\delta\tau(x_n)} = -\frac{\delta^n\Gamma_{WZ}[\delta,\tau]}{\delta\tau(x_1)\ldots\delta\tau(x_n)}$$

Limit on anomalous dilaton self-interactions

Most important

$$\mathcal{I}_n(x_1,\dots,x_n)=0\,,\quad n\geq 5$$

Constraint (holds in general even dimensions):
n-dilaton anomalous interactions **vanish identically in 4 dimensions for** $n>4$.

Why ?
Direct consequence of the 4-derivative structure of the anomaly $\sim R^2$

$$\begin{aligned}
\hat{R}^{\mu}{}_{\nu\rho\sigma} &= R^{\mu}{}_{\nu\rho\sigma}+g_{\nu\rho}\left(\frac{\nabla_\sigma\partial^\mu\tau}{\Lambda}+\frac{\partial^\mu\tau\,\partial_\sigma\tau}{\Lambda^2}\right)-g_{\nu\sigma}\left(\frac{\nabla_\rho\partial^\mu\tau}{\Lambda}+\frac{\partial^\mu\tau\,\partial_\rho\tau}{\Lambda^2}\right)\\
&+\delta^{\mu}{}_{\sigma}\left(\frac{\nabla_\rho\partial_\nu\tau}{\Lambda}+\frac{\partial_\nu\tau\,\partial_\rho\tau}{\Lambda^2}\right)-\delta^{\mu}{}_{\rho}\left(\frac{\nabla_\sigma\partial_\nu\tau}{\Lambda}+\frac{\partial_\nu\tau\,\partial_\sigma\tau}{\Lambda^2}\right)+\left(\delta^{\mu}{}_{\rho}\,g_{\nu\sigma}-\delta^{\mu}{}_{\sigma}\,g_{\nu\rho}\right)\frac{(\partial\tau)^2}{\Lambda^2}\\
\hat{R}_{\mu\nu} &= R_{\mu\nu}-g_{\mu\nu}\left(\frac{\Box\tau}{\Lambda}-(d-2)\,\frac{(\partial\tau)^2}{\Lambda^2}\right)-(d-2)\left(\frac{\nabla_\mu\partial_\nu\tau}{\Lambda}+\frac{\partial_\mu\tau\,\partial_\nu\tau}{\Lambda^2}\right)\\
\hat{R} &\equiv \hat{g}^{\mu\nu}\,\hat{R}_{\mu\nu}=e^{\frac{2\tau}{\Lambda}}\left[R-2\,(d-1)\,\frac{\Box\tau}{\Lambda}+(d-1)\,(d-2)\,\frac{(\partial\tau)^2}{\Lambda^2}\right]
\end{aligned}$$

Energy-momentum tensor Green functions for conformal field theories

$$
\begin{aligned}
\langle T^{\mu_1\nu_1}(x_1)\dots T^{\mu_n\nu_n}(x_n)\rangle &\equiv 2^n \left.\frac{\delta^n\Gamma[g]}{\delta g_{\mu_1\nu_1}(x_1)\dots\dots\delta g_{\mu_n\nu_n}(x_n)}\right|_{g_{\mu\nu}=\eta_{\mu\nu}} \\
&\equiv [\Gamma[g]]^{\mu_1\nu_1\dots\mu_n\nu_n}(x_1,\dots x_n)
\end{aligned}
$$

$$
g_{\mu\nu}\langle T^{\mu\nu}\rangle_s \equiv \langle T\rangle_s = \mathcal{A}[g]
$$

It defines **an open hierarchy**

$$
\begin{aligned}
\langle T(k_1)\ \dots\ T(k_{n+1})\rangle &= 2^n\,[\sqrt{g}\,\mathcal{A}[g]]^{\mu_1\dots\mu_n}_{\mu_1\dots\nu_n}(k_1,\dots k_{n+1}) \\
&\quad - 2\sum_{i=1}^{n}\langle T(k_1)\dots T(k_{i-1})T(k_{i+1})\dots T(k_{n+1}+k_i)\rangle
\end{aligned}
$$

Iterate it and end with $n-1$ metric variations of the anomaly functional for a n-point correlator ! The difficulty grows very rapidly...

Recurrence relations: the idea

Anything else interesting from the Wess-Zumino effective action?
Yes, if you take **a closer look from another point of view...**

An equivalent expression of $\Gamma_{WZ}[g,\tau]$ is given by its **perturbative expansion** for a conformally flat background $\hat{g}_{\mu\nu} = \eta_{\mu\nu} e^{-2\tau/\Lambda}$ in $1/\Lambda$...

$$
\begin{aligned}
\hat{g}_{\mu\nu} &= \eta_{\mu\nu}\, e^{-2\,\tau/\Lambda} = \eta_{\mu\nu} \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} \left(\frac{\tau}{\Lambda}\right)^n \\
\hat{\Gamma}_{\rm ren}[\delta,\tau] - \Gamma_{\rm ren}[\delta,\tau] &= \frac{1}{2!\,\Lambda^2} \int d^d x_1 d^d x_2 \, \langle T(x_1) T(x_2) \rangle \, \tau(x_1)\tau(x_2) \\
&- \frac{1}{3!\,\Lambda^3} \Bigg[\int d^d x_1 d^d x_2 d^d x_3 \, \langle T(x_1) T(x_2) T(x_3) \rangle \, \tau(x_1)\tau(x_2)\tau(x_3) \\
&+ 6 \int d^d x_1 d^d x_2 \, \langle T(x_1) T(x_2) \rangle \, (\tau(x_1))^2 \tau(x_2) \Bigg] + \ldots
\end{aligned}
$$

Recurrence relations: how to solve an infinite hierarchy

Clue: anomalous dilaton vertices as linear combinations of traced Green functions of the energy-momentum tensor !

$$
\begin{aligned}
\mathcal{I}_2(k_1,-k_1) &= \frac{1}{\Lambda^2}\langle T(k_1)T(-k_1)\rangle \\
\mathcal{I}_3(k_1,k_2,k_3) &= -\frac{1}{\Lambda^3}\Big[\langle T(k_1)T(k_2)T(k_3)\rangle \\
&\quad + 2\Big(\langle T(k_1)T(-k_1)\rangle + \langle T(k_2)T(-k_2)\rangle + \langle T(k_3)T(-k_3)\rangle\Big)\Big] \\
&\dots
\end{aligned}
$$

There must be exact matching for consistency: thoroughly checked in 2, 4 and 6 dimensions !!!

$$
\begin{aligned}
\langle T(k_1)T(-k_1)\rangle &= -4\,\beta_a\,{k_1}^4 \\
\langle T(k_1)T(k_2)T(k_3)\rangle &= 8\Big[-\Big(\beta_a+\beta_b\Big)\Big(k_1^2\,k_2\cdot k_3 + k_2^2\,k_1\cdot k_3 + k_3^2\,k_1\cdot k_2\Big) \\
&\quad + \beta_a\Big(k_1^4+k_2^4+k_3^4\Big)\Big] \\
&\dots
\end{aligned}
$$

and so on, ad infinitum...everything purely algebraic !

Summary and conclusions

- Just as in QCD, triangle diagrams feature anomaly poles, suggesting the presence of new scalar effective states
- Wess-Zumino effective actions are the natural way to provide an infrared description of a system affected by an anomaly
- Weyl gauging is a straightforward method to predict the structure of Wess-Zumino conformal anomaly actions in full generality
- Is the dilaton an effective state pointing towards new physics or is it just a mathematical artifice ?
- An infinite hierarchy of recurrence relations stems from consistency requirements on the effective action