2. Dark Matter production: the standard lore

freeze-out, WIMPs, Boltzmann equation...

Pasquale D. Serpico

ISAPP - Belgirate 24 July 2014

2. An Introduction to Dark Matter

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If "The Boss" said so...

Recap and Plan of this lecture

Observationally

Current determination (Planck 2013, 68% CL)

$\Omega_c h^2 = 0.120 \pm 0.003$, i.e. $\Omega_c \sim 0.27$

Phenomenologically

$$\Omega_X h^2 = 2.74 \times 10^8 \left(\frac{M_X}{\mathrm{GeV}}\right) Y_0$$

Goal: *compute the current value of number to entropy density ratio, Y_0*

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Goal: *compute the current value of number to entropy density ratio, Y_0*

- We shall first provide a heuristic argument for the simplest (yet powerful!) toy-model evolution equation for *Y*
- We shall use this equation in different regimes to elucidate different classes (not all!) of DM candidates
- We'll come back to a "microscopic" derivation/interpretation of the equation we started with.
- Some generalizations will be briefly discussed.

Caveat: *matching Ω_X is one condition for a good DM candidate, not the only one! Remember lecture I (collisionless, right properties for LSS structures...)*

Boltzmann equation for DM relic density computation

Assume that binary interactions of our particle X are present with species of the thermal bath

$$X\,X \leftrightarrow (\text{thermal bath particles})$$

If interaction rate **Γ=n σ v** very slow wrt Hubble rate **H**, # of particles conserved covariantly, i.e.

$$\frac{dn}{dt} + 3H\,n = 0 \Rightarrow n \propto a^{-3}$$

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If interaction rate **Γ>> H**, # of particles follows equilibrium, e.g. for non-relativistic particles

$$n_{\text{eq}} = g \left(\frac{m\, T}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right)$$

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The following equation has the right limiting behaviours

$$\frac{dn}{dt} + 3H\,n = -\langle\sigma v\rangle[n^2 - n_{\text{eq}}^2]$$

must be quadratic, for binary processes

for now, symbolic only

Rewriting in terms of Y and x

$$\frac{dn}{dt} + 3H\,n = -\langle \sigma v \rangle [n^2 - n_{\rm eq}^2]$$

$$\frac{dY}{dt} = -s\langle \sigma v \rangle [Y^2 - Y_{\rm eq}^2]$$

B. W. Lee and S. Weinberg, "Cosmological Lower Bound on Heavy Neutrino Masses," PRL 39, 165 (1977).

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Define $x{=}m/T$ (m *arbitrary mass, either* M_X *or not*); for an iso-entropic expansion one has

$$\frac{d}{dt}(a^3 s) = 0 \Longrightarrow \frac{d}{dt}(a\,T) = 0 \Longrightarrow \frac{d}{dt}(a/x) = \frac{\dot a}{x} - \frac{a}{x^2}\dot x = 0 \Longrightarrow \frac{dx}{dt} = H\,x$$

$$\frac{dY}{dx} = -\frac{x\,s\langle\sigma v\rangle}{H(T=m)}[Y^2 - Y_{\rm eq}^2]$$

Rewriting in terms of *Y* and *x*

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$$\frac{dY}{dx} = -\frac{x\, s\langle\sigma v\rangle}{H(T=m)}[Y^2 - Y_{\rm eq}^2]$$

More in general (arbitrary *s(t)* and *H(t)*):

$$\frac{dY}{dx} = -\sqrt{45\pi} M_{\rm Pl}\, m \frac{h_{\rm eff}(x)\langle\sigma v\rangle}{\sqrt{g_{\rm eff}(x)}\, x^2}\left(1 - \frac{1}{3}\frac{d\log h_{\rm eff}}{d\log x}\right)(Y^2 - Y_{\rm eq}^2)$$

M. Srednicki, R. Watkins and K. A. Olive, "Calculations of Relic Densities in the Early Universe," Nucl. Phys. B 310, 693 (1988)

P. Gondolo and G. Gelmini, "Cosmic abundances of stable particles: Improved analysis," Nucl. Phys. B 360, 145 (1991).

Freeze-out

The previous equation is a *Riccati equation*: no closed form solution exist!

Approximate analytical solutions exist for different hypotheses/regimes

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For $h_{\rm eff}$ ~ const., we can re-write

$$\frac{x}{Y_{\rm eq}}\frac{dY}{dx} = -\frac{\Gamma_{\rm eq}}{H}\left[\left(\frac{Y}{Y_{\rm eq}}\right)^2 - 1\right] \quad \text{with} \quad \Gamma_{\rm eq} = \langle\sigma v\rangle n_{\rm eq}$$

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If Γ_{eq} >> H the particle starts from equilibrium condition at sufficiently small x (high-T), when relativistic. Crucial variable to determine the Y_{final} is the freeze-out epoch x_F from condition

$$\Gamma_{\rm eq}(x_F) = H(x_F)$$

Relativistic freeze-out

$$\Gamma_{\rm eq}(x_F) = H(x_F)$$

If the solution to this condition yields $x_F \ll 1$, then *(Lecture 1)*

$$n_{\rm eq} = g\frac{\zeta(3)}{\pi^2}T^3 \times \left\{1(\mathrm{B}), \frac{3}{4}(\mathrm{F})\right\}$$

comoving abundance stays constant, and independent of *x (if dof do not change)*

$$Y(x_F) = 0.28\frac{g \times \{1(\mathrm{B}), 3/4(\mathrm{F})\}}{h_{\rm eff}(x_F)}$$

Today's abundance of such a relativistic freeze-out relic is thus

$$\Omega_X h^2 = 0.0762 \times \left(\frac{M_X}{\mathrm{eV}}\right)\frac{g \times \{1(\mathrm{B}), 3/4(\mathrm{F})\}}{h_{\rm eff}(x_F)}$$

Relativistic freeze-out

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For the neutrino case, $h_{\rm eff}$=10.75, g×{ }=3/2, thus

$$\Omega_\nu h^2 \simeq \frac{\sum m_\nu}{94\,\mathrm{eV}}$$

Inconsistent with DM for current upper limits!

Freeze-out: non-relativistic case

to determine x_F

$$\Gamma_{\rm eq}(x_F) = H(x_F)$$

namely

$$\frac{g\langle\sigma v\rangle}{(2\pi)^{3/2}} M_X^3\, x_F^{-3/2} e^{-x_F} = \sqrt{\frac{4\pi^3}{45} g_{\rm eff}}\, \frac{M_X^2}{x_F^2\, M_{\rm Pl}}$$

i.e.

$$x_F^{1/2} e^{-x_F} = \sqrt{\frac{4\pi^3}{45} g_{\rm eff}}\, \frac{(2\pi)^{3/2}}{M_{\rm Pl} M_X\, g\, \langle\sigma v\rangle}$$

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Thus one obtains

$$Y(x_F) = \frac{n(x_F)}{s(x_F)} = \frac{g}{h_{\rm eff}} \frac{45}{2\pi^2 (2\pi)^{3/2}} x_F^{3/2} e^{-x_F}$$

which also writes

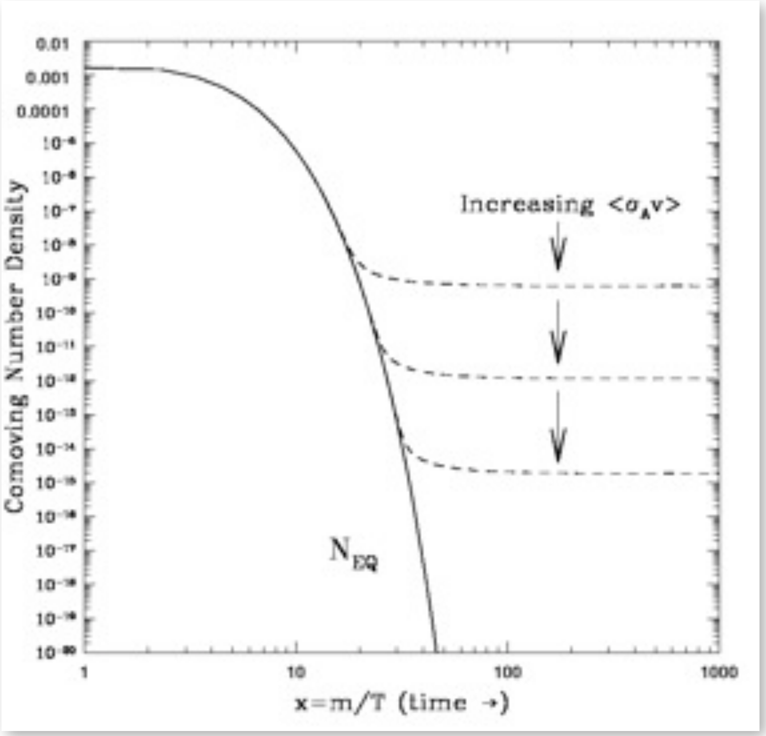
(Note the important result ***Y(x_F)~ 1/<σv>***)

$$Y(x_F) = \sqrt{\frac{45\, g_{\rm eff}}{\pi}}\, \frac{x_F}{h_{\rm eff} M_{\rm Pl}\, M_X\, \langle\sigma v\rangle} = \mathcal{O}(1) \frac{x_F}{M_{\rm Pl}\, M_X\, \langle\sigma v\rangle}$$

Non-relativistic freeze-out: interpretation

$$Y(x_F) \simeq \mathcal{O}(1)\,\frac{x_F}{M_{\rm Pl}\,M_X\,\langle\sigma v\rangle}$$

makes sense, in the Boltzmann suppressed tail: The more it interacts, the later it decouples, the fewer particles around.



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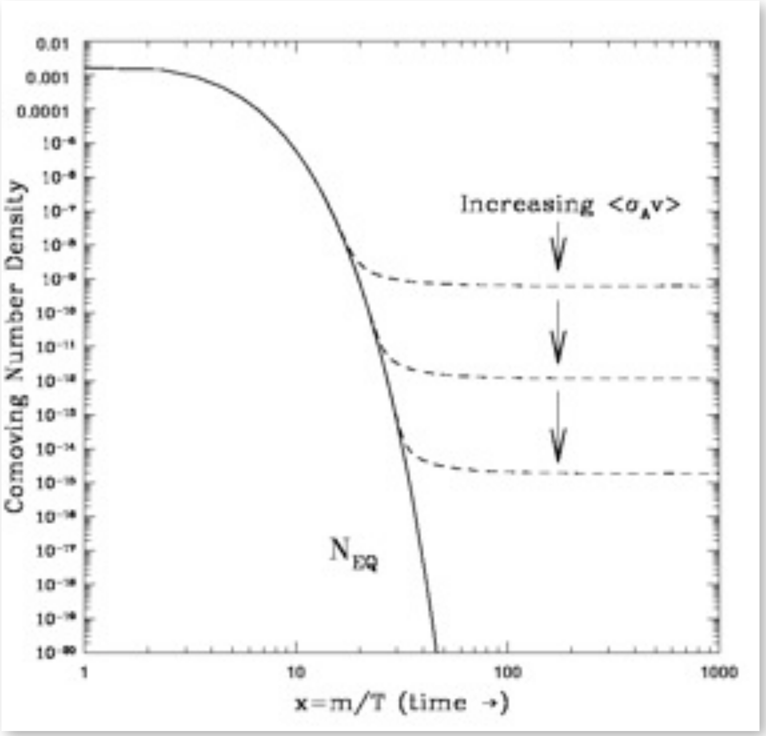

Also, plugging numbers (typically x_F~30), one has

$$\Longrightarrow \Omega_X h^2 \simeq \frac{0.1\,\mathrm{pb}}{\langle\sigma v\rangle}$$

dimensionally, for electroweak scale masses and couplings, one gets the right value!

$$\langle\sigma v\rangle \sim \frac{\alpha^2}{m^2} \simeq 1\,\mathrm{pb}\left(\frac{200\,\mathrm{GeV}}{m}\right)^2$$

But the pre-factor depends from widely different cosmological parameters (Hubble parameter, CMB temperature) and the Planck scale. Is this match simply a coincidence?

Dubbed sometimes “Weakly Interacting Massive Particle” (WIMP) Miracle

Exercise

Apply the previous formalism to baryons, with m_b~1 GeV & $<\sigma v>\sim 1/m_\pi^2$

What is the current energy density of baryons?

Is this a plausible mechanism behind their abundance?

WIMPs & Particle Physics Models: caveats

- Sometimes, one interprets it very strongly as “Dark Matter” favours (or implies) new weakly interacting particles (at electroweak scale). Beware of pushing that too far!

- I would rather interpret the other way around: the appeal of TeV scale new physics models is greater if they can “elegantly” solve the DM nature puzzle. But disproving the former (e.g. weak scale natural SUSY) should not be taken as a “punch” to DM itself!

- Within PP models, can be used to constrain parameters of the theory or theories themselves (as in original Lee-Weinberg model): theories predicting too large relic values for a (meta)stable candidate are disfavoured/excluded.

- Requiring a WIMP DM candidate has even been used as guideline in TeV-scale BSM model-building! (e.g. split SUSY, Minimal Dark Matter...)

- In actual models, often many particles and parameters contribute. And the final results may not be “as elegant” or “as natural” as the previous toy model.

Care should be taken when one deals with...

- cohannihilations with other particle(s) close in mass
- resonant annihilations*
- thresholds*

K. Griest and D. Seckel, "Three exceptions in the calculation of relic abundances," Phys. Rev. D 43, 3191 (1991).

* i.e., whenever ***σ(s)*** is a strongly varying function of the center-of-mass energy ***s***
(one recently popular example is the "Sommerfeld Enhancement")

For a pedagogical overview of generalization in presence of coannihilations (and decays), see

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Nowadays, relic density calculations have reached a certain degree of sophistication and are automatized with publicly available software. But if you have a theory with "unusual" features... better to check!

MicrOMEGAs: a code for the calculation of Dark Matter Properties including the **relic density**, direct and **indirect rates** in a general supersymmetric model and other models of New Physics

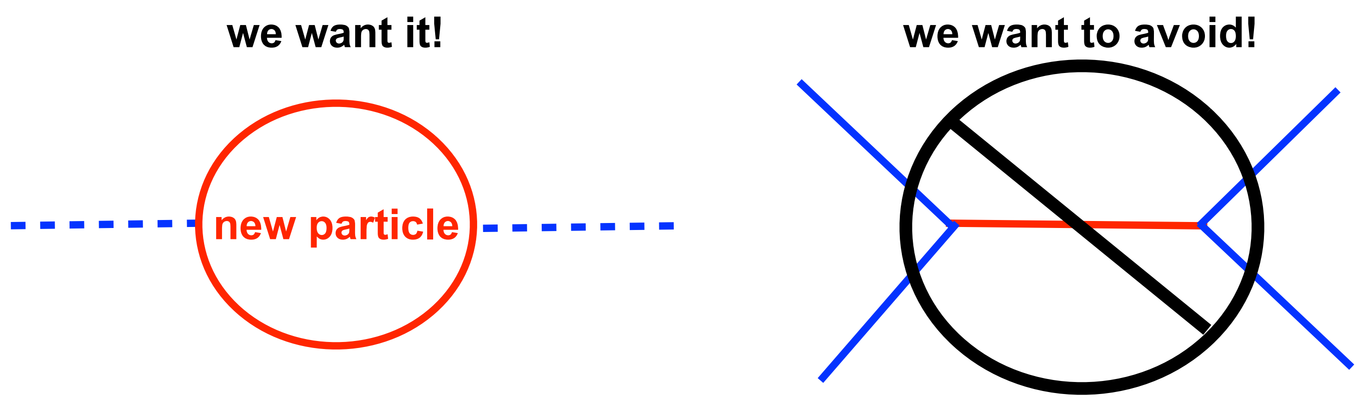
http://lapth.cnrs.fr/micromegas/

Dark SUSY

http://www.physto.se/~edsjo/darksusy/

Link with colliders

- If one has a strong prior for new TeV scale physics (~with ew. strength coupling) due to the hierarchy problem, precision ew data (e.g. from LEP) suggest that tree-level couplings SM-SM-BSM should be avoided!



- Straightforward solution (not unique!) is to impose a discrete "parity" symmetry e.g.: SUSY R-parity, K-parity in ED, T-parity in Little Higgs. New particles only appear in pairs!

  ➡ Automatically makes lightest new particle stable!
  ➡ May have other benefits (e.g. respect proton stability bounds...)

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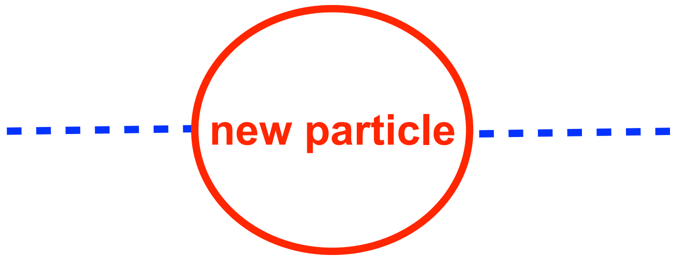
we want it!



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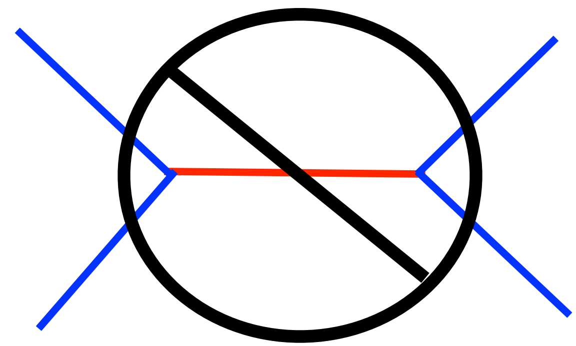

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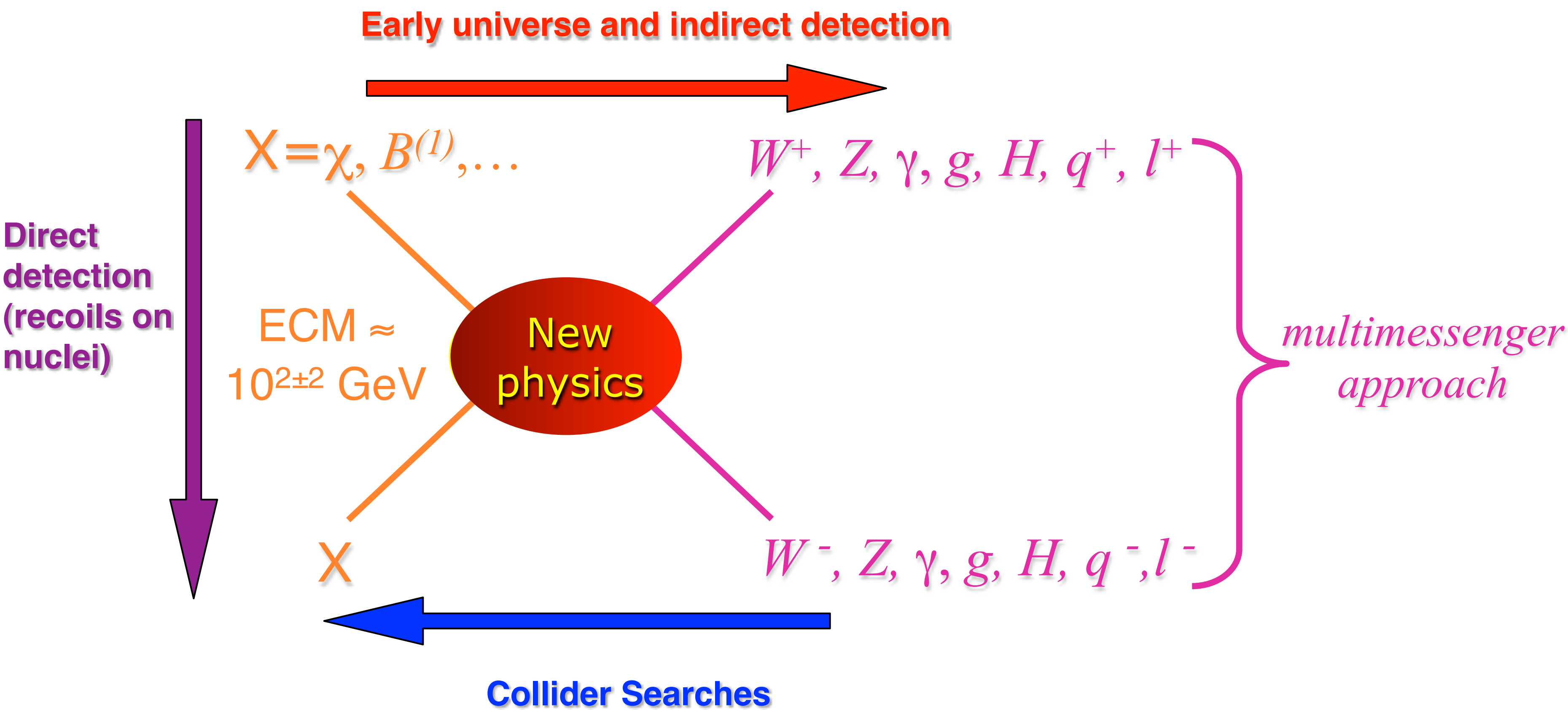
In a sense, some WIMP DM (too few? too much?) is "naturally" expected for consistency of the currently favored framework for BSM physics at EW scale.

Beware of the reverse induction:

LHC is now testing this paradigm, but if no new physics is found at EW scale it is at best the WIMP scenario to be disfavored, not the "existence of DM"

WIMP (not generic DM!) "discovery program"

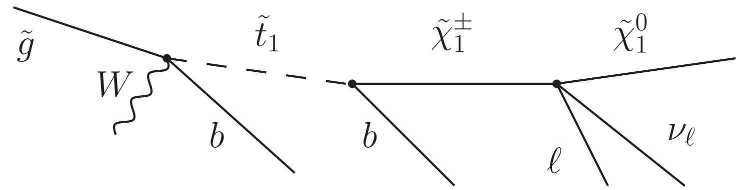


- ✓ demonstrate that astrophysical DM is made of particles (locally, via DD; remotely, via ID)
- ✓ Possibly, create DM candidates in the controlled environments of accelerators
- ✓ Find a consistency between properties of the two classes of particles. Ideally, we would like to calculate abundance and DD/ID signatures → link with cosmology/test of production

DM@colliders: The model-dependent way

Dark Matter studies at LHC are mostly model-dependent.

Either one can limit oneself to processes involving "chains" ending with large $\not{E}$, which allow at most to check if a "stable" particle (on detector scale!) has been produced, and in some cases to constrain its mass (scale).



$$M_{\rm eff} = \sum_i p_T^{{\rm jet},i} + \sum_i p_T^{{\rm lep},i} + E_T^{\rm miss}$$

For a review, Barr & Lester 1004.2732

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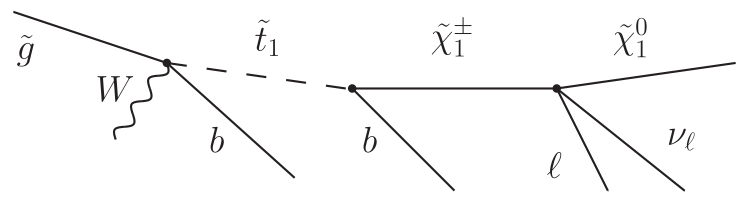

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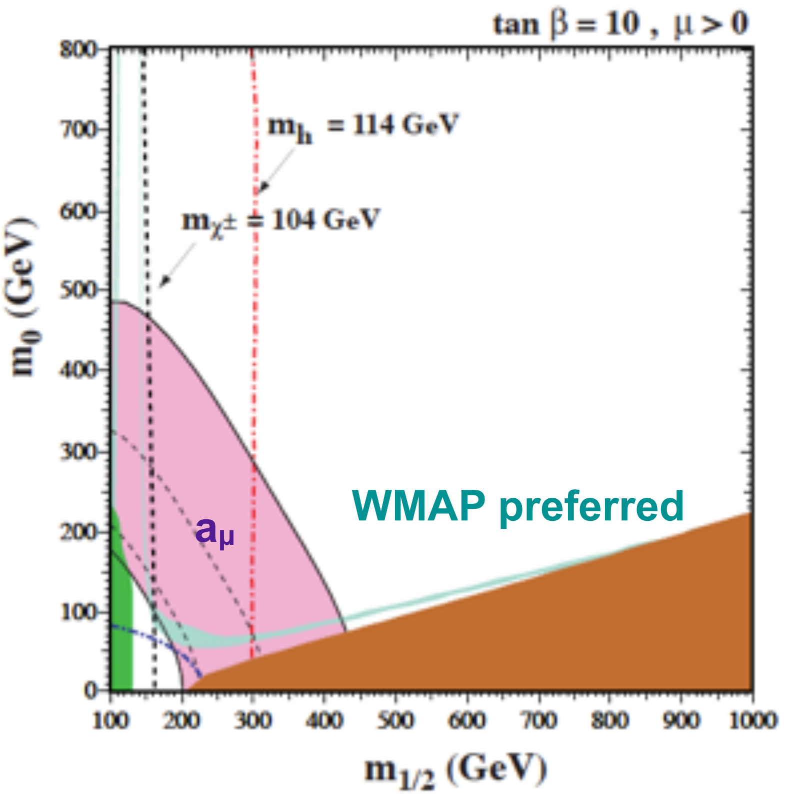
Alternative Strategy: Pick "benchmark" models (e.g. in CMSSM), derive bounds on DM from bounds on "observable" object and theoretical relations, with plots e.g. in m_0-$m_{1/2}$ for different tan β... hope to learn "generic lessons"



For a review, Ellis & Olive 1001.3651
(results now outdated...)

DM production at colliders, EFT approach

From the “WIMP paradigm” it follows that one can produce DM “as in the early universe”, via

$$(SM)(SM) \to XX$$

- Main problem: the dominating channel $(SM)(SM) \to XX$ is obviously invisible.
- One may consider the “large $\not{E}$” channel $(SM)(SM) \to XXY$ with $Y = \gamma$, jet(s) unavoidably produced at least by initial state leptons/quarks.

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- One may consider the "large $\not{E}$" channel (SM)(SM) → XXY with Y= γ, jet(s) unavoidably produced at least by initial state leptons/quarks.

- One can parameterize DM-SM interactions in an EFT approach. E.g., for a Dirac fermion:

$$\mathcal{L} = \mathcal{L}_{SM} + i\bar{X}\gamma^\mu\partial_\mu X - M_X\bar{X}X + \sum_q \sum_{i,j} \frac{G_{qij}}{\sqrt{2}} \left[\bar{X}\Gamma_i^X X\right]\left[\bar{q}\Gamma_q^j q\right]$$

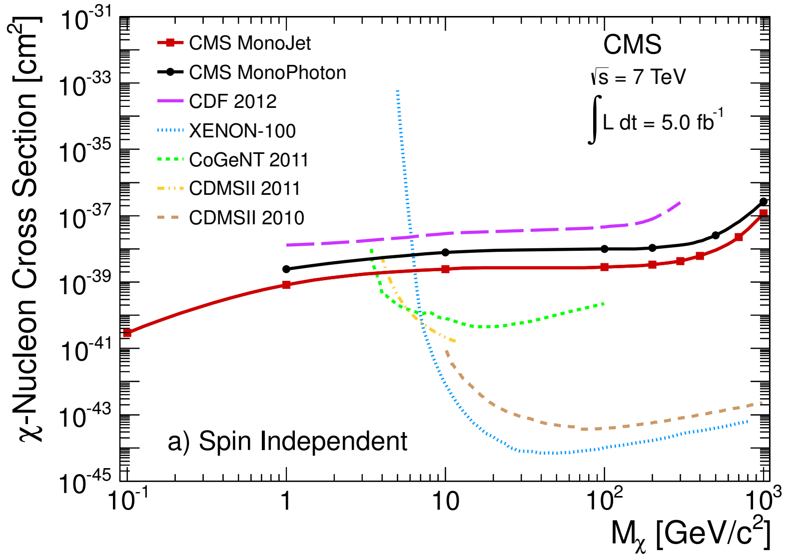
- Map the effective operators into signatures of missing energy+jet(s) as well as DD cross sections. Remarkable bounds already now!

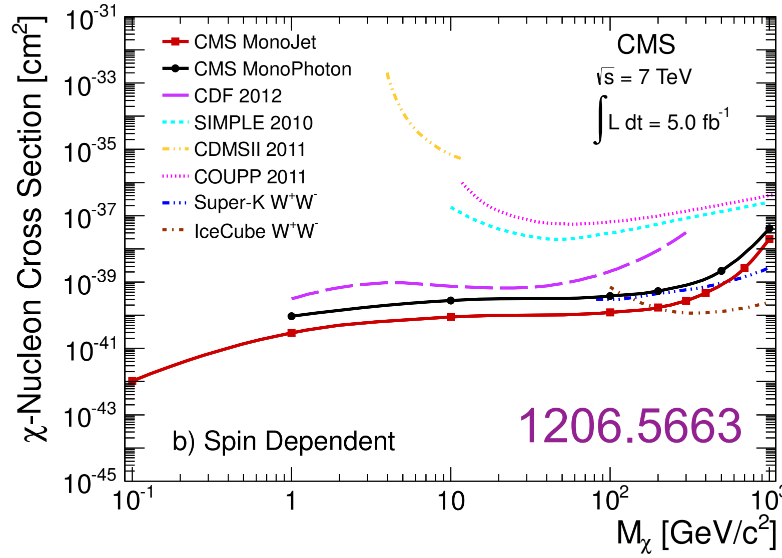
- Of course breaks down when/if BSM physics at low scale is present, hence it is complementary to explicit models (troublesome already @ LHC-7 TeV!)

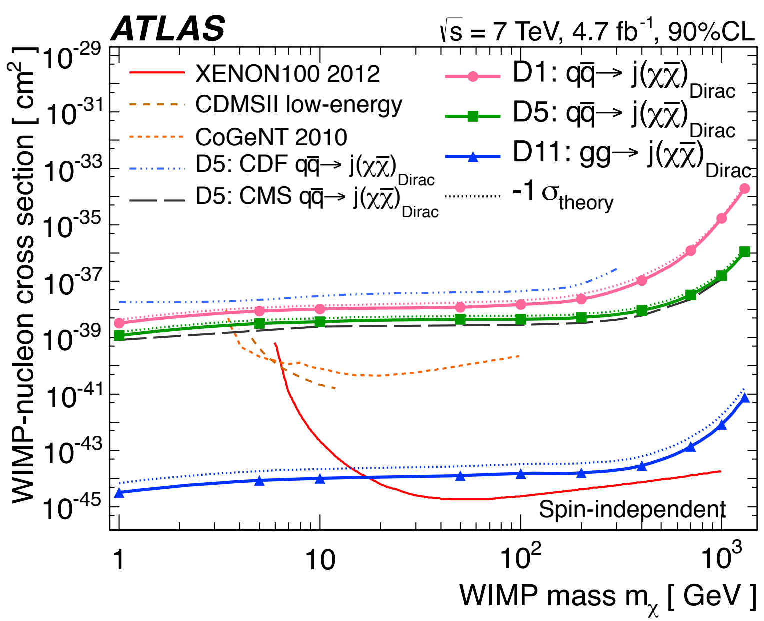
Incomplete list:

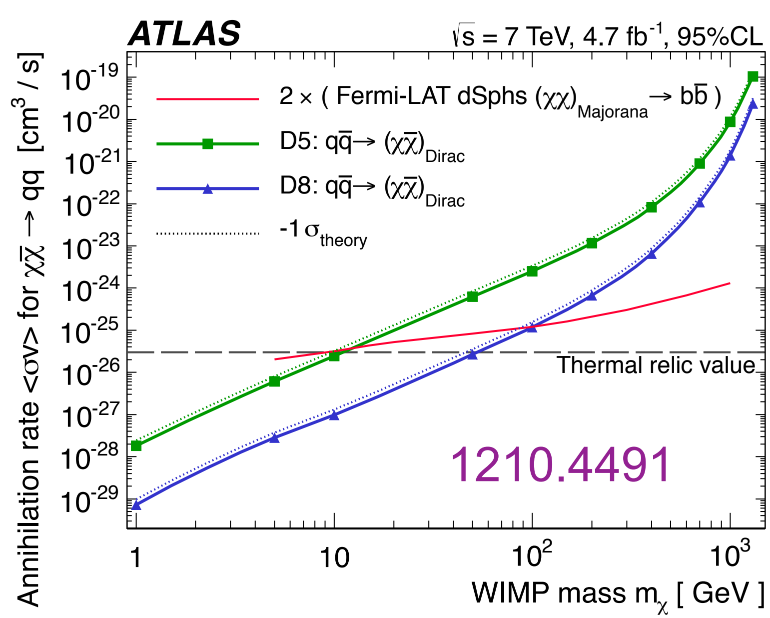
Beltran, Hooper, Kolb, Krusberg, Tait, 1002.4137 Bai,Fox, Harnik, 1005.3797
Goodman et al, 1005.1286 (majorana)   Goodman et al, 1008.1783 (dirac, scalar)
M. Buckley, 1104.1429 (EFT for asymmetric DM) ...

Well practiced at LHC...









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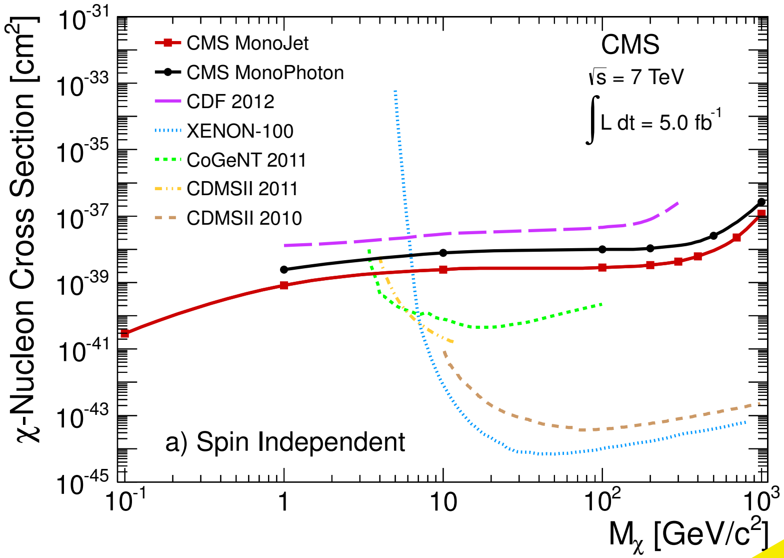
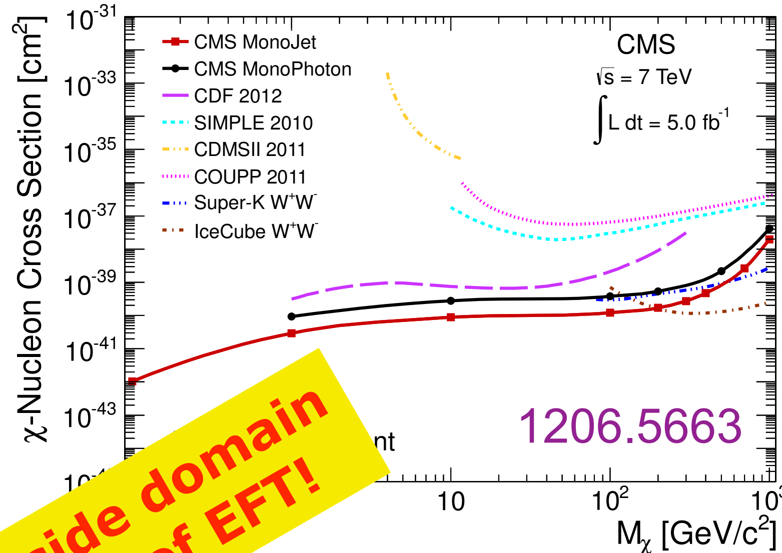
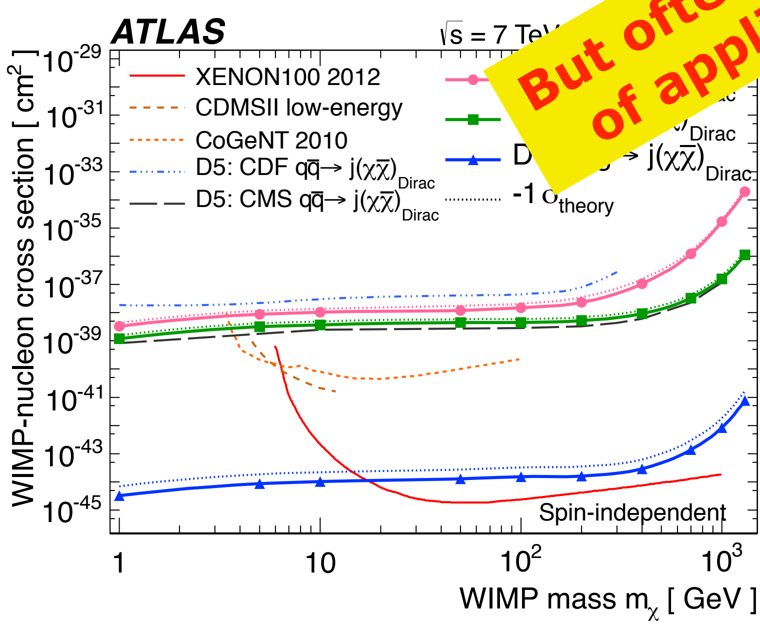
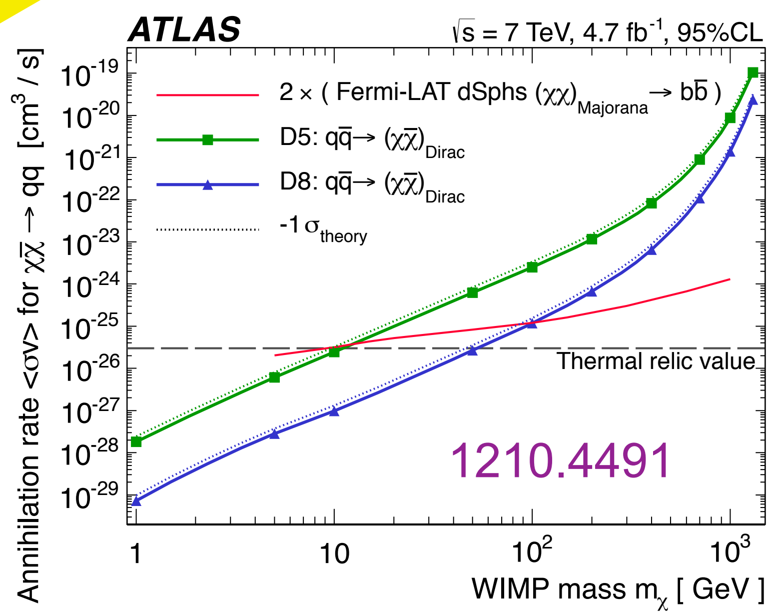




But often outside domain of applicability of EFT!





Freeze-in

- We assumed that at small x ($T >> m$), RHS$\rightarrow$ 0, i.e. Y follows it's equilibrium value
- If, however, DM extremely weakly coupled, some production can take place via $ff \rightarrow XX$ but Y may never attain equilibrium. In this case:

$$\frac{dY}{dx} = \frac{x s(x) \langle \sigma v \rangle}{H(m)} Y_{\mathrm{eq},f}^2$$

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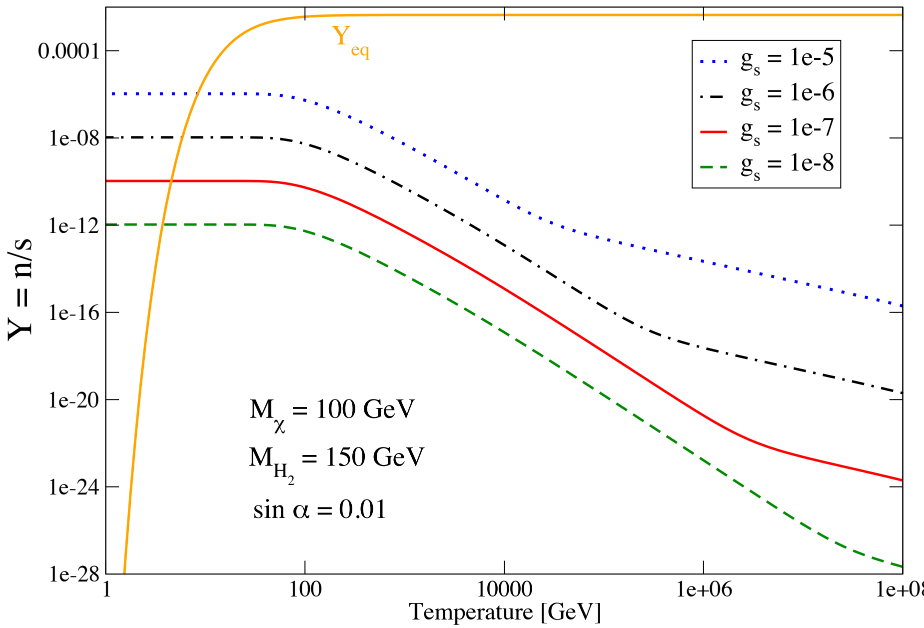
Assuming negligible initial abundance (otherwise it's not produced via freeze-in!)

$$Y_\infty \simeq \int_{x_0}^{\infty} dx' \frac{x' s(x')\langle\sigma v\rangle}{H(m)} Y^2_{\mathrm{eq},f}$$

Note that now

$$Y_\infty \propto \langle\sigma v\rangle$$

- Requires typically small couplings (harder to test...)
- It is more model dependent



M. Klasen and C. E. Yaguna, "Warm and cold fermionic dark matter via freeze-in," JCAP 1311, 039 (2013)

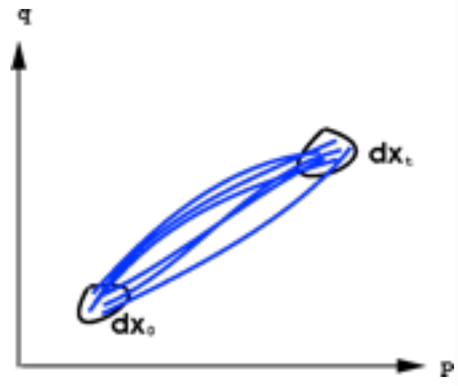
Boltzmann equation

Boltzmann equation

Start from Boltzmann equation for the phase-space distr. function f

along trajectories of hamiltonian flow

$$\frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$



In absence of collision, volume in phase space preserved, otherwise some non-vanishing RHS, depending on f only under some assumption (molecular chaos...)

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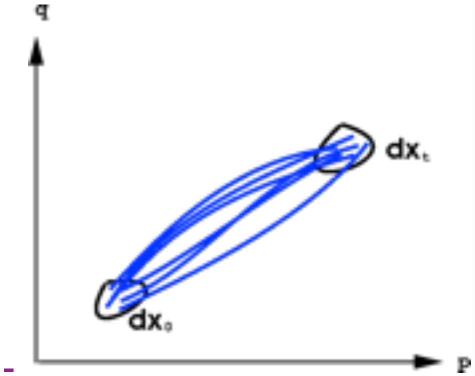

In absence of collision, volume in phase space preserved, otherwise some non-vanishing RHS, depending on f only under some assumption (molecular chaos...)

Using the EOM, this is equivalent to:

$$m\frac{\partial f}{\partial t} + \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

which we can rewrite symbolically as (Liouville operator acting at the LHS)

$$\hat{L}[f] = \hat{C}[f]$$

At RHS, the Collisional operator accounts for sources or sinks of particles in phase space. Since these are typically quantum phenomena, most likely you rather encountered it written down in “relativistic/quantum realm” courses

Boltzmann equation in GR

In relativistic case, similar relation along geodesics

Liouville operator $$\hat{L}[f] = \hat{C}[f]$$ Collisional operator

$$\hat{L}[f] = \frac{df}{d\lambda}(x^\mu(\lambda), p^\mu(\lambda))$$

in general, affine parameter λ to parametrize world-line

$$\hat{L}[f] = \frac{\partial f}{\partial x^\mu}\frac{d\,x^\mu}{d\lambda} + \frac{\partial f}{\partial p^\mu}\frac{d\,p^\mu}{d\lambda} = \hat{C}[f]$$

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Just like in classical theory the derivative of momentum is proportional to the “Force” (~ gradient of potential) in GR it can be expressed in terms of first-derivative of the metric $g_{\mu\nu}$, via the so-called Christoffel symbols (no need to be more specific, here)

$$\hat{L} \to p^\mu \frac{\partial}{\partial x^\mu} - p^\alpha p^\beta \Gamma^\mu_{\alpha\beta}\frac{\partial}{\partial p^\mu}$$

Boltzmann equation in GR

thanks to homogeneity and isotropy in FLRW (cosmological principle)

$$f(x^\mu, p^\mu, t) = f(E, t) \qquad \hat{L} \to E\left(\frac{\partial}{\partial t} - \frac{\dot{a}}{a} p \frac{\partial}{\partial p}\right)$$

compare with the classical operator

$$\hat{L}[f] = m\frac{\partial f}{\partial t} + \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

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Now, let us take, for the specific case of FLRW metric:

$$\frac{\hat{L}[f]}{E} = \frac{\hat{C}[f]}{E}$$

And let’s check that we obtain our “heuristic” equation
for relic calculations, when we integrate over the energy.

This will also provide a “microscopic” expression for the *C*

Left-hand side...

Integrate over phase space

$$g\int \frac{d^3\vec{p}}{(2\pi)^3 E}\hat{L}[f] = \frac{g}{(2\pi)^3}\int d^3\vec{p}\left(\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}p\frac{\partial f}{\partial p}\right) = \frac{dn}{dt} + 3H\,n = s\,\frac{dY}{dt}$$

recognize perhaps (twice) the relativistic invariant phase-space

integrate 2nd term by parts: *f* vanishes at boundary, deriving p^3 get factor 3...

$$Y \equiv n/s$$

where we introduced as customary the comoving density & entropy density

$$a^3\,s = const.$$

if relativistic d.o.f. do not change (isoentropic expansion)

$$\frac{dY}{dt} = \frac{d}{dt}\left(\frac{n}{s}\right) = \frac{d}{dt}\left(\frac{n\,a^3}{s\,a^3}\right) = \frac{1}{s\,a^3}\frac{d}{dt}\left(n\,a^3\right) = \frac{1}{s\,a^3}\left(a^3\frac{dn}{dt} + 3a^2\dot{a}\,n\right) = \frac{1}{s}\left(\frac{dn}{dt} + 3H\,n\right)$$

...Right-hand side

$$\frac{g}{(2\pi)^3}\int \hat{C}[f_a]\frac{d^3\vec{p}_a}{E_a} =$$

assumes T-invariance

+ bosons
- fermions

$$d\Pi_a \equiv g_a \frac{d^3\vec{p}_a}{2E_a(2\pi)^3}$$

a,b=WIMP
1,2=SM (light) particles

factor 1/2 to avoid double counting when we integrate over all momenta

$$-\int d\Pi_a d\Pi_b d\Pi_1 d\Pi_2 (2\pi)^4 \delta^{(4)}(p_a + p_b - p_1 - p_2)|\mathcal{M}|^2[f_a f_b(1 \pm f_1)(1 \pm f_2) - f_1 f_2(1 \pm f_a)(1 \pm f_b)]$$

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~ ok for non-relativistic particles (in absence of bose cond. or degeneracy)

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Thermal equilibrium &
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detailed balance (enforce E-conservation)

thermally averaged annihilation cross section

$$\langle \sigma v\rangle \equiv \frac{1}{n_{\rm eq}^2}\int d\Pi_a d\Pi_b d\Pi_1 d\Pi_2 |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(p_a+p_b-p_1-p_2) f_a^{\rm eq} f_b^{\rm eq}$$

Do we ever "need" full Boltzmann equation?

I mean, apart from microscopic formula to compute relevant cross-sections?

Depending on the DM candidate, retaining the full dependence from the momentum can be crucial. Notable example: **sterile neutrinos**

We saw that neutrinos "almost work" as DM candidate.

A better candidate would:

- contribute more to energy density
- be "colder"

↓

Add a ***more massive neutrino with weaker than weak interaction***
(decouples earlier/more "non-relativistic")

Preliminary: 1 slide on see-saw...

Add at least 1 SM singlet, mixing with at least 1 active ν, plus its Majorana mass term

$$\delta\mathcal{L} = \bar{N} i \partial_\mu \gamma^\mu N - \lambda_\ell H \bar{N} L^\ell - \frac{M}{2} \bar{N}^c N + h.c.$$

after EW breaking can write mass matrix for L,R components in the compact form

$$\begin{pmatrix} 0 & \lambda_\ell v \\ \lambda_\ell v & M \end{pmatrix}$$

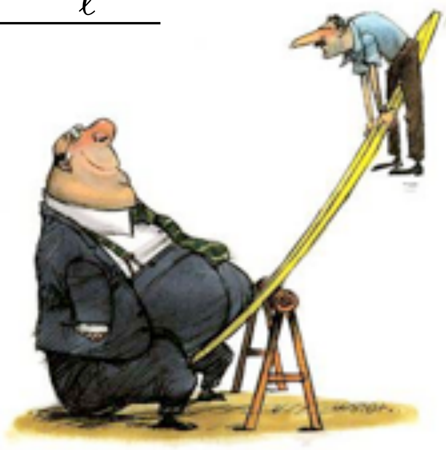
whose eigenvalues are

$$\mu_\pm = \frac{M \pm \sqrt{M^2 + 4\lambda_\ell^2 v^2}}{2}$$

If $M \gg \lambda_\ell v$ → **seesaw mechanism**

$$\mu_+ \simeq M$$

$$\mu_- \simeq -\frac{(\lambda_\ell v)^2}{M}$$



Dodelson-Widrow warm sterile neutrino

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In the previous framework, for a small mixing and keV masses, say

$$\theta \sim \lambda_\ell v/M \sim 10^{-5} \qquad M \sim 10\,\mathrm{keV}$$

The lightest active neutrino has sub-eV mass (Ok) and the "heavy" one is produced *via oscillations*, suppressed by the small mixing.

$$\left[\frac{\hat{C}}{E}\right] \sim \Gamma_{\rm int} \sim \Gamma_w \times \theta^2$$

Remarkable that parameters can be chosen "right"!

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$$\left(\frac{\partial}{\partial t} - HE\frac{\partial}{\partial E}\right) f_S(E,t) = \left[\frac{1}{2}\sin^2(2\theta_M(E,t))\;\Gamma(E,t)\right] f_A(E,t)$$

under some approx., one can compute the non-thermal spectrum analytically

$$\frac{f_S}{f_A} = \frac{7.7}{g_*^{1/2}}\left(\frac{\mu}{\text{eV}}\right)^2\left(\frac{\text{keV}}{M}\right)\; y\int_x^\infty \frac{dx'}{(1+y^2x'^2)^2}$$

$x \sim T^3/M$
$y \sim E/T$

Extra complications & features

- The mixing matrix gets modified in the medium (mixing in matter, see PP lectures).
- The spectrum can be “quasi-thermal” or relatively far from equilibrium one. ν_s’s are “relatively warmer” candidates, free-streaming length comparable with dwarf-Galaxies Jeans mass length: can suppress **non-linear** structures at sub-kpc scales
- With ν/anti-ν asymmetry, *resonant* production can happen (enhancement of lower-energy part) on their self-refraction potential. Corresponding DM “closer to cold DM”.

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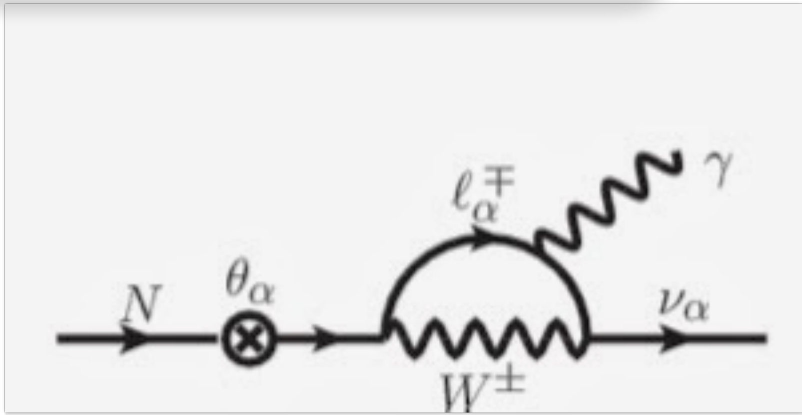
Some features:

- can be searched for via X-ray line (rare loop-suppressed decay)
- can be embedded in a "minimal extension" of the SM with 3 right-handed neutrinos (two GeV-ish ones explaining baryon asymmetry...)

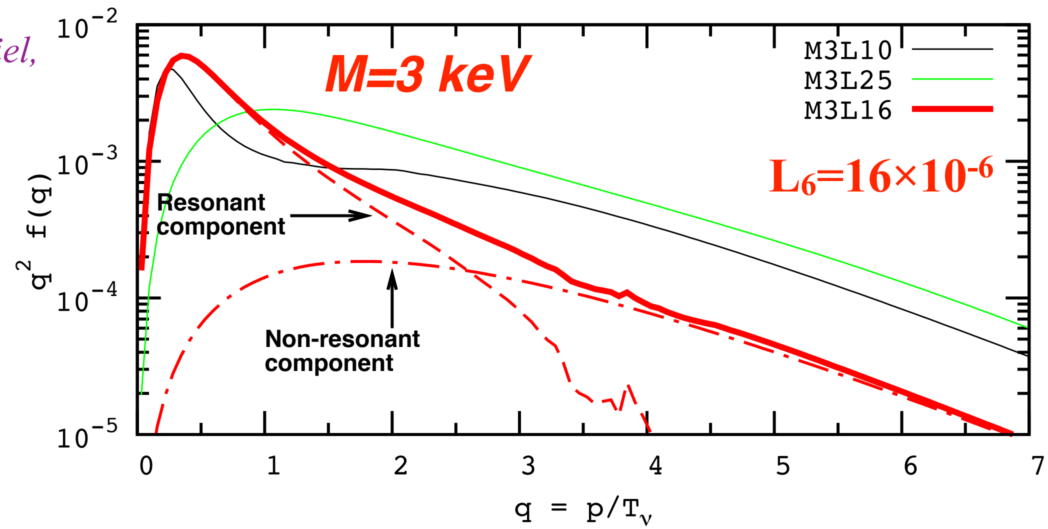
Note: no physics <u>above</u> the electroweak scale is required

for a review, A. Boyarsky, O. Ruchayskiy and M. Shaposhnikov, Ann. Rev. Nucl. Part. Sci. 59, 191 (2009)

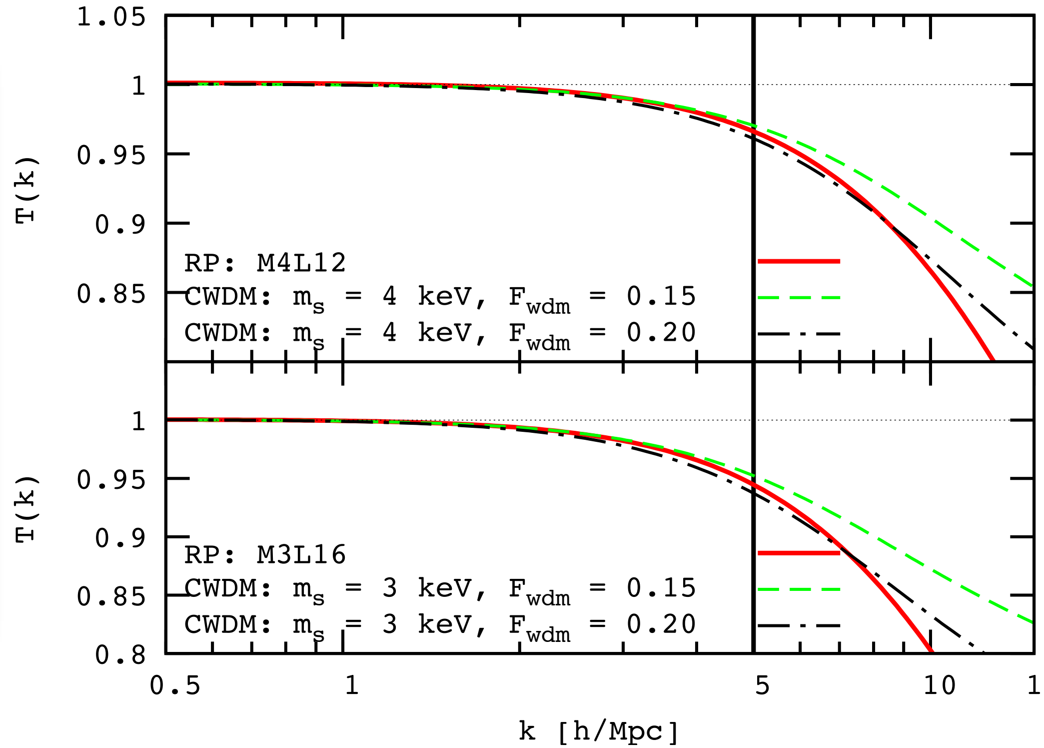


Those “details” matter!

A. Boyarsky, J. Lesgourgues, O. Ruchayskiy and M. Viel, “Realistic sterile neutrino dark matter with keV mass does not contradict cosmological bounds,” PRL 102, 201304 (2009) [arXiv:0812.3256].



Momentum distribution should be calculated for different choices of particle parameters (mixing, asymmetry, mass...)



The *momentum shape* influences the *spatial* power-spectrum, again computed numerically.

$$T = \sqrt{P_{\nu_s}(k)/P_{\Lambda CDM}(k)}$$

Main feature: cutoff beyond some k (“free-streaming” effect)

Free streaming length estimate

$$\lambda_{FS} = a(t) \int_{t_F}^{t} \frac{dt'}{a(t)} \sqrt{\langle v^2 \rangle}$$

Divide integral in pieces, with key times:

t_{NR}: time at which the particle becomes non relativistic, i.e. 3 $T_X \sim M_X$, before which v~1; after that, it scales as 1/a

t_{EQ}: time of matter-radiation equality, a(t) changes regime.

What comes first depends on the model details. If we assume **$t_{NR} < t_{EQ}$**

If I did not make mistakes:

$$\lambda_{FS}^{\mathrm{com}} = \frac{\lambda_{FS}}{a} = \frac{2\, c\, t_{NR}}{a_{NR}} \left[\frac{5}{2} + \log \frac{a_{\mathrm{EQ}}}{a_{NR}} \right]$$

or, numerically:

$$\lambda_{FS}^{\mathrm{com}} = \frac{\lambda_{FS}}{a} \simeq \mathrm{Mpc} \left(\frac{\mathrm{keV}}{m_\nu} \right)$$

But one has a “mix” of species, actual observable is *P(k) ... one needs to solve Boltzmann eq.*

Conclusions / 2

- We have introduced the Boltzmann equation to describe species evolution & DM production

- For most applications (including e.g. neutrino contribution to DM!) their integrated form is sufficient.

- This includes the popular WIMP class of candidates, rich in collider, direct and indirect signatures and thus extremely well studies.

- We saw at least one alternative to WIMP freeze-out: freeze-in

- In some cases, momentum-dependent equations are needed: case of sterile neutrino, which in many respects is one of the minimal scenarios to extend the SM while obtaining a DM candidate.