Entanglement in Quantum Field Theory

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University of Pisa, November 2013

Outline

- Quantum entanglement in general and its quantification
- Path integral approach
- Entanglement entropy in 1+1-dimensional CFT
- Higher dimensions
- Mixed states and negativity

Work largely carried out with Pasquale Calabrese (Pisa) and Erik Tonni (Trieste)

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SPECIAL ISSUE: ENTANGLEMENT ENTROPY IN EXTENDED QUANTUM SYSTEMS

INTRODUCTION

REVIEWS

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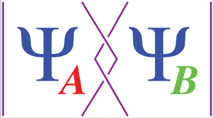
Special issue

Entanglement entropy in extended quantum systems

Guest Editors: Pasquale Calabrese, John Cardy and Benjamin Doyon



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Quantum Entanglement (Bipartite, Pure State)

- quantum system in a pure state $|\Psi\rangle$, density matrix $\rho = |\Psi\rangle\langle\Psi|$
- $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Alice can make unitary transformations and measurements only in A, Bob only in the complement B
- in general Alice's measurements are entangled with those of Bob
- example: two spin-$\frac{1}{2}$ degrees of freedom

$$|\psi\rangle = \cos\theta \,|\uparrow\rangle_A |\downarrow\rangle_B + \sin\theta \,|\downarrow\rangle_A |\uparrow\rangle_B$$

Measuring bipartite entanglement in pure states

- Schmidt decomposition:

$$|\Psi\rangle = \sum_j c_j |\psi_j\rangle_A \otimes |\psi_j\rangle_B$$

with $c_j \geq 0$, $\sum_j c_j^2 = 1$.

- one quantifier of the amount of entanglement is the entropy

$$S_A \equiv -\sum_j |c_j|^2 \log |c_j|^2 = S_B$$

- if $c_1 = 1$, rest zero, $S = 0$ and $|\Psi\rangle$ is unentangled
- if all c_j equal, $S \sim \log \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$ – maximal entanglement

- equivalently, in terms of Alice's reduced density matrix:

$$\rho_A \equiv \mathrm{Tr}_B \, |\Psi\rangle\langle\Psi|$$

$$S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A = S_B$$

- the von Neumann entropy: similar information is contained in the Rényi entropies

$$S_A{}^{(n)} = (1-n)^{-1} \log \mathrm{Tr}_A \, \rho_A{}^n$$

- $S_A = \lim_{n \to 1} S_A{}^{(n)}$

- other measures of entanglement exist, but entropy has several nice properties: additivity, convexity, . . .

- it increases under Local Operations and Classical Communication (LOCC)

- it gives the amount of classical information required to specify ρ_A (important for numerical computations)
- it gives a basis-independent way of identifying and characterising quantum phase transitions
- in a relativistic QFT the entanglement in the vacuum encodes all the data of the theory (spectrum, anomalous dimensions, $\ldots$)

Entanglement entropy in QFT

In this talk we consider the case when:

- the degrees of freedom are those of a local relativistic QFT in large region $\mathcal{R}$ in $\mathbb{R}^d$
- the whole system is in the vacuum state $|0\rangle$
- A is the set of degrees of freedom in some large (compact) subset of $\mathcal{R}$, so we can decompose the Hilbert space as

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

- in fact this makes sense only in a cut-off QFT (e.g. a lattice), and some of the results will in fact be cut-off dependent

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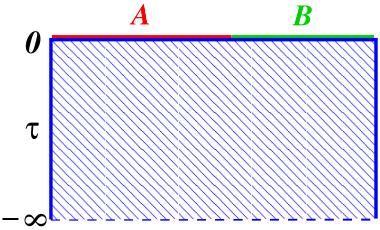
- in fact this makes sense only in a cut-off QFT (e.g. a lattice), and some of the results will in fact be cut-off dependent
- How does S_A depend on the size and geometry of A and the universal data of the QFT?

Rényi entropies from the path integral ($d = 1$)



- wave functional $\Psi(\{a\}, \{b\})$ is proportional to the conditioned path integral in imaginary time from $\tau = -\infty$ to $\tau = 0$:

$$\Psi(\{a\}, \{b\}) = Z_1^{-1/2} \int_{a(0)=a, b(0)=b} [da(\tau)][db(\tau)]\, e^{-(1/\hbar) S[\{a(\tau)\}, \{b(\tau)\}]}$$

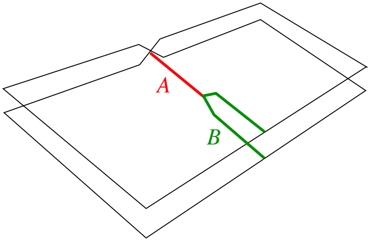
where $S = \int_{-\infty}^{0} L\big(a(\tau), b(\tau)\big)\, d\tau$

- similarly $\Psi^*(\{a\}, \{b\})$ is given by the path integral from $\tau = 0$ to $+\infty$

Example: $n = 2$

$$\rho_A(a_1, a_2) = \int db\, \Psi(a_1, b)\Psi^*(a_2, b)$$

$$\mathrm{Tr}_A\, \rho_A^2 = \int da_1 da_2 db_1 db_2\, \Psi(a_1, b_1)\Psi^*(a_2, b_1)\Psi(a_2, b_2)\Psi^*(a_1, b_2)$$



$$\mathrm{Tr}_A\, {\rho_A}^2 = Z(\mathcal{R}_2)/Z_1^2$$

where $Z(\mathcal{R}_2)$ is the euclidean path integral (partition function) on an 2-sheeted conifold $\mathcal{R}_2$

- in general

$$\mathrm{Tr}_A\, {\rho_A}^n = Z(\mathcal{R}_n)/Z_1^n$$

where the half-spaces are connected as

to form $\mathcal{R}_n$.

- conical singularity of opening angle $2\pi n$ at the boundary of A and B on $\tau = 0$

- if space is 1d and A is an interval (u, v) (and B is the complement) then $Z(\mathcal{R}_n)$ can be thought of as the the correlation function of twist operators:

$$Z(\mathcal{R}_n)/Z_1^n = \langle \mathcal{T}(u)\mathcal{T}(v)\rangle$$

- we can either think of this as the partition function on a conifold, or consider n copies of the QFT (with fields ϕ_j) on a single-sheeted surface with $\mathcal{T}$ acting as orbifold points in the target space: on taking z in a closed contour around u

$$\phi_j(z)\mathcal{T}(u) \to \phi_{j+1}(z)\mathcal{T}(u) \qquad (\text{mod } n)$$

- these have similar properties to other local operators e.g.
- in a massless QFT (a CFT)

$$\langle\mathcal{T}(u)\mathcal{T}(v)\rangle \sim |u-v|^{-2\Delta_n}$$

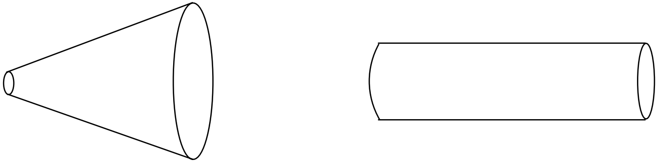
- in a massive QFT,

$$\langle\mathcal{T}\rangle \sim m^{\Delta_n} \quad \text{and} \quad \langle\mathcal{T}(u)\mathcal{T}(v)\rangle - \langle\mathcal{T}\rangle^2 \sim \mathrm{e}^{-2m|u-v|}$$

- main result for $d = 1$

$$\Delta_n = (c/12)(n - 1/n)$$

where c is the central charge of the UV CFT



- consider a single cone of radius R and opening angle α
- $w = \log z$ maps this into a cylinder of length $\log R$ and circumference α

$$\frac{Z_{\rm cone}(2\pi n)}{Z_{\rm cone}(2\pi)^n} = \frac{Z_{\rm cyl}(2\pi n)}{Z_{\rm cyl}(2\pi)^n} \sim \frac{e^{\pi c \log R/12\pi n}}{(e^{\pi c \log R/12\pi})^n} \sim R^{-\Delta_n}$$

- from this we see for example that for a single interval A of length ℓ [Holzhey, Larsen, Wilczek 1994]

$$S_A \sim -\left.\frac{\partial}{\partial n}\right|_{n=1} \ell^{-2\Delta_n} = (c/3)\log(\ell/\epsilon)$$

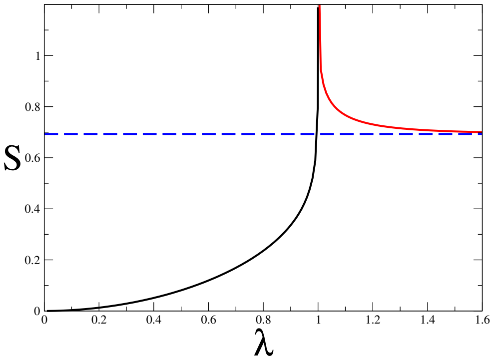
- note this is much less than the entanglement in a *typical* state which is $O(\ell)$
- many more universal results, eg finite-temperature cross-over between entanglement and thermodynamic entropy ($\beta = 1/k_B T$):

$$\begin{aligned} S_A &= (c/3)\log\left((\beta/\pi)\sinh(\pi\ell/\beta)\right) \\ &\sim (c/3)\log\ell \qquad \text{for } \ell \ll \beta \\ &\sim \pi c\ell/3\beta \qquad \text{for } \ell \gg \beta \end{aligned}$$

Massive QFT in 1+1 dimensions

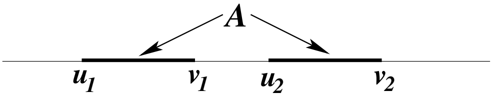
- for 2 intervals $A = (-\infty, 0)$ and $B = (0, \infty)$

$$S_A \sim (c/6)\,\log(1/m \cdot \epsilon)$$



1+1-dimensional Ising model, $m \propto |\lambda - 1|$

Two intervals



- in general there is no simple result since $Z(\mathcal{R}_n)$ depends on the moduli of the conifold
  but we can use an operator product expansion

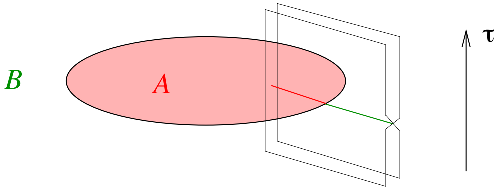
$$\mathcal{T}(u)\mathcal{T}(v) = \sum_{\{k_j\}} C_{\{k_j\}}(u-v)\prod_{j=1}^{n}\Phi_{k_j}\left(\tfrac{1}{2}(u+v)_j\right)$$

  in terms of a complete set of local operators Φ_{k_j}
- this gives the Rényi entropies as an expansion in powers

$$\sum_{\{k_j\}} C^2_{\{k_j\}}\eta^{\sum_j \Delta_{k_j}} \quad \text{where} \quad \eta = \frac{(u_1-v_1)(u_2-v_2)}{(u_1-u_2)(v_1-v_2)}$$

- the $C_{\{k_j\}}$ encode all the data of the CFT

Higher dimensions $d > 1$



- the conifold $\mathcal{R}_n$ is now $\{2d \text{ conifold}\} \times \{\text{boundary } \partial A\}$

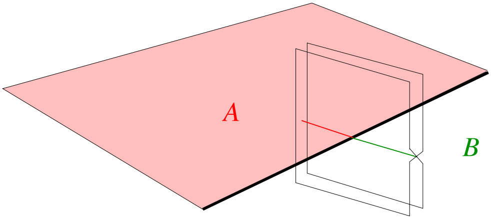
$$\log Z(\mathcal{R}_n) \sim \text{Vol}(\partial A) \cdot \epsilon^{-(d-1)}$$

- this is the 'area law' in 3+1 dimensions [Srednicki 1992]
- coefficient is non-universal
- for even $d+1$ there are interesting corrections

$$\text{Vol}(\partial A)\, m^2 \log(m\epsilon)\,, \quad \log(R_A/\epsilon)$$

whose coefficients are related to curvature anomalies of the CFT and are universal

- e.g. in 3+1 dimensions, if A is the interior of a sphere S^2, we can make a conformal mapping so the boundary of A becomes $\mathbb{R}^2$

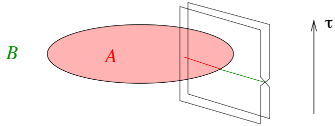


- in cylindrical coordinates (ρ, θ, x_3, x_4)

$$\langle T_{\rho\rho} \rangle \propto \frac{(1 - 1/n^4)a}{\rho^4} \qquad \text{`}a\text{-anomaly'}$$

$$\epsilon(\partial/\partial\epsilon) \log Z(\mathcal{R}_n) = n \int \langle T_{\rho\rho} \rangle \rho d\rho d\theta dx_3 dx_4 \sim \epsilon^{-2} \times \mathrm{Area}(\partial A)$$

- but when we map back to the sphere



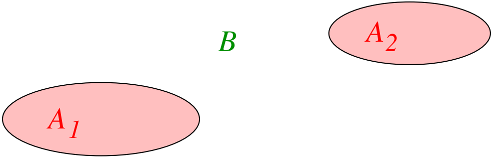
$$\langle T_{\rho\rho}\rangle \propto a\,(1-1/n^4)\left(\frac{1}{\rho^4}+\frac{1}{R_A^2\rho^2}+\cdots\right)$$

$$\epsilon(\partial/\partial\epsilon)\log Z(\mathcal{R}_n) \sim \epsilon^{-2}\times(4\pi R_A^2) + \text{universal } O(1)$$

$$S_A^{(n)} \sim \epsilon^{-2}\mathrm{Area}(\partial A) + \#\,a\,(n-1/n^3)\log(R_A/\epsilon)$$

[Casini/Huerta, Fursaev/Soludukhin,…]

Mutual Information of multiple regions



- the non-universal ‘area’ terms cancel in

$$I^{(n)}(A_1, A_2) = S^{(n)}_{A_1} + S^{(n)}_{A_2} - S^{(n)}_{A_1 \cup A_2}$$

- this mutual Rényi information is expected to be universal depending only on the geometry and the data of the CFT
- e.g. for a free scalar field in 3+1 dimensions [JC 2013]

$$I^{(n)}(A_1, A_2) \sim \frac{n^4 - 1}{15 n^3 (n-1)} \left(\frac{R_1 R_2}{r_{12}^2} \right)^2$$

Negativity

- however, mutual information does not correctly capture the quantum entanglement between A_1 and A_2, e.g. it also includes classical correlations at finite temperature
- more generally we want a way of quantifying entanglement in a mixed state $\rho_{A_1 \cup A_2}$
- one computable measure is negativity [Vidal, Werner 2002]
- let $\rho^{T_2}_{A_1 \cup A_2}$ be the *partial* transpose:

$$\rho^{T_2}_{A_1 \cup A_2}(a_1, a_2; a_1', a_2') = \rho_{A_1 \cup A_2}(a_1, a_2'; a_1', a_2)$$

- $\mathrm{Tr}\, \rho^{T_2}_{A_1 \cup A_2} = 1$, but it may now have negative eigenvalues λ_k

$$\text{Log-negativity} \qquad \mathcal{N} = \log \mathrm{Tr} \left| \rho^{T_2}_{A_1 \cup A_2} \right| = \log \sum_k |\lambda_k|$$

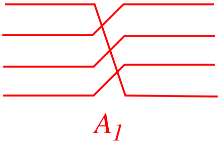
- if this is > 0 there are negative eigenvalues. This is an entanglement measure with nice properties, including increasing under LOCC – for a pure state $\mathcal{N} = S^{(1/2)}_{A_1}$

Negativity in QFT

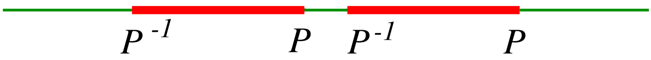
- 'replica trick'

$$\mathrm{Tr}\,(\rho_{A_1\cup A_2}^{T_2})^n = \sum_k \lambda_k^n = \sum_k |\lambda_k|^n \quad \text{if } n \text{ is even}$$

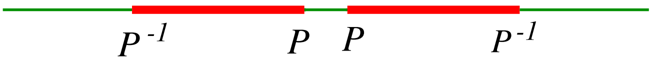
- analytically continue to $n = 1$ to get $\sum_k |\lambda_k|$ (!!)
- we can compute $\mathrm{Tr}\,(\rho_{A_1\cup A_2}^{T_2})^n$ by connecting the half-spaces in the opposite order along A_2:



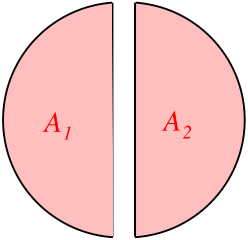
- going around each conical singularity corresponds to a cyclic permutation P_n or P_n^{-1} of the n sheets:
- for $\rho_{A_1 \cup A_2}$



- for $\rho^{T_2}_{A_1 \cup A_2}$



$$
\begin{aligned}
P_n^2 &\cong P_n \quad (n \text{ odd}) \to Id \quad \text{for } n \to 1 \\
&\cong P_{n/2} \otimes P_{n/2} \quad (n \text{ even}) \to P_{1/2} \otimes P_{1/2} \quad \text{for } n \to 1
\end{aligned}
$$



- so for example for $d > 1$ for 2 large regions a finite distance apart

  $\mathcal{N}(A_1, A_2) \propto$ Area of common boundary between A_1 and A_2

- $\mathcal{N}$ appears to decay exponentially with separation of the regions, even in a CFT

Other Related Stuff

- ‘quantum quenches’ where the system is prepared in a state $|\psi\rangle$ which is not an eigenstate of hamiltonian: how does entanglement (and correlation functions) behave?
- topological phases in 2 (and higher) spatial dimensions - entanglement entropy distinguishes these in absence of local order parameter [Kitaev/Preskill and many others]
- ’entanglement spectrum’ of the eigenvalues of $\log\rho_A$ [Haldane]
- entanglement in random states [Nadal/Majumdar]
- holographic computation of entanglement using AdS/CFT [Ryu/Takayanagi and many others]
- entanglement and RG flows [Casini/Huerta, Soludukhin,. . .]