The Superconformal Bootstrap Program

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Based on work with
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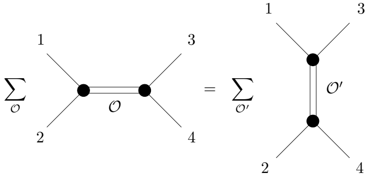
In recent years, explosion of results for SuperConformal Field Theories in $d > 2$.

- A huge list of new models, mostly with no Lagrangian description.
- A hodgepodge of techniques (localization, large N integrability, AdS/CFT). Powerful but with limitations.

Time is ripe for a more systematic approach.
Bootstrap philosophy: abstract operator algebra, obeying general consistency requirements from symmetries, unitarity and crossing.



Basic Framework

Viewpoint: A general Conformal Field Theory hasn't much to do with "fields" (of the kind you write in a Lagrangian).
We'll think more abstractly. A CFT is defined by a set of local operators,

$$\{\mathcal{O}_k(x)\},$$

and by their correlation functions

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle .$$

Local operators can be multiplied. Operator product expansion,

$$\text{OPE}: \quad \mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k c_{12k}(x)\mathcal{O}_k(0) .$$

This is a true operator equation. The sum converges.
The identity operator $\mathbf{1}$ and a (unique) stress tensor $T_{\mu\nu}$ are part of $\{\mathcal{O}_k(x)\}$.

Note: this definition does not capture non-local observables (*e.g.* Wilson loops) or constraints from non-trivial geometries.

Local operators $\mathcal{O}_{\Delta,\ell,f}$ are labeled by a conformal dimension Δ,

$$\mathcal{O}_{\Delta,\ell,f}(\lambda x) = \lambda^{-\Delta}\mathcal{O}_{\Delta,\ell,f}(x)\,,$$

a Lorentz representation ℓ and possibly a flavor quantum number f.

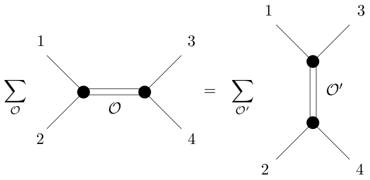
The CFT data $\{(\Delta_i, \ell_i, f_i)\,, c_{ijk}\}$ completely specify the theory. All correlators can be computed by taking successive operator products till $\langle \mathbf{1} \rangle \equiv 1$.

In principle, the classification and construction of CFTs is reduced to a very contrained algebraic problem. Consistent CFT data are very rigid!

Famous success story in $d = 2$, where the conformal group is the infinite dimensional group of holomorphic maps $z \to f(z)$, and many models have been exactly solved. (But still very far from a complete classification).

The Bootstrap

Old idea (Polyakov, ...): use internal consistency conditions to fix the CFT data. Taking operator products in different orders must give the same result. In 4pt,



Taking the four external operators to be identical scalars φ,

$$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\varphi} x_{34}^{2\Delta_\varphi}} \sum_{\mathcal{O}} (C_{\varphi\varphi\mathcal{O}})^2 G_{\mathcal{O}}(u,v)\,,$$

where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ $(x_{ij} \equiv x_i - x_j)$ are conformal invariant cross-ratios.

- The sum is over primary operators only, which obey $[K_\mu, \mathcal{O}_{primary}(0)] = 0$.
- The conformal block $G_{\mathcal{O}}(u,v)$ encodes the contribution of the primary $\mathcal{O}$ and of its whole tower of descendants $\{\partial^n \mathcal{O}\}$ and is completely fixed by kinematics.

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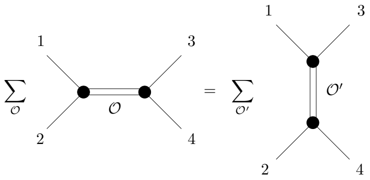

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The modern numerical bootstrap

Crossing symmetry sum rule

$$\sum_{(\Delta,\ell)\neq(0,0)} a_{\Delta,\ell} \underbrace{\left[v^{\Delta_\varphi} G_\Delta^{(\ell)}(u,v) - u^{\Delta_\varphi} G_\Delta^{(\ell)}(v,u)\right]}_{F_\Delta^{(\ell)}(u,v)} = \underbrace{u^{\Delta_\varphi} - v^{\Delta_\varphi}}_{I(u,v)} \, .$$

Unitarity ($d = 4$): $a_{\Delta,\ell} \equiv (C_{\varphi\varphi\mathcal{O}_\Delta^{(\ell)}})^2 \geqslant 0$, $\Delta \geqslant \ell + 2$ for $\ell \neq 0$, $\Delta \geqslant 1$ for $\ell = 0$.

(Rattazzi Rychkov Tonni Vichi) :
use this equation to constrain the space of CFT data.

For example, consider a trial spectrum with $\Delta \geqslant \bar{\Delta}_\ell$ for operators of spin ℓ.
If there exists a linear functional χ such that

$$\chi \cdot F_\Delta^{(\ell)}(u,v) \geqslant 0 \quad \text{when } \Delta \geqslant \bar{\Delta}_\ell$$

$$\chi \cdot I(u,v) < 0$$

that trial spectrum is ruled out.

Applying linear programming methods one can systematically carve out whole regions of the putative CFT spectrum. Surprisingly powerful!

[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi, PRD **86**, 025022]

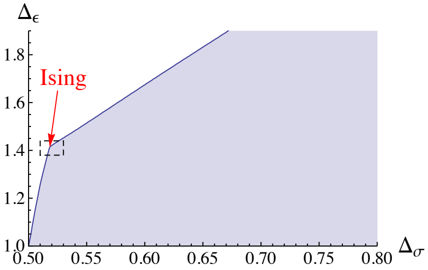


Figure : Exclusion plot in the subspace of $d = 3$ CFT data $(\Delta_\sigma, \Delta_\epsilon)$.

Remarkably, interesting theories sit at special places of the exclusion plots. Why? When do theories saturate bounds? When do they sit at kinks?

Increasing evidence that bounds are “real”, *i.e.* correspond to actual crossing symmetric, unitary $4pt$ functions.

Two sorts of questions

What is the space of consistent SCFTs in various dimensions?

- 32 Qs: plausibly, complete catalogues in $d = 3$, $d = 4$ and $d = 6$.
- 16 Qs: proposed catalogue in in $d = 6$, beginning of a classification scheme in $d = 4$ (class S, $\ldots$)
- 8 Qs: wide open.
  E.g. Conjectural landscape of AdS_4 string vacua $\leftrightarrow$ $d = 3$ SCFTs.

Can we bootstrap concrete models of special interest?

The bootstrap should be particularly powerful for models that are uniquely cornered by a few discrete data.
It is the only method presently available for finite N, non-Lagrangian theories, such as the $6d$ (2,0) theory.

Do the conformal bootstrap equations in dimension $d > 2$ admit a solvable truncation in the case of superconformal field theories?

A priori, there are two primary scenarios in which the constraints of crossing symmetry are nontrivial, yet solvable:

(I) *Meromorphic (and rational) conformal field theories in $d = 2$*
(II) *Topological quantum field theories.*

(I) is realized in $\mathcal{N} \geqslant 2$ theories in $d = 4$ and in $(2, 0)$ theories in $d = 6$.
This will be our focus.
(II) is realized in $\mathcal{N} \geqslant 4$ theories in $d = 3$.

The Superconformal Bootstrap Program

The bootstrap of $d = 4$, $\mathcal{N} \geqslant 2$ SCFTs can be organized into two steps:

1. The bootstrap for a protected subsector of BPS operators ("minibootstrap")
2. The full-fledged bootstrap for generic operators.

Indeed, crossing-symmetry constraints for a BPS 4pt function neatly split into

1. Equations that describe intermediate BPS operators.
   They can be solved analytically.
2. Equations that describe intermediate non-BPS operators.
   They can be analyzed numerically.

Step (1) serves as essential input for Step (2).
Step (1) is captured by carving out a $2d$ chiral algebra inside the $4d$ SCFT.
(Infinite dimensional Virasoro and W-symmetries!).

In this talk, we'll focus on $\mathcal{N} = 4$ SCFTs, and mostly on Step (2).

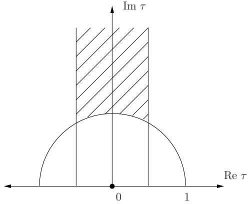
The $\mathcal{N} = 4$ landscape: old fashioned Lagrangian QFT

Unique $\mathcal{N} = 4$ multiplet with spins $\leqslant 1$: (A_μ, λ_A, X_i), $A = 1, \ldots 4$, $i = 1, \ldots 6$.

All fields must be in the adjoint representation of a gauge group G.
$SU(4)$ R-symmetry, but no flavor symmetry.

If $G = G_1 \times G_2$, theory factorizes, so we can take G to be $U(1)$ (free theory!) or one of the simple compact Lie groups.

Unique Lagrangian with complexified gauge coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ that does not run.
Conjecture (S-duality): $SL(2, \mathbb{Z})$ transformations of τ are an exact symmetry.



Local operators $\{\mathcal{O}_i\}$ identified at small g with gauge-invariant composites: $\mathrm{Tr}\, X_i X_j$, $\mathrm{Tr}\, F^2$, etc.

Much progress in the $N \to \infty$ limit of $SU(N)$ theory (with $\lambda \equiv g^2 N$ fixed) from integrability and AdS/CFT.

Virtually nothing known about non-susy observables at finite N.

The $\mathcal{N} = 4$ landscape: abstract CFT

Natural conjecture: no exotics!
The only $\mathcal{N} = 4$ SCFTs are the $\mathcal{N} = 4$ Yang-Mills theories.

Compatible with simple facts from $\mathcal{N} = 4$ representation theory:

- Stress tensor $T_{\mu\nu}$ belongs to a short multiplet whose bottom component is $\mathcal{O}_{\mathbf{20'}}$, a scalar operator of $\Delta = 2$ in the $\mathbf{20'}$ irrep of $SU(4)_R$.
  $T_{\mu\nu} = Q^2 \tilde{Q}^2 \mathcal{O}_{\mathbf{20'}}$

  The same multiplet contains as top component a complex scalar $\mathcal{O}_\tau = Q^4 \mathcal{O}_{\mathbf{20'}}$, which generates exactly marginal deformations.

  Viceversa, any exactly marginal operator that preserves $\mathcal{N} = 4$ susy must be the top component of the $\mathbf{20'}$ multiplet.

  One stress tensor $\leftrightarrow$ one-dimensional conformal manifold, as in $\mathcal{N} = 4$ SYM.

- Flavor symmetries completely forbidden (no room in any supermultiplet for a conserved current, except for the $SU(4)$ R-symmetry).

- Conformal anomalies $a \equiv c$ (Ward identities).
  In Lagrangian SYM, $a = c = \frac{\dim G}{4}$.

The $\mathcal{N} = 4$ superconformal bootstrap Beem, L.R., van Rees

Natural to start from the universal 4pt function of the stress tensor multiplet,

$$\langle \mathcal{O}^{I_1}_{\mathbf{20'}}(x_1)\mathcal{O}^{I_2}_{\mathbf{20'}}(x_2)\mathcal{O}^{I_3}_{\mathbf{20'}}(x_3)\mathcal{O}^{I_4}_{\mathbf{20'}}(x_4)\rangle = \frac{A^{I_1 I_2 I_3 I_4}(u,v)}{x_{12}^4 x_{34}^4} \ .$$

$\mathbf{20'} \times \mathbf{20'} = \mathbf{1} + \mathbf{15} + \mathbf{20'} + \mathbf{84} + \mathbf{105} + \mathbf{175}$: a priori six functions of u and v, but susy Ward identities allow to reduce them to:

1. two meromorphic protected functions $f_1(z)$, $f_2(z)$,
2. one unprotected function $\mathcal{G}(u,v)$. Here $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$.

Eden Petkou Schubert Sokatchev, Dolan Osborn, ...

Remarkably, crossing symmetry implies:

1. a set of equations involving f_1 and f_2 only – these are the bootstrap equations of the chiral algebra. There is unique family of solutions parametrized by the central charge a. Plugging back f_i, one derives
2. a single crossing symmetry equation for the unprotected part

$$\sum_{\Delta,\ell} a_{\Delta,\ell} F_{\Delta,\ell}(u,v) = F^{\text{short}}(u,v;a) \ ,$$

where $F^{\text{short}}(u,v;a)$ is a complicated but completely known function. The sum is over the intermediate unprotected superconformal primaries, which are constrained by Ward identities to be $SU(4)_R$ singlets. $\ell = 0, 2, 4, \ldots$ is the spin, $\Delta \geqslant \ell + 2$ the conformal dimension.

Formally very similar to the basic bootstrap sum rule for identical scalar operators, with $F^{\text{short}}(u,v;a)$ replacing $I(u,v)$ (contribution of the identity). Rattazzi Rychkov Tonni Vichi

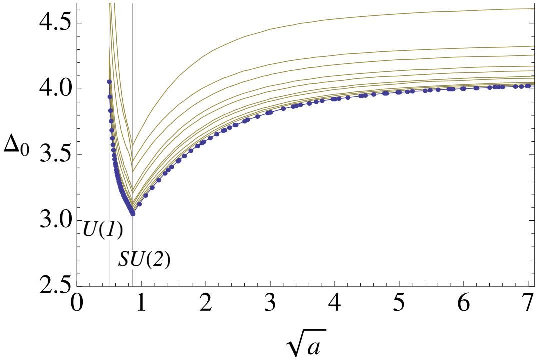


Figure : Bounds for the scaling dimension of the leading twist unprotected operator of spin zero. The bounds are displayed as a function of the (square root of the) central charge a. The best bound is shown in blue.

Note: kink at $a = 3/4$ is part of the input.
$F^{\text{short}}(u, v; a)$ has non-analytic behavior (continuous but not differentiable) at $a = 3/4$. For $a < 3/4$ unitarity forces the introduction of higher spin currents.

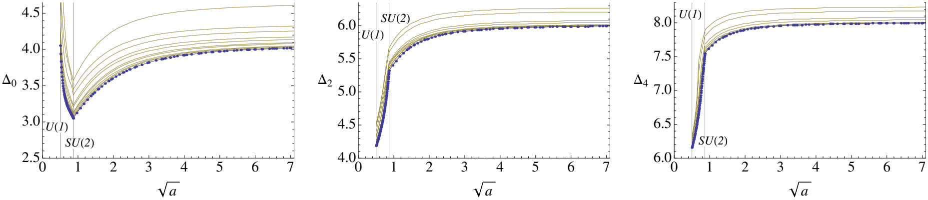


Figure : Bounds for the scaling dimension of the leading twist unprotected operator of spin $\ell = 0, 2, 4$. The bounds are displayed as a function of the (square root of the) central charge a. The best bound is shown in blue.

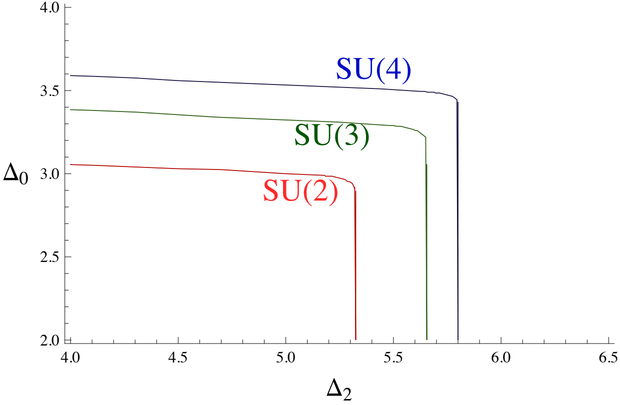


Figure : Exclusion plots in the space of spin zero and spin two leading twist gaps Δ_0 and Δ_2, for central charges $a = 3/4$, $a = 2$ and $a = 15/4$, corresponding to $\mathcal{N} = 4$ SYM with gauge groups $SU(2)$, $SU(3)$ and $SU(4)$ respectively.

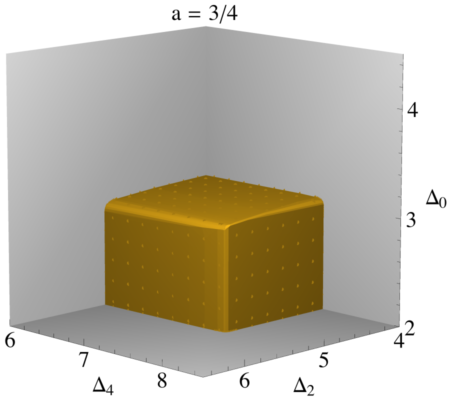


Figure : Exclusion plot in the space of leading twist gaps Δ_0, Δ_2, and Δ_4. The central charge $a = 3/4$ corresponding to $\mathcal{N} = 4$ SYM with gauge group $SU(2)$. The region outside of the "cube" is excluded.

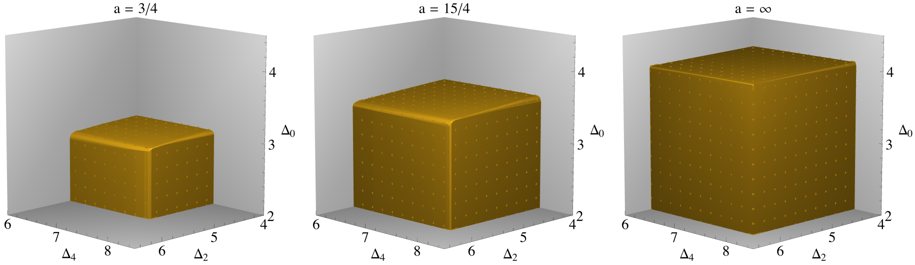


Figure : Exclusion plots in the space of leading twist gaps Δ_0, Δ_2, and Δ_4. The central charge $a = 3/4$, $a = 15/4$ and $a = \infty$ are shown, corresponding to $\mathcal{N} = 4$ SYM with gauge group $SU(2)$, $SU(4)$ and $SU(\infty)$.

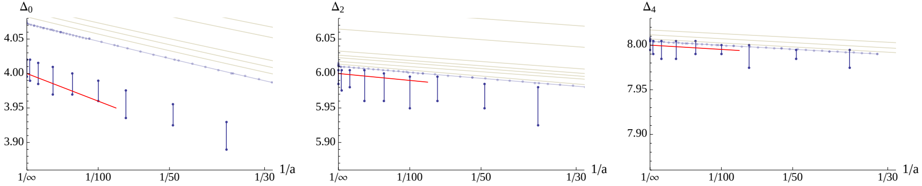
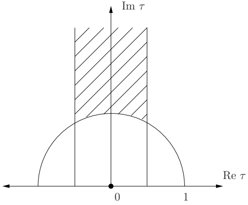


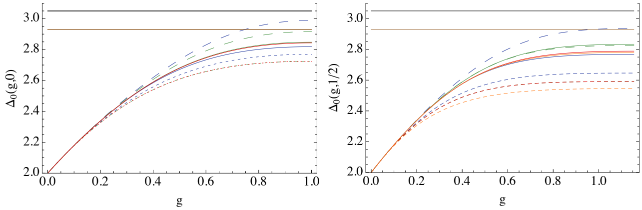
Figure : Estimates for twist gap Δ_ℓ for $\ell = 0, 2, 4$ that characterize the corners of the exclusion "cubes" at large central charge. Uncertainty is due to the smoothing of the cube. Superimposed in red are the results for planar $\mathcal{N} = 4$ SYM in the limit of infinite 't Hooft coupling; $\Delta_0 \approx 4 - \frac{4}{a}$, $\Delta_2 \approx 6 - \frac{1}{a}$, and $\Delta_4 \approx 8 - \frac{12}{25a}$.

For large a, the bounds appear to be saturated by $AdS_5 \times S^5$ supergravity.

Conjecture: the bounds are saturated also for finite a, on some point of the conformal manifold. Which one?
The cubic exclusion plots suggest simultaneous maximization of Δ_ℓ. This can occur naturally at either of the orbifold points:
$\tau_2 \equiv i$, fixed by $\tau \to -1/\tau$;
$\tau_3 \equiv e^{i\pi/3}$, fixed by $\tau \to (\tau-1)/\tau$.



We tested this idea by "S-duality invariant" resummation of perturbative results. (C. Beem, L.R., B. van Rees, A. Sen)



$\theta = \mathbf{0}$ $\theta = \pi$

Resummation results for Konishi for $SU(2)$ gauge group.Small dash: 2 loops. Large dash: 3 loops. Continuous: 4 loops. Different colors: different schemes.

Meromorphic correlators in $d = 4$, $\mathcal{N} = 2$ SCFTs

Fix a plane $\mathbb{R}^2 \subset \mathbb{R}^4$, parametrized by complex coordinates $(z, \bar{z})$.

Claim : Any $\mathcal{N} = 2$ SCFT contains a subsector $\mathcal{A}_\chi = \{\mathcal{O}_i(z_i, \bar{z}_i)\}$ of protected local operators, with meromorphic correlation functions,

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\, \mathcal{O}_2(z_2, \bar{z}_2) \ldots \mathcal{O}_n(z_n, \bar{z}_n) \rangle = R(z_i)\,.$$

Rationale: $\mathcal{A}_\chi$ is defined by the cohomology of a nilpotent $\mathbb{Q}$, of the form

$$\mathbb{Q} = \mathcal{Q} + \mathcal{S}\,.$$

where $\mathcal{Q}$ is a Poincaré and $\mathcal{S}$ a conformal supercharge.
The $\bar{z}$ dependence turns out to be $\mathbb{Q}$-exact.

Richer structure than the $\mathcal{N} = 1$ chiral ring because of z dependence.

χ : 4d $\mathcal{N} = 2$ SCFT $\longrightarrow$ 2d Chiral Algebra.

Some universal properties:

- Virasoro enhancement of $\mathfrak{sl}(2)$, with $T(z)$ arising from a component of the $SU(2)_R$ conserved current, $T(z) := [\mathcal{J}_R(z, \bar{z})]_{\mathbb{Q}}$, with

$$c_{2d} = -12\, c_{4d}\,,$$

  where c_{4d} is one of the conformal anomaly coefficient.

- Affine symmetry enhancement of global flavor symmetry, with $J(z)$ arising from the moment map operator, $J(z) := [M(z, \bar{z})]_{\mathbb{Q}}$, with

$$k_{2d} = -\frac{k_{4d}}{2}\,.$$

- Generators of the $4d$ Higgs branch $\Rightarrow$ generators of the chiral algebra.

  Higgs branch relations encoded in null states of the chiral algebra!
  (Crucial that k_{2d} takes special negative levels).

Prospects

Minibootstrap:

- For a given theory $\mathcal{T}$, develop systematic tools to characterize $\chi[\mathcal{T}]$ as $\mathcal{W}$ algebra.
- Classification of SCFTs related to classification of special chiral algebras.
- Add non-local operators.
  Particularly interesting in $d = 6$, where it should lead to AGT.

Maxibootstrap:

- (2, 0) bootstrap: in progress, stay tuned.
- Exploration of landscape of $\mathcal{N} = 2$ models, especially non-Lagrangian ones.
- More $\mathcal{N} = 4$.

Neat interplay of striking mathematical physics and numerical experiments.