Gravitational electric-magnetic duality

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Gravitational duality is also thought to be relevant to the so-called problem of "hidden symmetries".

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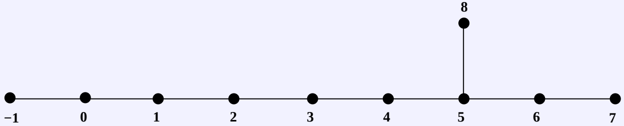

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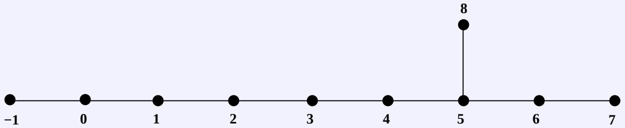

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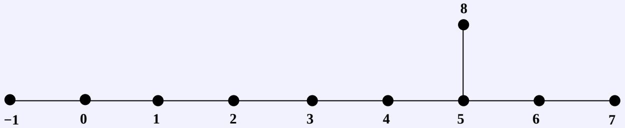

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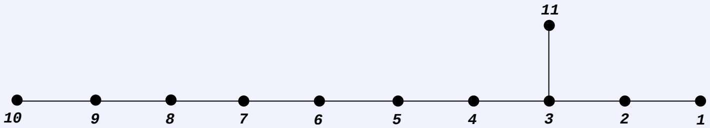

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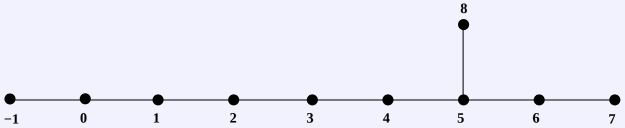

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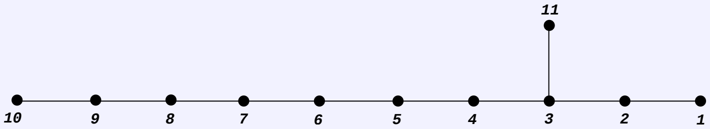

which contains electric-magnetic gravitational duality,

might be a “hidden symmetry" of maximal supergravity or of an appropriate extension of it.

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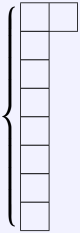

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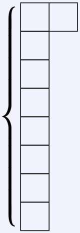

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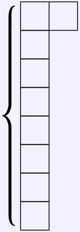

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Understanding gravitational duality is thus important in this context.

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- finally conclude and mention some open questions.

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\mathbf{E} &\rightarrow \cos\alpha\, \mathbf{E} + \sin\alpha\, \mathbf{B} \\
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are not symmetries of the theory and in particular are not symmetries of the equations of motion. There is no contradiction with Noether's theorem.

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Theorem : There is no variation of A_μ that yields the above duality transformations of the field strengths.

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Once A_μ is introduced, $dF = 0$ is an identity. But $dF' = 0$ does not hold off-shell (unless α is a multiple of π but then F and its dual are not mixed), where F' is the new field strength $\cos\alpha\, F - \sin\alpha\, {}^*F$ after duality rotation. Hence there is no A' such that $F' = dA'$.

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It follows from this theorem that **it is meaningless to ask whether the Maxwell action $S[A_\mu]$ is invariant under the above duality transformations of the field strengths since there is no variation of A_μ that yields these variations.**

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In the same way, it is meaningless to ask whether the Maxwell action is invariant under hyperbolic rotations (and conclude that it is, yielding an apparent contradiction with the non-invariance of the equations of motion), because the same argument indicates that there is no variation of the vector potential that gives the hyperbolic rotations off-shell for the field strength.

Explicit form of the duality transformation of the vector potential under em duality

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As we just argued, this is the best one can hope for.

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We shall choose the duality transformations of the vector potential such that these non-local terms are non-local only in space, where the inverse Laplacian $\triangle^{-1}$ can be given a meaning. Terms non local in time (and $\Box^{-1}$) are much more tricky. We also use the gauge ambiguity in the definition of δA_μ in such a way that $\delta A_0 = 0$.

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Similarly, $\delta E_i = -\delta \dot{A}_i = \epsilon B_i - \epsilon \triangle^{-1} (\epsilon_{ijk} \partial^j \partial_\mu F^{\mu k})$.

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It is therefore a genuine Noether symmetry (with Noether charge etc).

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If, besides the standard "magnetic" vector potential defined through,

$$\vec{B} \equiv \vec{B}_1 = \vec{\nabla} \times \vec{A}_1,$$

one introduces an additional vector potential $\vec{A}_2$ through,

$$\vec{E} \equiv \vec{B}_2 = \vec{\nabla} \times \vec{A}_2,$$

one may rewrite the standard Maxwell action in terms of the two potentials A^a as

$$S = \frac{1}{2} \int dx^0 d^3x \left(\epsilon_{ab} \vec{B}^a \cdot \dot{\vec{A}}^b - \delta_{ab} \vec{B}^a \cdot \vec{B}^b \right).$$

Here, ϵ_{ab} is given by $\epsilon_{ab} = -\epsilon_{ba}$, $\epsilon_{12} = +1$.

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To conclude : the "proof" using the standard form of the em duality transformations that the second order Maxwell action $S[A_\mu] = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}$ is not invariant under duality transformations is simply incorrect because it is based on a form of the duality transformations that is inconsistent with the existence of the dynamical variable A_μ. A consistent set of transformations leaves the action invariant.

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The action is invariant under rotations in the $(1,2)$ plane of the vector potentials ("electric-magnetic duality rotations") because ϵ_{ab} and δ_{ab} are invariant tensors.

The action is also invariant under the gauge transformations,

$$\vec{A}^a \longrightarrow \vec{A}^a + \vec{\nabla}\Lambda^a.$$

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Deser-Teitelboim 1976, Deser-Gomberoff-Henneaux-Teitelboim 1997

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The analysis can be extended to several abelian vector fields ($U(n)$-duality invariance), as well as to vector fields appropriately coupled to scalar fields ($Sp(n, \mathbb{R})$-duality invariance), with the same conclusions.

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In this formulation, Poincaré invariance is not manifest, however. More on this later.

Duality invariance of the Einstein equations

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The Riemann tensor

$$R_{\lambda\mu\rho\sigma} = -\frac{1}{2}\left(\partial_\lambda\partial_\rho h_{\mu\sigma} - \partial_\mu\partial_\rho h_{\lambda\sigma} - \partial_\lambda\partial_\sigma h_{\mu\rho} + \partial_\mu\partial_\sigma h_{\lambda\rho}\right)$$

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This transformation rotates the Schwarzschild mass into the Taub-NUT parameter N.

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Is this also a symmetry of the Pauli-Fierz action ?

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It is obtained by starting from the first-order (Hamiltonian) action and solving the constraints.

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Both are invariant under

$$\delta Z^a_{ij} = \partial_i \xi^a_j + \partial_j \xi^a_i + 2\epsilon^a \delta_{ij}$$

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$$H = \int d^3x \left(4 R^a_{ij} R^{bij} - \frac{3}{2} R^a R^b \right) \delta_{ab}.$$

Here, R^a_{ij} is the Ricci tensor constructed out of the prepotential Z^a_{ij}.

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We start with the p-form case.

Twisted self-duality for p-forms

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- The Maxwell action

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- The general solution to the equation of motion is $^*F = dB$ for some $(D-p-2)$-form B. One calls the original form A the "electric potential" and the form B the "magnetic potential".

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- Conversely, assume that one has a p-form A and a $(D-p-2)$-form B, the curvatures $F=dA$ and $H=dB$ of which are dual to one another,

$$^{*}F=H, \quad (-1)^{(p+1)(D-1)-1}\,{}^{*}H=F,$$

Twisted self-duality for p-forms

Gravitational electric-magnetic duality

Marc Henneaux

Introduction

Electromagnetism in $D = 4$

Gravitational duality in $D = 4$ (linearized gravity)

Twisted self-duality for p-forms

Twisted self-duality for (linearized) Gravity

Duality invariance and spacetime covariance

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where

$$\mathscr{F}=\begin{pmatrix}F\\H\end{pmatrix}, \quad \mathscr{S}=\begin{pmatrix}0 & (-1)^{(p+1)(D-1)-1}\\ 1 & 0\end{pmatrix}.$$

One refers to these equations as the twisted self-dual formulation of Maxwell's equations because $\mathscr{S}$ introduces a “twist".

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- The answer is positive (Henneaux-Teitelboim, Sen-Schwarz, Bunster-Henneaux).
- While the corresponding action is duality-symmetric, it lacks manifest space-time covariance (but of course it is covariant).

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with

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- A and B are not only duality conjugate, but also canonically conjugate.

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- Gauge invariance is manifest.

Twisted self-duality for gravity

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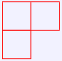
- For definiteness, consider $D = 5$. In that case, the “dual graviton" is described by a tensor $T_{\alpha\beta\gamma}$ of mixed symmetry type described by the Young tableau



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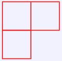

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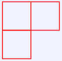

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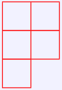

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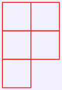

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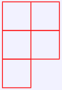

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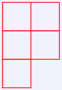

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$$E_{\alpha_1\alpha_2\beta} = 0$$

for the "Ricci tensor" $E_{\alpha_1\alpha_2\beta}$.

Twisted self-duality for gravity

The Einstein equations $R_{\mu\nu}=0$ for the Riemann tensor $R_{\mu\nu\alpha\beta}[h]$ imply that the dual Riemann tensor $E_{\beta_1\beta_2\beta_3\rho_1\rho_2}$, defined by

$$
\begin{aligned}
E_{\beta_1\beta_2\beta_3\rho_1\rho_2} &= \frac{1}{2!}\epsilon_{\beta_1\beta_2\beta_3\alpha_1\alpha_2}R^{\alpha_1\alpha_2}{}_{\rho_1\rho_2} \\
R_{\alpha_1\alpha_2\rho_1\rho_2} &= -\frac{1}{3!}\epsilon_{\alpha_1\alpha_2\beta_1\beta_2\beta_3}E^{\beta_1\beta_2\beta_3}{}_{\rho_1\rho_2}
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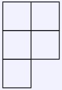 .

Here, $h_{\alpha\beta}$ is the spin-2 (Pauli-Fierz) field.

Twisted self-duality conditions for gravity

Furthermore, (i) the tensor $E_{\beta_1\beta_2\beta_3\rho_1\rho_2}$ obeys the differential identities $\partial_{[\beta_0}E_{\beta_1\beta_2\beta_3]\rho_1\rho_2}=0$, $E_{\beta_1\beta_2\beta_3[\rho_1\rho_2,\rho_3]}=0$ that guarantee the existence of a tensor $T_{\alpha\beta\mu}$ such that

$$E_{\beta_1\beta_2\beta_3\rho_1\rho_2}=E_{\beta_1\beta_2\beta_3\rho_1\rho_2}[T];$$

and (ii) the field equations for the dual tensor $T_{\alpha\beta\mu}$ are satisfied.

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- Let $h_{\mu\nu}$ and $T_{\alpha\beta\mu}$ be tensor fields of respective Young symmetry types $\yng(2)$ and $\yng(2,1)$, and let $R_{\alpha_1\alpha_2\rho_1\rho_2}[h]$ and $E_{\beta_1\beta_2\beta_3\rho_1\rho_2}[T]$ be the corresponding gauge-invariant curvatures.

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The “twisted self-duality conditions", which express that E is the dual of R (we drop indices)

$$R = - \,^*E, \quad E = \,^*R,$$

or, in matrix notations,

$$\mathfrak{R} = \mathscr{S}\,^*\mathfrak{R},$$

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imply that $h_{\mu\nu}$ and $T_{\alpha\beta\mu}$ are both solutions of the linearized Einstein equations and the Curtright equations,

$$R_{\mu\nu} = 0, \quad E_{\mu\nu\alpha} = 0.$$

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- One can define electric and magnetic fields for $h_{\alpha\beta}$ and $T_{\alpha\beta\gamma}$. The twisted self-duality conditions are equivalent to $\mathscr{B}_{ijrs}[T] = \mathscr{E}_{ijrs}[h]$, $\mathscr{B}_{ijr}[h] = -\mathscr{E}_{ijr}[T]$.

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- (iii) Insert the solution of the constraints back into the action.

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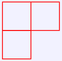
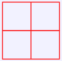
- The prepotential  is at the same time the prepotential for the Pauli-Fierz field h_{ij} and for the momentum π^{ijk} conjugate to the Curtright field, while the prepotential  is at the same time the prepotential for the Curtright field T_{ijk} and for the momentum π^{ij} conjugate to the Pauli-Fierz field.
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- The details can be found in Bunster-Henneaux-Hörtner 2013.

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THANK YOU !