A combined estimate of the KLOE08, KLOE10 and KLOE12 ISR measurements

S. E. Müller

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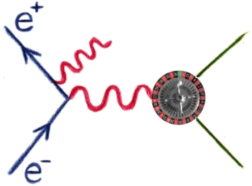

Radio MonteCarLOW Meeting
LNF Frascati, 13-14 September 2013

The KLOE data sets

- KLOE05: 60 points between 0.35 and 0.95 GeV^2, based on 141.4 pb^{-1} of data taken in 2001[a]
- KLOE08: 60 points between 0.35 and 0.95 GeV^2, based on 240.0 pb^{-1} data taken in 2002[b]
- KLOE10: 75 points between 0.1 and 0.85 GeV^2, based on 232.6 pb^{-1} data taken in 2006[c] with $\sqrt{s} = 1.00$ GeV
- KLOE12: 60 points between 0.35 and 0.95 GeV^2, based on 240.0 pb^{-1} data taken in 2002[d], normalized to muons

[a] Phys. Lett. B**606** (2005) 12
[b] Phys. Lett. B**670** (2009) 285
[c] Phys. Lett. B**700** (2011) 102
[d] Phys. Lett. B**720** (2013) 336

The KLOE data sets

- KLOE05: 60 points betw… .95 GeV^2, based on 141.4 … ta taken in 2001[a]

  Superseded by KLOE08!

- KLOE08: 60 points between 0.35 and 0.95 GeV^2, based on 240.0 pb^{-1} data taken in 2002[b]

- KLOE10: 75 points between 0.1 and 0.85 GeV^2, based on 232.6 pb^{-1} data taken in 2006[c] with $\sqrt{s} = 1.00$ GeV

- KLOE12: 60 points between 0.35 and 0.95 GeV^2, based on 240.0 pb^{-1} data taken in 2002[d], normalized to muons

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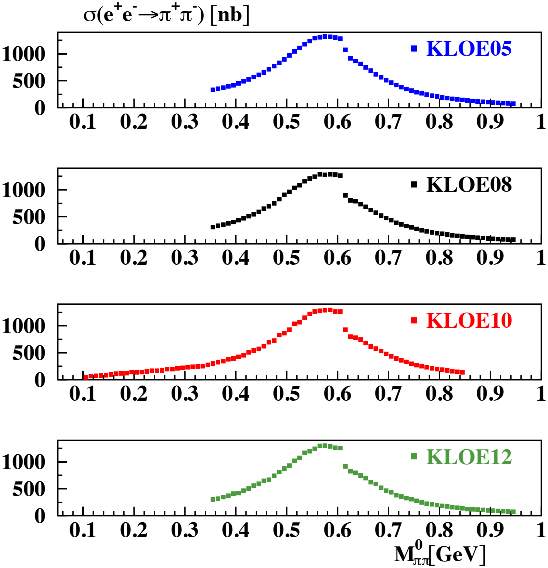

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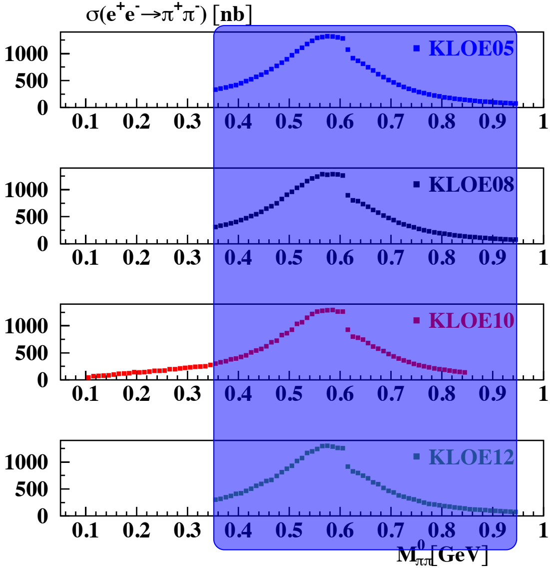

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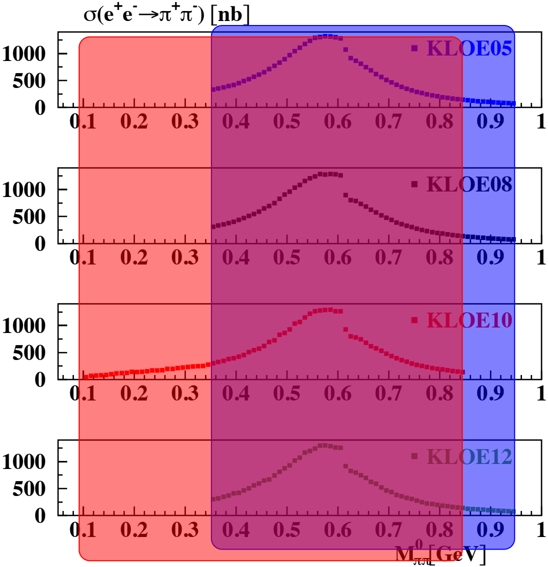

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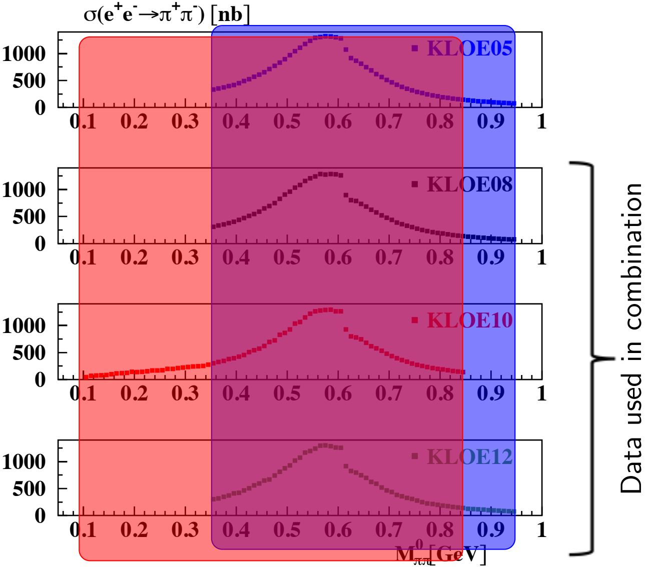

BLUE: Best linear unbiased estimate[1]

We have 195 $y_i = \{y_1, \ldots, y_{195}\}$ measurements for the 85 observables with true value $X_\alpha = \{X_1, \ldots, X_{85}\}$ (the 85 bins between 0.1 and 0.95 GeV^2).

Find the Best Linear Unbiased Estimates $\hat{x}_\alpha$ of the 85 observables such that

A the $\hat{x}_\alpha$ are a linear combination of the input measurements y_i

B the $\hat{x}_\alpha$ are unbiased estimates of the true values X_α

C the $\hat{x}_\alpha$ minimize the variance $var(\hat{x}_\alpha) \equiv cov(\hat{x}_\alpha, \hat{x}_\alpha)$ (they are the best estimates)

[1] A. Valassi, NIM A500 (2003) 391; L. Lyons and D. Gibaut, NIM A270 (1988) 110

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BLUE: Best linear unbiased estimate

- condition A gives

$$\hat{x}_\alpha = \sum_{i=1}^{195} \lambda_{\alpha i} y_i \equiv \sum_{\beta=1}^{85} \sum_{i=1}^{195} \lambda_{\alpha i} \mathscr{U}_{i\beta} y_i \tag{1}$$

- condition B gives the normalization constraints

$$\sum_{i=1}^{195} \lambda_{\alpha i} \mathscr{U}_{i\beta} = \delta_{\alpha\beta} \tag{2}$$

- condition C requires to determine the linear weights $\lambda_{\alpha i}$ such that they minimize the variances of the $\hat{x}_\alpha$:

$$var(\hat{x}_\alpha) \equiv cov(\hat{x}_\alpha, \hat{x}_\alpha) = \sum_{i=1}^{195} \sum_{j=1}^{195} \lambda_{\alpha i} \mathscr{M}_{ij} \lambda_{\alpha j}, \tag{3}$$

  where $\mathscr{M}_{ij}$ is the (195,195) covariance matrix of the 195 measurements.

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BLUE: Best linear unbiased estimate

Task: Determine the $\lambda_{\alpha i}$ defined by condition A by minimizing the variances in condition C under the constraints given by condition B!
Introducing the Lagrange multipliers $K_{\alpha\beta}$, the object to minimize becomes

$$\left[var(\hat{x}_\alpha) + 2\sum_{\gamma=1}^{85} K_{\alpha\gamma}(\delta_{\alpha\gamma} - \sum_{j=1}^{195} \lambda_{\alpha j}\mathscr{U}_{j\gamma}) \right] \quad (4)$$

Differentiating with respect to $K_{\alpha\gamma}$ and $\lambda_{\alpha j}$ gives

$$\begin{cases} \delta_{\alpha\beta} - \sum_{j=1}^{195} \lambda_{\alpha j}\mathscr{U}_{j\beta} = 0 & \forall\alpha, \forall\beta, \\ (\sum_{j=1}^{195} \mathscr{M}_{ij}\lambda_{\alpha j}) - (\sum_{\gamma=1}^{85} K_{\alpha\gamma}\mathscr{U}_{i\gamma}) = 0 & \forall\alpha, \forall i \end{cases}$$

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BLUE: Best linear unbiased estimate

The system of linear equations is solved by

$$K_{\alpha\gamma} = (\mathscr{U}^T \mathscr{M}^{-1} \mathscr{U})^{-1}_{\alpha\gamma} \tag{5}$$

$$\lambda_{\alpha i} = \sum_{\beta=1}^{85} (\mathscr{U}^T \mathscr{M}^{-1} \mathscr{U})^{-1}_{\alpha\gamma} (\mathscr{U}^T \mathscr{M}^{-1})_{\beta i} \tag{6}$$

$$cov(\hat{x}_\alpha, \hat{x}_\beta) = (\mathscr{U}^T \mathscr{M}^{-1} \mathscr{U})^{-1}_{\alpha\beta} \tag{7}$$

Knowing the two matrices $\mathscr{U}_{i\alpha}$ and $\mathscr{M}_{ij}$ allows to determine the linear coefficients $\lambda_{\alpha i}$ and therefore the Best Unbiased Linear Estimates $\hat{x}_\alpha$.

The $\mathscr{U}_{i\alpha}$ matrix

is a $(\underbrace{195}_{\text{rows}} \times \underbrace{85}_{\text{cols}})$ matrix linking the measurements y_i to the observables X_α

$$\mathscr{U}_{i\alpha} = \begin{cases} 1 & \text{if } y_i \text{ is a measurement of } X_\alpha, \\ 0 & \text{if } y_i \text{ is not a measurement of } X_\alpha, \end{cases}$$

The $\mathscr{U}_{i\alpha}$ matrix

$$\mathscr{U} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
 & & & & & & & & & & & & & & & & & \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
 & & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}$$

$\rightarrow$ bins in $M^2_{\pi\pi}$ (0.1 - 0.95 GeV^2)

The $\mathscr{U}_{i\alpha}$ matrix

KLOE08 data (highlighted top block)

$$
\mathscr{U} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
 & & & & & & & & & & & & & & & & & \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
 & & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1
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0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\ \text{KLOE08 data} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
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0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\ \text{KLOE10 data} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1
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$$

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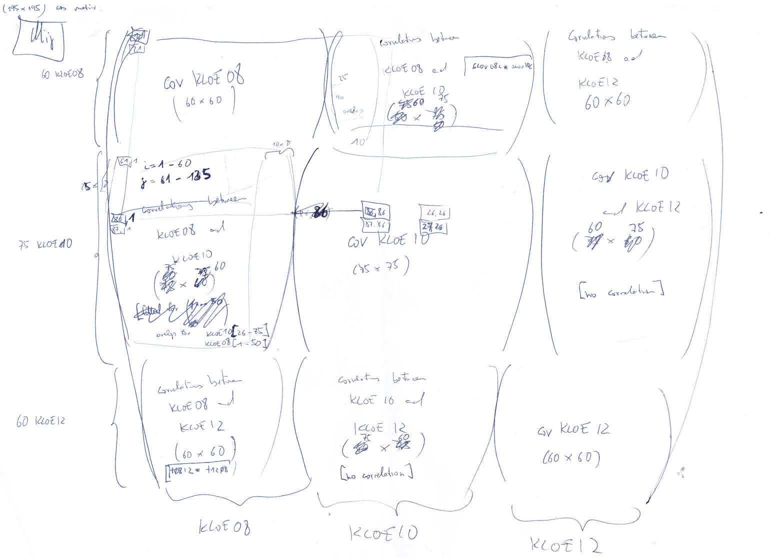
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \multicolumn{6}{c}{\text{KLOE08 data}} & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \multicolumn{6}{c}{\text{KLOE10 data}} & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \multicolumn{6}{c}{\text{KLOE12 data}} & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1
\end{pmatrix}
$$

$\rightarrow$ bins in $M^2_{\pi\pi}$ (0.1 - 0.95 GeV^2)

The $\mathscr{M}_{ij}$ matrix

is the covariance matrix for the $60 + 75 + 60 = 195$ data points.



The $\mathscr{M}_{ij}$ matrix

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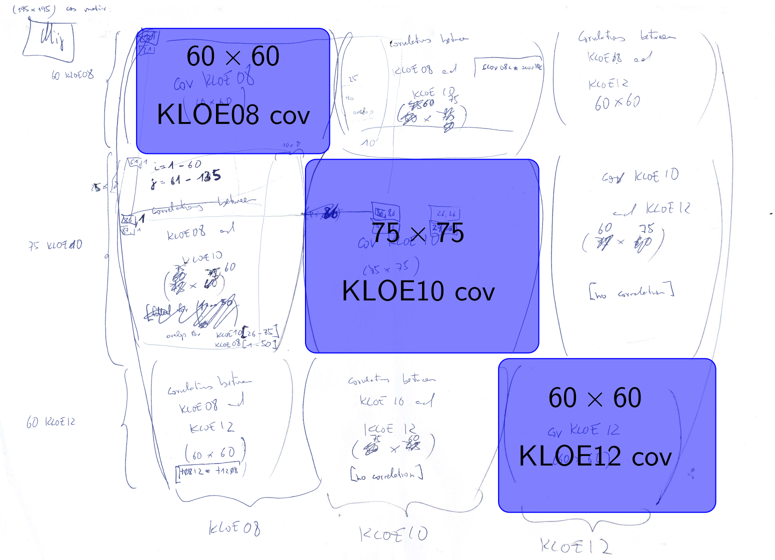

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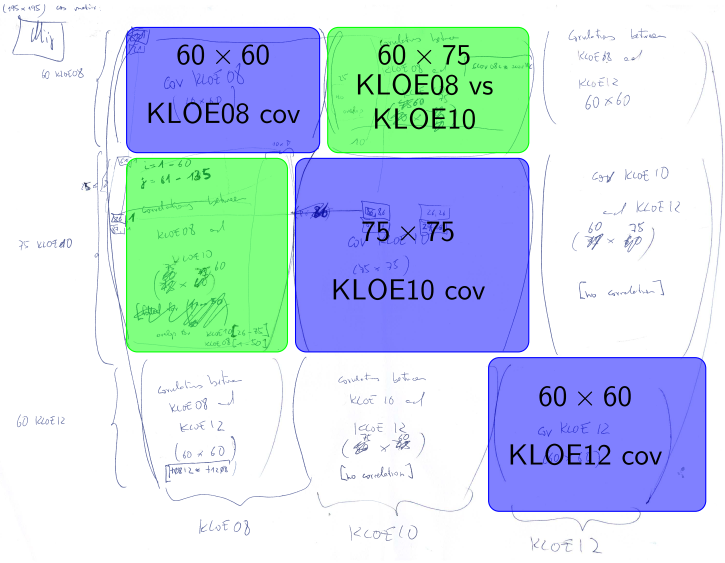

The $\mathscr{M}_{ij}$ matrix

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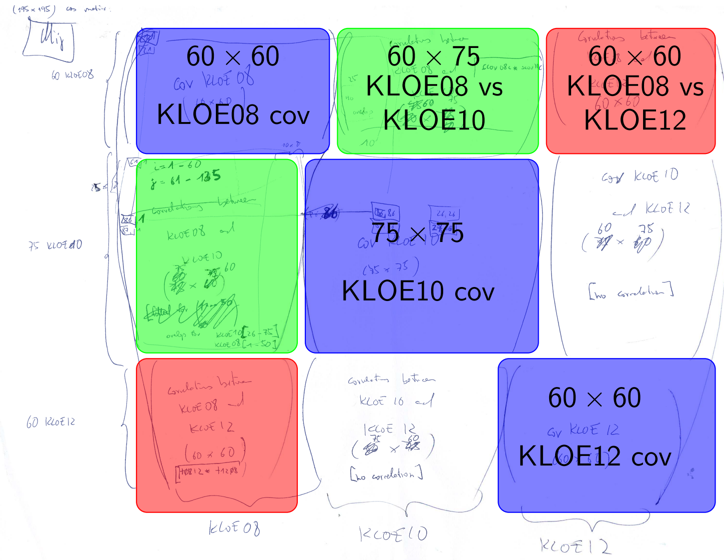

The $\mathscr{M}_{ij}$ matrix

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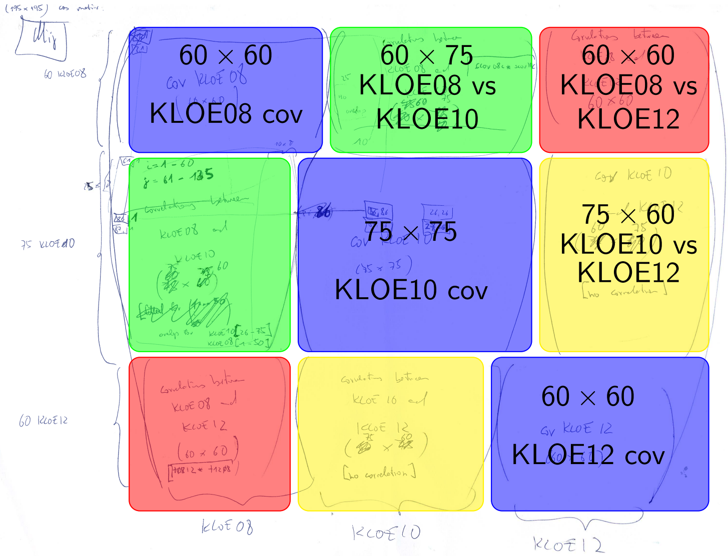

A caveat: normalization errors

G. D'Agostini (NIM A346 (1994) 306):
Normalization errors (e.g. errors on scale factors) can create a bias when fitting correlated data

The problem of finding the linear unbiased estimates of minimum variance for the 85 observables X_α is equivalent to the problem of finding the estimates $\hat{x}_\alpha$ minimizing the quantity

$$S = \sum_{i=1}^{195} \sum_{j=1}^{195} [y_i - (\mathscr{U}\hat{x})_i]\, \mathscr{M}_{ij}^{-1}\, [y_j - (\mathscr{U}\hat{x})_j] \tag{8}$$

However, only the free parameters $\hat{x}_\alpha$ are varied (within the errors) to find the minimum of S. But in the case of a normalization error, also the elements of $\mathscr{M}_{ij}$ should be scaled accordingly when varying the $\hat{x}_\alpha$. Therefore, normalization errors lead to a bias in this method.

A caveat: normalization errors

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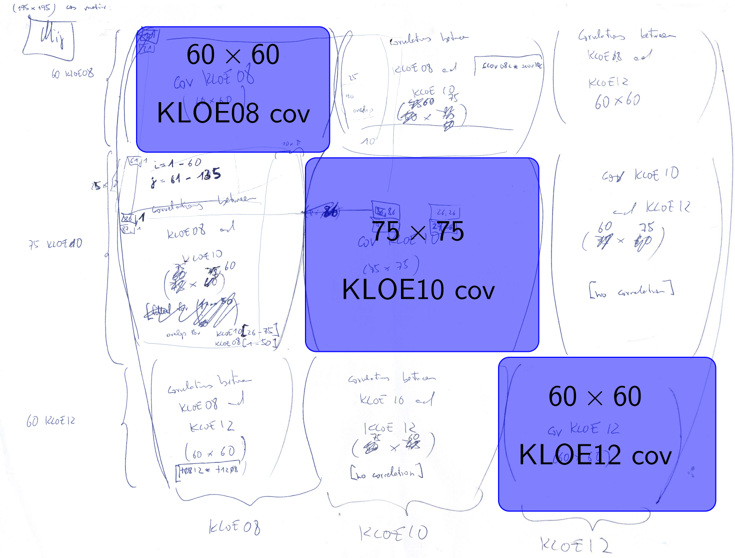
Way out: Two separate covariance matrices:

- $\mathscr{M}_{ij}^{\mathrm{stat}}$ which contains the statistical uncertainties and is used to find the $\hat{x}_\alpha$
- $\mathscr{M}_{ij}^{\mathrm{syst}}$ which contains all the normalization errors, gives $cov^{\mathrm{syst}}(\hat{x}_\alpha, \hat{x}_\beta) = \sum_{i=1}^{195} \sum_{j=1}^{195} \lambda_{\alpha i} \mathscr{M}_{ij}^{\mathrm{syst}} \lambda_{\beta j}$, which can then be added to $cov^{\mathrm{stat}}(\hat{x}_\alpha, \hat{x}_\beta)$:

$$cov^{\mathrm{tot}}(\hat{x}_\alpha, \hat{x}_\beta) = cov^{\mathrm{stat}}(\hat{x}_\alpha, \hat{x}_\beta) + cov^{\mathrm{syst}}(\hat{x}_\alpha, \hat{x}_\beta) \quad (8)$$

The $\mathscr{M}_{ij}$ matrix: The 3 diagonal blocks



The $\mathscr{M}_{ij}$ matrix: The 3 diagonal blocks

- $\mathscr{M}_{ij}^{\mathrm{stat}}$: Take the data files provided at the KLOE ppg webpage for each analysis `http://www.lnf.infn.it/kloe/ppg/`

- $\mathscr{M}_{ij}^{\mathrm{syst}}$: For each contribution to the systematic error (given at the webpage), construct the corresponding contribution to the matrix element via $\delta_{ij}^{xxxx} = +1 \cdot \delta_i^{xxxx}\delta_j^{xxxx}$, and add the contributions:

$$\mathscr{M}_{ij}^{\mathrm{syst}} = \delta_i^{\mathrm{trig}}\delta_j^{\mathrm{trig}} + \delta_i^{\mathrm{Bkg}}\delta_j^{\mathrm{Bkg}} + \delta_i^{\mathrm{Lumi}}\delta_j^{\mathrm{Lumi}} + \ldots \tag{9}$$

Exception: Errors on unfolding for KLOE08 and KLOE10 - one can assume them to be fully anti-correlated, and therefore it contributes with $\delta_{ij,i\neq j}^{Unf} = -1 \cdot \delta_i^{Unf}\delta_j^{Unf}$ to the off-diagonal elements.

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- $\mathscr{M}_{ij}^{\mathrm{stat}}$: Take the data files provided at the KLOE ppg webpage for each analysis `http://www.lnf.infn.it/kloe/ppg/`

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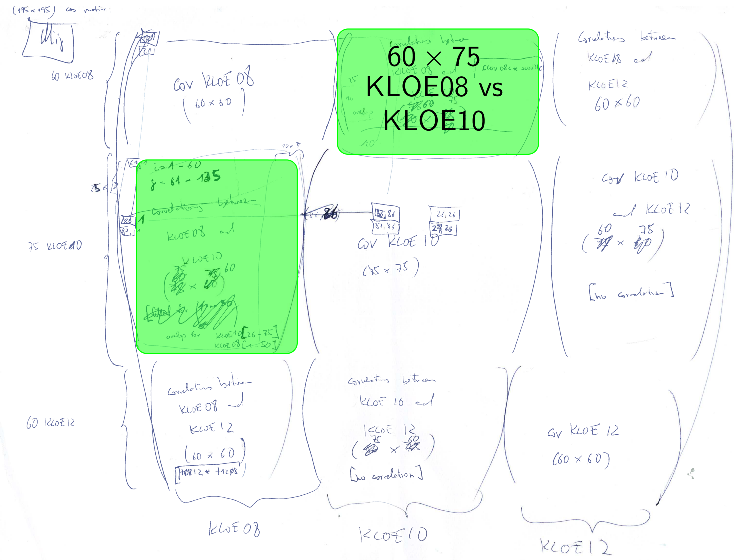
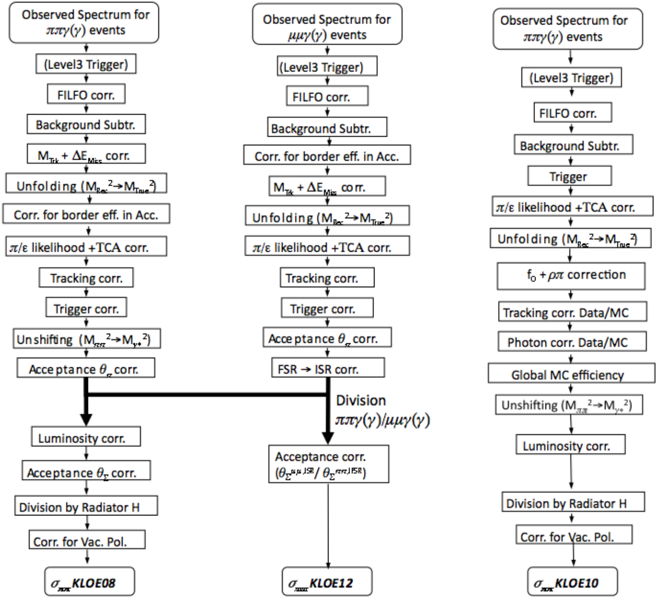
  Exception: Errors on unfolding for KLOE08 and KLOE10 - one can assume them to be fully anti-correlated, and therefore it contributes with $\delta_{ij,i\neq j}^{Unf} = -1 \cdot \delta_i^{Unf}\delta_j^{Unf}$ to the off-diagonal elements.

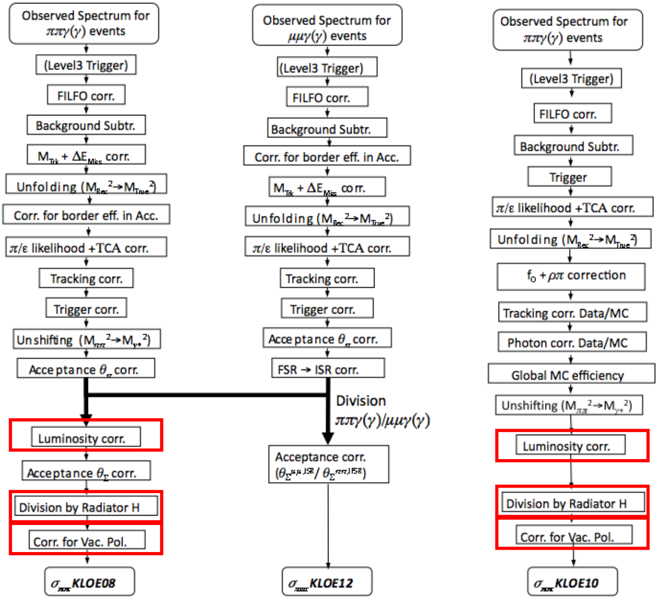
The $\mathscr{M}_{ij}$ matrix: The KLOE08 - KLOE10 correlation block



The $\mathscr{M}_{ij}$ matrix: The KLOE08 - KLOE10 correlation block



The $\mathscr{M}_{ij}$ matrix: The KLOE08 - KLOE10 correlation block



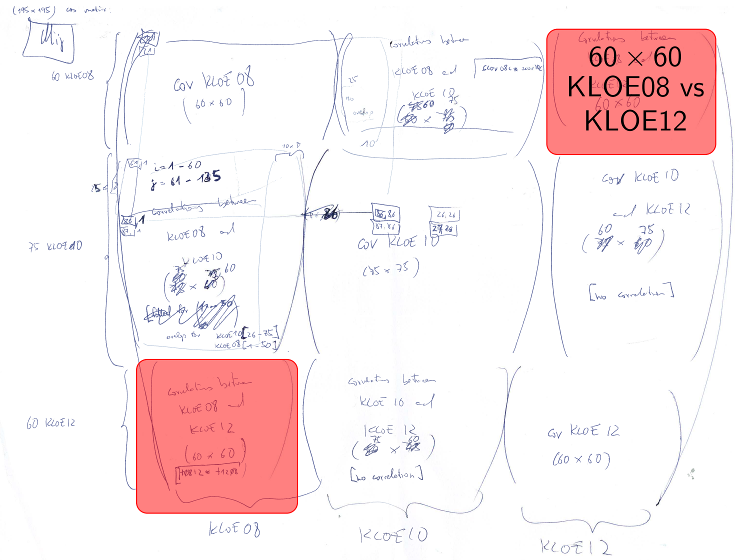
The $\mathscr{M}_{ij}$ matrix: The KLOE08 - KLOE10 correlation block

- $\mathscr{M}_{ij}^{\mathrm{stat}}$: In this case, there are no contribution to the statistical covariance matrix.
- $\mathscr{M}_{ij}^{\mathrm{syst}}$: The following contributions to the systematic uncertainty are common to both analyses:
  - uncertainty on luminosity evaluation (0.3%)
  - uncertainty on radiator function (0.5%)
  - uncertainty on vacuum polarization correction (from F. Jegerlehner's routine)

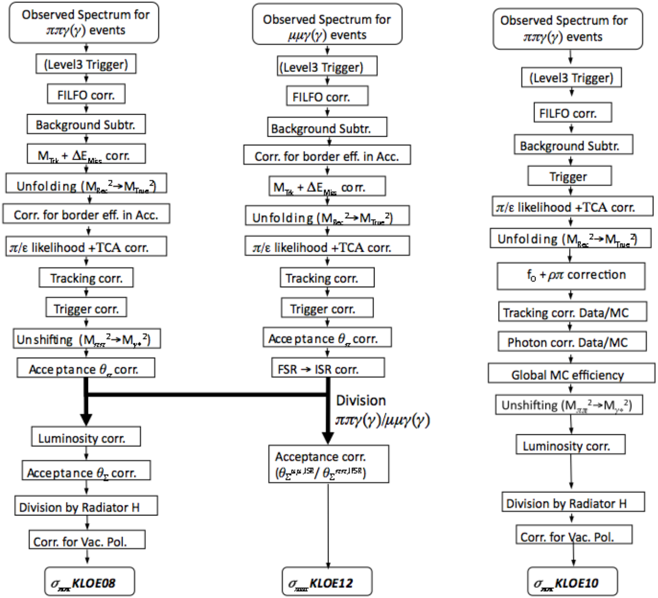
  Therefore, the elements of $\mathscr{M}_{ij}^{\mathrm{syst}}$ are constructed like this:

$$\mathscr{M}_{ij}^{\mathrm{syst}} = \delta 08_i^{\mathrm{Lumi}} \delta 10_j^{\mathrm{Lumi}} + \delta 08_i^{\mathrm{Radiator}} \delta 10_j^{\mathrm{Radiator}} + \delta 08_i^{\mathrm{VacPol}} \delta 10_j^{\mathrm{VacPol}} \tag{10}$$

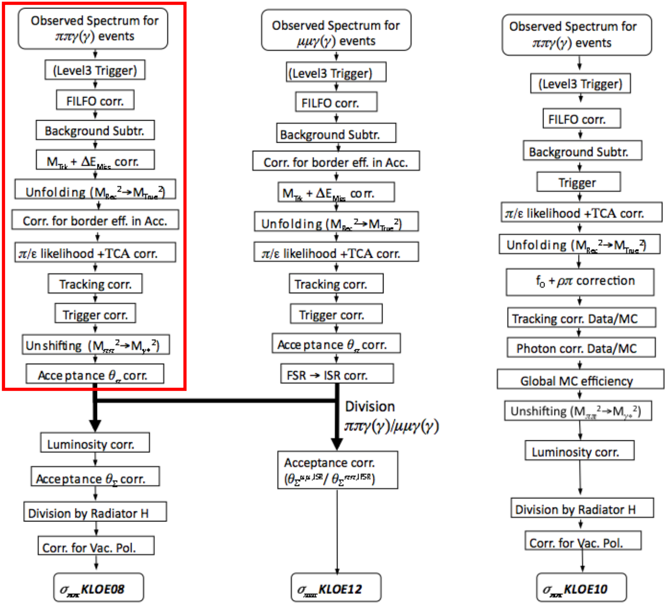
The $\mathscr{M}_{ij}$ matrix: The KLOE08 - KLOE12 correlation block



The $\mathscr{M}_{ij}$ matrix: The KLOE08 - KLOE12 correlation block



The $\mathscr{M}_{ij}$ matrix: The KLOE08 - KLOE12 correlation block



The $\mathscr{M}_{ij}$ matrix: The KLOE08 - KLOE12 correlation block

- $\mathscr{M}_{ij}^{\mathrm{stat}}$: "Intermediate" covariance matrix $\mathscr{M}_{ij}^{\mathrm{int}}$ up to the acceptance correction for θ_π for the KLOE08 analysis, then propagate this matrix towards $\sigma_{\pi\pi}^{08}$ and $\sigma_{\pi\pi}^{12}$ by applying the remaining correction factors:

$$\mathscr{M}_{ij}^{08} = \frac{\mathscr{M}_{ij}^{\mathrm{int}}}{(0.01\mathrm{GeV}^2)(\mathrm{Acc}_{\theta_\Sigma})(\int \mathcal{L}dt)(H(s_\pi, s))\delta_{VP}} \tag{11}$$

$$\mathscr{M}_{ij}^{12} = \frac{\mathscr{M}_{ij}^{\mathrm{int}}}{N_{\mu\mu\gamma}^{\mathrm{vis}}}(\mathrm{Acc}\frac{\theta_\Sigma^{\mu\mu\gamma,ISR}}{\theta_\Sigma^{\pi\pi\gamma,ISR}}) \times \mathrm{kin.\,factors} \tag{12}$$

$$\mathscr{M}_{ij}^{\mathrm{stat}} = \sqrt{\mathscr{M}_{ij}^{08}\mathscr{M}_{ij}^{12}} \tag{13}$$

The $\mathscr{M}_{ij}$ matrix: The KLOE08 - KLOE12 correlation block

- $\mathscr{M}_{ij}^{\mathrm{syst}}$: The following contributions to the systematic uncertainty are common to both analyses:
  - uncertainty from L3 trigger in KLOE08 analysis (0.1%)
  - uncertainty on background evaluation of the KLOE08 $\pi\pi\gamma$ analysis (tabulated in KLOE08 publication)
  - uncertainty on cuts on M_{trk} and ΔE_{miss} in KLOE08 analysis (0.2%)
  - uncertainty on unfolding in KLOE08 analysis (only in 5 bins between $0.58 - 0.63$ GeV^2), considered fully anti-correlated
  - uncertainty on tracking efficiency in KLOE08 analysis (0.3%)
  - uncertainty on trigger efficiency in KLOE08 analysis (0.1%)
  - uncertainty on acceptance in KLOE08 analysis (tabulated in KLOE08 publication)

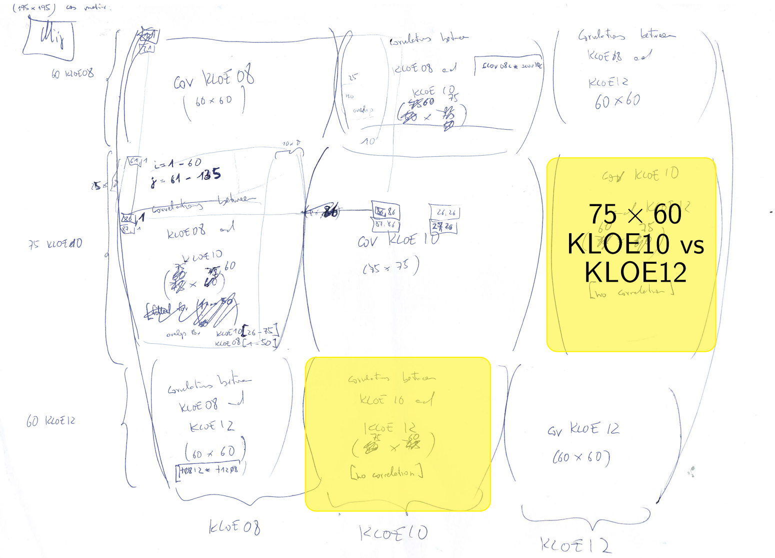
The $\mathscr{M}_{ij}$ matrix: The KLOE08 - KLOE12 correlation block

- $\mathscr{M}_{ij}^{\rm syst}$: The following contributions to the systematic uncertainty are common to both analyses:
  - uncertainty from L3 trigger in KLOE08 analysis (0.1%)
  - uncertainty on background evaluation of the KLOE08 $\pi\pi\gamma$ analysis (tabulated in KLOE08 publication)
  - uncertainty on cuts on $M_{\rm trk}$ and $\Delta E_{\rm miss}$ in KLOE08 analysis (0.2%)
  - uncertainty on unfolding in KLOE08 analysis (only in 5 bins between $0.58 - 0.63$ GeV^2), considered fully anti-correlated
  - uncertainty on tracking efficiency in KLOE08 analysis (0.3%)
  - uncertainty on trigger efficiency in KLOE08 analysis (0.1%)
  - uncertainty on acceptance in KLOE08 analysis (tabulated in KLOE08 publication)

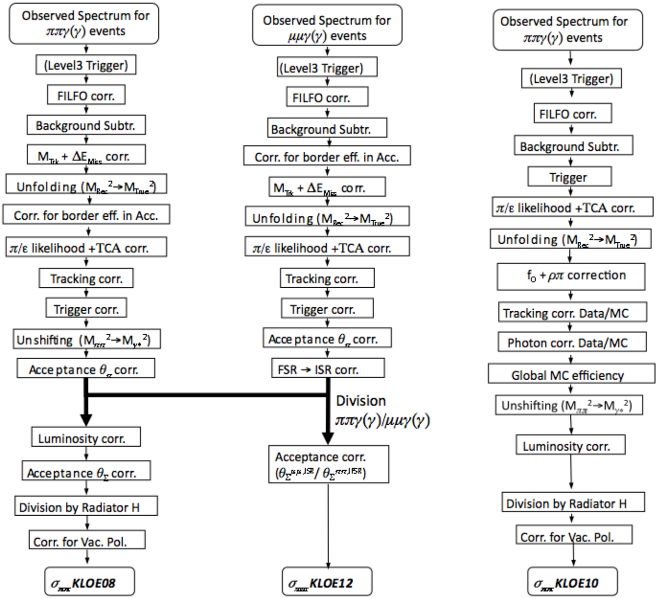
Therefore, the elements of $\mathscr{M}_{ij}^{\rm syst}$ are constructed like this:

$$\begin{aligned}\mathscr{M}_{ij}^{\rm syst} &= \delta 08_i^{\rm L3}\delta 12_j^{\rm L3} + \delta 08_i^{\rm bg}\delta 12_j^{\rm bg} + \delta 08_i^{\rm TM,\Delta E}\delta 12_j^{\rm TM,\Delta E} + \\ &+\delta 08_i^{\rm trk}\delta 12_j^{\rm trk} + \delta 08_i^{\rm trg}\delta 12_j^{\rm trg} + \delta 08_i^{\rm acc}\delta 12_j^{\rm acc} + \\ &+\varrho_{\rm unf}\delta 08_i^{\rm unf}\delta 12_j^{\rm unf}\end{aligned}$$

The $\mathscr{M}_{ij}$ matrix: The KLOE10 - KLOE12 correlation block



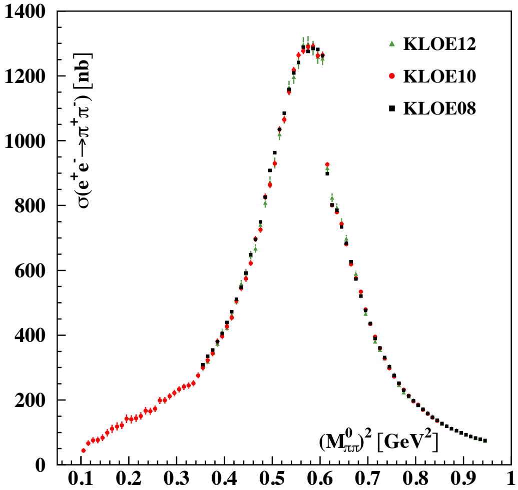
The $\mathscr{M}_{ij}$ matrix: The KLOE10 - KLOE12 correlation block



The $\mathscr{M}_{ij}$ matrix: The KLOE10 - KLOE12 correlation block

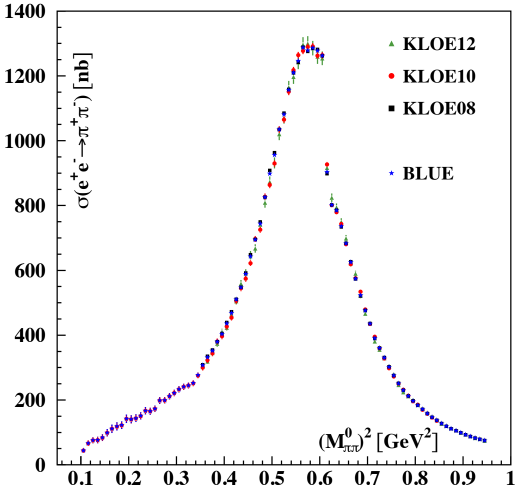
- $\mathscr{M}_{ij}^{\mathrm{stat}}$: In this case, there are no contributions to the statistical covariance matrix.
- $\mathscr{M}_{ij}^{\mathrm{syst}}$: There are no contributions to the systematic uncertainty that are common to both analyses!

The result



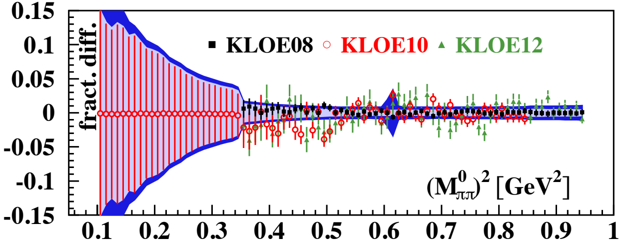
1400
1200
1000
800
600
400
200
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$\sigma(e^+e^- \to \pi^+\pi^-)$ [nb]
$(M^0_{\pi\pi})^2$ [GeV^2]
KLOE12
KLOE10
KLOE08

The result



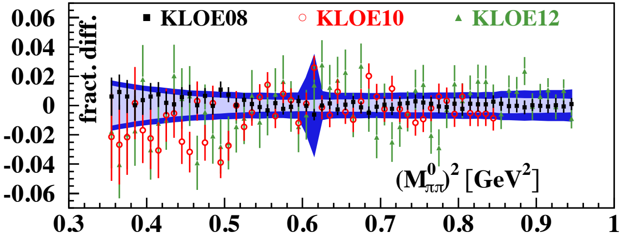
1400
1200
1000
800
600
400
200
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$\sigma(e^+e^- \to \pi^+\pi^-)$ [nb]
$(M^0_{\pi\pi})^2$ [GeV^2]
KLOE12
KLOE10
KLOE08
BLUE

The result



0.15
0.1
0.05
0
-0.05
-0.1
-0.15
fract. diff.
KLOE08
KLOE10
KLOE12
$(M^0_{\pi\pi})^2$ [GeV^2]
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

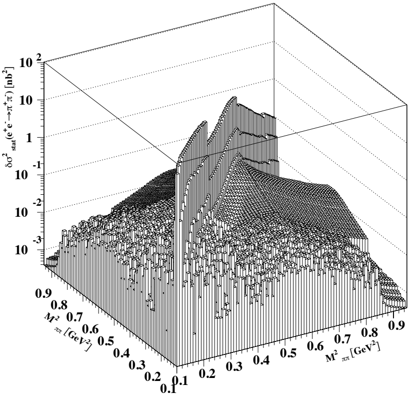
The result



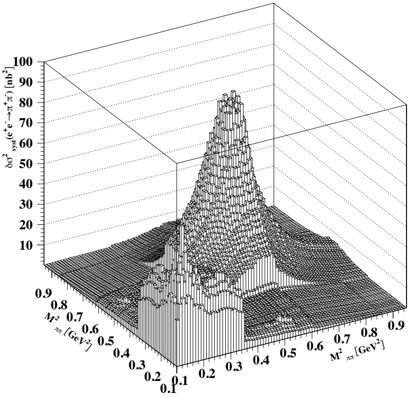
KLOE08
KLOE10
KLOE12
fract. diff.
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$(M^0_{\pi\pi})^2$ [GeV^2]

The covariance matrices for the BLUE values

stat. covariance matrix:



syst. covariance matrix:



To obtain the total covariance matrix, simply add the two.

Dispersion integral for $a_\mu^{\rm had}$

Evaluating $a_\mu^{\rm had}$ between $s_{\rm min} = 0.1$ GeV^2 and $s_{\rm max} = 0.95$ GeV^2 with a binned data set of $n = 85$ points with binwidth $\Delta s = 0.01$ GeV^2:

$$a_\mu^{\rm had}[s_{\rm min}, s_{\rm max}] = \frac{1}{4\pi^3}\int_{s_{\rm min}}^{s_{\rm max}} \sigma^{\rm had}(s)K(s)\,ds \simeq \frac{1}{4\pi^3}\sum_{i=1}^{n}\sigma_i^{\rm had}K_i\Delta s$$

With $c_i \equiv \partial a_\mu^{\rm had}/\partial\sigma_i = (1/4\pi^3)K_i\Delta s$ we obtain for the uncertainty on $a_\mu^{\rm had}$

$$\left(\sigma_{a_\mu^{\rm had}}\right)^2 = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} c_\alpha c_\beta \left[cov^{\rm tot}(\hat{x}_\alpha,\hat{x}_\beta)\right]_{\alpha\beta}$$

$$= \underbrace{\sum_{\alpha=1}^{n} c_\alpha^2 var(\hat{x}_\alpha)}_{\substack{\alpha=\beta,\\ \text{diagonal term}}} + \underbrace{\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \alpha\neq\beta}}^{n} c_\alpha c_\beta \left[cov^{\rm tot}(\hat{x}_\alpha,\hat{x}_\beta)\right]_{\alpha\beta}}_{\substack{\alpha\neq\beta,\\ \text{off}-\text{diagonal term}}}$$

Dispersion integral for a_μ^{had}

Evaluating a_μ^{had} between $s_{\mathrm{min}} = 0.1$ GeV^2 and $s_{\mathrm{max}} = 0.95$ GeV^2 with a binned data set of $n = 85$ points with binwidth $\Delta s = 0.01$ GeV^2:

$$a_\mu^{\mathrm{had}}[s_{\mathrm{min}}, s_{\mathrm{max}}] = \frac{1}{4\pi^3}\int_{s_{\mathrm{min}}}^{s_{\mathrm{max}}} \sigma^{\mathrm{had}}(s)K(s)\,ds \simeq \frac{1}{4\pi^3}\sum_{i=1}^{n}\sigma_i^{\mathrm{had}}K_i\Delta s$$

With $c_i \equiv \partial a_\mu^{\mathrm{had}}/\partial\sigma_i = (1/4\pi^3)K_i\Delta s$ we obtain for the uncertainty on a_μ^{had}

$$\left(\sigma_{a_\mu^{\mathrm{had}}}\right)^2 = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} c_\alpha c_\beta \left[cov^{\mathrm{tot}}(\hat{x}_\alpha, \hat{x}_\beta)\right]_{\alpha\beta}$$

$$= \underbrace{\sum_{\alpha=1}^{n} c_\alpha^2 var(\hat{x}_\alpha)}_{\substack{\alpha=\beta,\\ \text{diagonal term}}} + \underbrace{\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \alpha\neq\beta}}^{n} c_\alpha c_\beta \left[cov^{\mathrm{tot}}(\hat{x}_\alpha, \hat{x}_\beta)\right]_{\alpha\beta}}_{\substack{\alpha\neq\beta,\\ \text{off}-\text{diagonal term}}}$$

Dispersion integral for a_μ^{had}

Evaluating $\mathrm{a}_\mu^{\mathrm{had}}$ between $s_{\mathrm{min}} = 0.1\ \mathrm{GeV}^2$ and $s_{\mathrm{max}} = 0.95\ \mathrm{GeV}^2$ with a binned data set of $n = 85$ points with binwidth $\Delta s = 0.01\ \mathrm{GeV}^2$:

$$\mathrm{a}_\mu^{\mathrm{had}}[s_{\mathrm{min}}, s_{\mathrm{max}}] = \frac{1}{4\pi^3}\int_{s_{\mathrm{min}}}^{s_{\mathrm{max}}} \sigma^{\mathrm{had}}(s)K(s)\,ds \simeq \frac{1}{4\pi^3}\sum_{i=1}^{n}\sigma_i^{\mathrm{had}} K_i \Delta s$$

With $c_i \equiv \partial \mathrm{a}_\mu^{\mathrm{had}}/\partial\sigma_i = (1/4\pi^3)K_i\Delta s$ we obtain for the uncertainty on $\mathrm{a}_\mu^{\mathrm{had}}$

$$\begin{aligned}\left(\sigma_{\mathrm{a}_\mu^{\mathrm{had}}}\right)^2 &= \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} c_\alpha c_\beta \left[cov^{\mathrm{tot}}(\hat{x}_\alpha, \hat{x}_\beta)\right]_{\alpha\beta} \\ &= \underbrace{\sum_{\alpha=1}^{n} c_\alpha^2 var(\hat{x}_\alpha)}_{\substack{\alpha=\beta,\\ \text{diagonal term}}} + \underbrace{\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \alpha\neq\beta}}^{n} c_\alpha c_\beta \left[cov^{\mathrm{tot}}(\hat{x}_\alpha, \hat{x}_\beta)\right]_{\alpha\beta}}_{\substack{\alpha\neq\beta,\\ \text{off-diagonal term}}}\end{aligned}$$

Results on $a_\mu^{\pi\pi}$:

Using a combination of KLOE08 & KLOE10 (KLOE Note 225):

$$\Delta a_\mu^{\pi\pi}[0.10 - 0.95\mathrm{GeV}^2] = (488.6 \pm 6.0) \times 10^{-10} \quad (11)$$

$$\Delta a_\mu^{\pi\pi}[0.35 - 0.85\mathrm{GeV}^2] = (379.1 \pm 2.9) \times 10^{-10} \quad (12)$$

Results on $a_\mu^{\pi\pi}$:

Using a combination of KLOE08 & KLOE10 (KLOE Note 225):

$$\Delta a_\mu^{\pi\pi}[0.10 - 0.95\mathrm{GeV}^2] = (488.6 \pm 6.0) \times 10^{-10} \quad (11)$$
$$\Delta a_\mu^{\pi\pi}[0.35 - 0.85\mathrm{GeV}^2] = (379.1 \pm 2.9) \times 10^{-10} \quad (12)$$

Using the BLUE for KLOE08, KLOE10 & KLOE12:

$$\Delta a_\mu^{\pi\pi}[0.10 - 0.95\mathrm{GeV}^2] = (488.6 \pm 5.7) \times 10^{-10} \quad (13)$$
$$\Delta a_\mu^{\pi\pi}[0.35 - 0.85\mathrm{GeV}^2] = (378.9 \pm 2.8) \times 10^{-10} \quad (14)$$

Results on $a_\mu^{\pi\pi}$:

Using a combination of KLOE08 & KLOE10 (KLOE Note 225):

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$$\Delta a_\mu^{\pi\pi}[0.35 - 0.85\mathrm{GeV}^2] = (379.1 \pm 2.9) \times 10^{-10} \quad (12)$$

Using the BLUE for KLOE08, KLOE10 & KLOE12:

$$\Delta a_\mu^{\pi\pi}[0.10 - 0.95\mathrm{GeV}^2] = (488.6 \pm 5.7) \times 10^{-10} \quad (13)$$
$$\Delta a_\mu^{\pi\pi}[0.35 - 0.85\mathrm{GeV}^2] = (378.9 \pm 2.8) \times 10^{-10} \quad (14)$$

Using the BLUE for KLOE08, KLOE10 & KLOE12, but correcting KLOE08 and KLOE10 for Fred Jegerlehner's 2012 corrections for vacuum polarization:

$$\Delta a_\mu^{\pi\pi}[0.10 - 0.95\mathrm{GeV}^2] = (487.8 \pm 5.7) \times 10^{-10} \quad (15)$$
$$\Delta a_\mu^{\pi\pi}[0.35 - 0.85\mathrm{GeV}^2] = (378.1 \pm 2.8) \times 10^{-10} \quad (16)$$

Summary and conclusion

- BLUE of KLOE08, KLOE10 and KLOE12 data have been evaluated
- Care needs to be applied in construction of covariance matrices (two separate matrices for statistical and systematic effects to avoid bias in construction of BLUE)
- Can't be done by people not familiar with the different analyses
- Still some cross checks needed, then result will be put on the KLOE ppg-webpage:
  `http://www.lnf.infn.it/kloe/ppg/`