Dualities Near the Horizon

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Based on collaborations with L.Borsten, MJ Duff, E.Orazi, S.Ferrara, M.Trigiante, A.Yeranyan,
arXiv:1102.4857, 1212.3254, 1305.2057

Ferrara, Dip. Fisica, October 23 2013

Summary

Maxwell-Einstein-Scalar Theories

Symmetric Scalar Manifolds :
Application to Supergravity and Extremal Black Hole Solutions

Attractor Mechanism, Effective Black Hole Potential

Duality Charge Orbits, Stability of Attractors,
Flat Directions and “Moduli Spaces”

The Matrix M and Freudenthal Duality

Horizon Freudenthal Duality and Attractor Mechanism
for 2-form Field Strengths

Groups “of type E_7”

Hints for Further Future Developments

Maxwell-Einstein-Scalar Theories

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2}g_{ij}(\varphi)\,\partial_\mu\varphi^i\partial^\mu\varphi^j + \frac{1}{4}I_{\Lambda\Sigma}(\varphi)\,F^\Lambda_{\mu\nu}F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}}R_{\Lambda\Sigma}(\varphi)\,\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}$$

D=4 Maxwell-Einstein-scalar system (with no potential)

[may be the bosonic sector of D=4 (ungauged) sugra]

$$H := \left(F^\Lambda, G_\Lambda\right)^T;$$

$$^*G_{\Lambda|\mu\nu} := 2\frac{\delta\mathcal{L}}{\delta F^{\Lambda|\mu\nu}}.$$

Abelian 2-form field strengths

static, spherically symmetric, asympt. flat, extremal BH

$$ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)}\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}\left(d\theta^2 + \sin\theta d\psi^2\right)\right]$$

$$\tau := -1/r$$

$$Q \; : \; = \int_{S^2_\infty} H = \left(p^\Lambda, q_\Lambda\right)^T;$$

$$p^\Lambda \; : \; = \frac{1}{4\pi}\int_{S^2_\infty} F^\Lambda, \; q_\Lambda = \frac{1}{4\pi}\int_{S^2_\infty} G_\Lambda.$$

dyonic vector of e.m. fluxes
(BH charges)

$$S_{D=1} = \int [(U')^2 + g_{ij}\varphi'^i\varphi'^j + e^{2U}V_{BH}(\varphi(\tau), \mathcal{Q})]d\tau$$

$$' \equiv \frac{d}{d\tau}$$

reduction D=4 →D=1 :effective 1-dimensional (radial) Lagrangian

$$V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\mathcal{Q},$$

BH effective potential

Ferrara,Gibbons,Kallosh

eoms

$$\begin{cases} \frac{d^2U}{d\tau^2} = e^{2U}V_{BH}; \\ \frac{d^2\varphi^i}{d\tau^2} = g^{ij}e^{2U}\frac{\partial V_{BH}}{\partial\varphi^j}. \end{cases}$$

in N=2 ungauged sugra, hyper mults. decouple, and we thus disregard them : scalar fields belong to vector mults.

Attractor Mechanism :

$$\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau\to-\infty}\varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$$

conformally flat geometry $AdS_2 \times S^2$ near the horizon

$$ds^2_{\text{B-R}} = \frac{r^2}{M^2_{\text{B-R}}}dt^2 - \frac{M^2_{\text{B-R}}}{r^2}(dr^2 + r^2 d\Omega)$$

near the horizon, the scalar fields are stabilized purely in terms of charges

$$S = \frac{A_H}{4} = \pi \left. V_{BH}\right|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T\mathcal{M}_H\mathcal{Q}$$

Bekenstein-Hawking entropy-area formula for extremal dyonic BH

Symmetric Scalar Manifolds

Let's specialize the discussion to theories with scalar manifolds which are **symmetric cosets G/H**

[N>2 : general, N=2 : particular, N=1 : need special cases]

H = isotropy group = linearly realized; scalar fields sit in an H-repr.

G = (global) electric-magnetic duality group
[in string theory : U-duality]

G is an *on-shell* symmetry of the Lagrangian

The 2-form field strengths (F,G) vector and the BH e.m. charges sit in a G-repr. **R** which is symplectic :

$$\exists ! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R};$$

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \equiv \mathcal{Q}_1^M \mathcal{Q}_2^N \mathbb{C}_{MN} = -\langle \mathcal{Q}_2, \mathcal{Q}_1 \rangle$$

$$\mathbb{C} = \begin{pmatrix} \mathbf{0}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbf{0}_n \end{pmatrix}$$

symplectic product

$$\begin{array}{ccc} G & \subset & Sp(2n, \mathbb{R}); \\ \mathbf{R} & = & \mathbf{2n} \end{array}$$

Gaillard-Zumino embedding
(generally maximal, but not symmetric)

Kac, Gaillard-Zumino

Symmetricity : algebraic def :

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k},$ Cartan decomposition of a Lie algebra g

h = compact Lie subalgebra

k = complementary of h in g

$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ from the definition of subalgebra

$[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$ by the adjoint action, h acts on k as a repr. whose real dim. is dim(G/H) (it holds in any coset G/H)

$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$ symmetricity condition; in gen. it simply holds $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{g}$

Symmetricity : differential def : :

$D_m R_{ijkl} = 0$ the Riemann tensor is covariantly constant

All symmetric scalar manifolds in supergravity are actually
(I)RGS = (**Irreducible) Riemannian Globally Symmetric Spaces**:

- strictly positive definite metric;
- *Einstein spaces*, with (constant) *negative* scalar curvature : $R_{ij} = \lambda g_{ij}$

❖ **symmetric** scalar manifolds of N=2, D=4 sugra

All special Kaehler of local type	$\frac{G_V}{H_V}$	r	$dim_{\mathbb{C}} \equiv n_V$
quadratic sequence $n \in \mathbb{N}$	$\frac{SU(1,n)}{U(1)\otimes SU(n)}$	1	n
$\mathbb{R} \oplus \Gamma_n,\ n \in \mathbb{N}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2)\otimes SO(n)}$	$2\ (n=1)$ $3\ (n \geqslant 2)$	$n+1$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_{6(-78)}\otimes U(1)}$	3	27
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{U(6)}$	3	15
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{S(U(3)\otimes U(3))} = \frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)}$	3	9

$$\textit{simplest}\ \text{example}: \frac{G_V}{H_V} = \frac{SL(2,\mathbb{R})}{U(1)}, r=1, \dim_{\mathbb{C}} = n_V = 1, J_3 = \mathbb{R}$$

$\mathcal{N}=2,\ D=4\ T^3$ model

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}}g_{k\bar{l}} - g_{i\bar{l}}g_{k\bar{j}} + C_{ikm}\overline{C}_{\overline{jlp}}g^{m\bar{p}}$$

$J_3^{\mathbb{A}}, \quad \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ is the *Jordan algebra* of degree 3 of Hermitian 3x3 matrices over the 4 *division algebras* of real (R), complex (C), quaternions (H), octonions (O)

$\Gamma_{m,n}$ is the Jordan algebra of degree 2 with a quadratic form with Lorentzian signature (m,n)

Jordan algebras were completely classified by Jordan, Von Neumann and Wigner in an attempt to generalize *Quantum Mechanics* beyond C

Gunaydin,Sierra,Townsend

They are related to the Magic Square of Freudenthal, Rosen and Tits

All Magic Squares of order 3 recently **classified and interpreted in sugra** in Cacciatori, Cerchiai, Marrani, **arXiv:1208.6153 [math-ph]**

J	$\mathrm{Aut}(J)$	$\mathrm{Str}_0(J)$	$\mathrm{Conf}(J)$	$\mathrm{QConf}(J)$
$\mathbb{R}$	1	1	$Sl(2,\mathbb{R})$	$G_{2(2)}$
$\mathbb{R} \oplus \Gamma_{n-1,1}$	$SO(n-1)$	$SO(n-1,1)$	$Sl(2) \times SO(n,2)$	$SO(n+2,4)$
$J_3^{\mathbb{R}}$	$SO(3)$	$Sl(3,\mathbb{R})$	$Sp(6)$	$F_{4(4)}$
$J_3^{\mathbb{C}}$	$SU(3)$	$Sl(3,\mathbb{C})$	$SU(3,3)$	$E_{6(+2)}$
$J_3^{\mathbb{H}}$	$USp(6)$	$SU^*(6)$	$SO^*(12)$	$E_{7(-5)}$
$J_3^{\mathbb{O}}$	F_4	$E_{6(-26)}$	$E_{7(-25)}$	$E_{8(-24)}$

attractor, stabilized configurations of scalar fields at the horizon of **extremal BH**

↓

critical points of the effective BH potential V_{BH}

→ **stability** of these critical pts. is crucial for the definition of attractor

determined by the signature of the $2n_V$ x $2n_V$ covariant Hessian matrix of V_{BH}

$$H_{ij}|_{\partial_\varphi V_{BH}=0} = [D_i D_j V_{BH}(\varphi, \mathcal{Q})]_{\partial_\varphi V_{BH}=0}$$

$$H_{ij}|_{\partial_\varphi V_{BH}=0} \geqslant 0$$

all *eigenvalues* are strictly positive → attractor

(local minimum of V_{BH})

$$H_{ij}|_{\partial_\varphi V_{BH}=0} \leqslant 0$$

all *eigenvalues* are strictly negative → repellor

(local maximum of V_{BH})

$$H_{ij}|_{\partial_\varphi V_{BH}=0} \gtrless 0$$

eigenvalues have any sign : some >0, some <0
(possibly, some = 0)
→ the crit. point is a flex point of V_{BH}

→The higher-order covariant ders. of V_{BH} (at its crit pts) have to be studied to check stability

Some general results :

in N=2, D=4 sugra, there are **no** massless Hessian modes **<u>at ½-BPS crit pts</u>** of V_{BH}

→ stability **OK**

Ferrara, Gibbons, Kallosh

<u>At non-BPS crit pts</u>: general result for <u>cubic</u> special Kaeheler geometries

$$\mathcal{F}(z) = d_{ijk} z^i z^j z^k, \; i,j,k = 1,...,n_V$$

split : $2n_V$ → [n_V +1 eigenvs >0] + [n_V -1 eigenvs =0 (massless Hessian modes)]

Tripathy, Trivedi

Vanishing eigenvalues (*i.e. massless Hessian modes*) are *ubiquitous* at non-BPS crit pts of V_{BH}, *whose actual **stability** must be checked*

symmetric scalar manifolds **G/H** (including symm. SKGs of N=2, D=4 sugra) :

The G-representation space R of the BH em charges gets **stratified**, under the action of G, in G-orbits (non-symmetric cosets **G/$\mathcal{H}$**).

Ferrara, Gunaydin

$\mathcal{H}$ is the **stabilizer** (isotropy) group of the orbit = symmetry of the charge configs., it relates equivalent BH charge configs

each G-orbit supports a class of crit. pts. of V_{BH} , corresponding to specific SUSY-preserving properties of the near-horizon geometry

[We will be considering the so-called "large" G-orbits, corresponding to extremal BHs with classical non-vanishing entropy]

When $\mathcal{H}$ is **non-compact**, there is a residual compact symmetry linearly acting on scalars, such that the scalars belonging to the ***"moduli space" $\mathcal{H}$/mcs($\mathcal{H}$)*** (symmetric submanifold of **G/H**) are **not** stabilized in terms of BH charges at the event horizon of the extremal BH

Ferrara, AM

The Attractor Mechanism is **inactive** on these unstabilized scalar fields, which are **flat directions** of V_{BH} at its critical points.

symmetric scalar manifolds **G/H** (cont'd) :

The **absence** of flat directions at **N=2 ½-BPS attractors** can thus be explained by the fact that the stabilizer of the ½-BPS orbit is **compact** : **$\mathcal{H}$=H/U(1)**, where H is the stabilizer of the scalar manifold **G/H**

The massless Hessian modes, ubiquitous at non-BPS crit pts of V_{BH}, are actually **all flat directions** of V_{BH} itself at the considered class of crit. pts.

In other words, *at each class of its crit pts*, V_{BH}, and thus the classical **Bekenstein-Hawking BH entropy**, <u>does **not** depend on a certain subset of the scalars</u>

Such a set of scalars is thus **not stabilized** at the BH event horizon. Nevertheless…

BH entropy is independent on all unstabilized scalars

Thus, the **flat directions** of V_{BH} at its critical points span various "***moduli spaces"***, related to the solutions of the *classical Attractor Eqs*.

❖ *"large"* **charge orbits** of symmetric N=2, D=4 sugras

Bellucci, Ferrara, Gunaydin, AM

	$\frac{1}{2}$-BPS orbits $\mathcal{O}_{\frac{1}{2}-BPS} = \frac{G}{H_0}$	non-BPS, $Z \neq 0$ orbits $\mathcal{O}_{non-BPS,Z\neq 0} = \frac{G}{\hat{H}}$	non-BPS, $Z = 0$ orbits $\mathcal{O}_{non-BPS,Z=0} = \frac{G}{\tilde{H}}$
Quadratic Sequence $(n = n_V \in \mathbb{N})$	$\frac{SU(1,n)}{SU(n)}$	$-$	$\frac{SU(1,n)}{SU(1,n-1)}$
$\mathbb{R} \oplus \Gamma_n$ $(n = n_V - 1 \in \mathbb{N})$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(2)\otimes SO(n)}$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(1,1)\otimes SO(1,n-1)}$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(2)\otimes SO(2,n-2)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1)\otimes SU(1,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

in N=2 : $$\left\{Q_\alpha^A, Q_\beta^B\right\} = \epsilon_{\alpha\beta} Z^{[AB]} = \epsilon_{\alpha\beta}\epsilon^{AB} Z$$

[T^3 model not considered]

❖ non-BPS Z<>0 **attractor moduli spaces** of symmetric N=2, D=4 sugras

Ferrara,AM

$\hat{h}$ =mcs $\hat{H}$

	$\frac{\hat{H}}{\hat{h}}$	r	$dim_{\mathbb{R}}$
$\mathbb{R} \oplus \Gamma_n$ $(n = n_V - 1 \in \mathbb{N})$	$SO(1,1) \otimes \frac{SO(1,n-1)}{SO(n-1)}$	$1(n=1)$ $2(n \geqslant 2)$	n
$J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	2	6
$J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	2	14
$J_3^{\mathbb{C}}$	$\frac{SL(3,C)}{SU(3)}$	2	8
$J_3^{\mathbb{R}}$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	2	5

They are nothing but the *real special* scalar manifolds of symmetric N=2, **<u>D=5</u>** sugras

[T^3 model not considered]

❖ non-BPS Z=0 **attractor moduli spaces** of symmetric N=2, D=4 sugras

Ferrara,AM

	$\frac{\tilde{H}}{\tilde{h}}=\frac{\tilde{H}}{\tilde{h}'\otimes U(1)}$	r	$dim_{\mathbb{C}}$
$Quadratic\ \ Sequence$ $(n=n_V\in\mathbb{N})$	$\frac{SU(1,n-1)}{U(1)\otimes SU(n-1)}$	1	$n-1$
$\mathbb{R}\oplus\Gamma_n$ $(n=n_V-1\in\mathbb{N})$	$\frac{SO(2,n-2)}{SO(2)\otimes SO(n-2)}, n\geqslant 3$	$1(n=3)$ $2(n\geqslant 4)$	$n-2$
$J_3^{\mathbb{O}}$	$\frac{E_{6(-14)}}{SO(10)\otimes U(1)}$	2	16
$J_3^{\mathbb{H}}$	$\frac{SU(4,2)}{SU(4)\otimes SU(2)\otimes U(1)}$	2	8
$J_3^{\mathbb{C}}$	$\frac{SU(2,1)}{SU(2)\otimes U(1)}\otimes\frac{SU(1,2)}{SU(2)\otimes U(1)}$	2	4
$J_3^{\mathbb{R}}$	$\frac{SU(2,1)}{SU(2)\otimes U(1)}$	1	2

Generally, they are non-special symmetric Kaehler manifolds

Thus, **all** non-degenerate crit pts of V_{BH} in *symmetric* N=2,D=4 sugras are **stable** (and thus determine extremal BH **attractors**):

✓ <u>with **no** flat directions at all</u> in ½-BPS class (indeed, the stabilizer $\mathcal{H}$ of the corresponding supporting charge orbits is **compact**);

✓ <u>with some flat directions</u>, spanning the related **moduli space** of **unstabilized** scalar degrees of freedom, in non-BPS (with Z<>0 and Z=0) classes.

What about N > 2 ?

In **N>2-extended**, D=4 sugras, <u>also</u> non-degenerate 1/N-BPS extremal BH attractors exhibit a related ***moduli space***.
The same reasoning as above can be made, because
all N>2-extended, D=4 sugras have **symmetric** scalar manifolds.

There are three classes of *non-degenerate* crit. Pts. of V_{BH} :

- ❑ 1/N-BPS;
- ❑ non-BPS with non-vanishing central charge matrix Z_{AB} (A,B=1,…,N);
- ❑ non-BPS with Z_{AB}=0.

Once again, **all** classes of crit pts of V_{BH} are **stable**, up to some ubiquitous **flat directions**, spanning the related symmetric **moduli spaces**.

scalar manifolds of N>2-extended, D=4 sugras

$\mathcal{N}$	$G_{\mathcal{N},4}/H_{\mathcal{N},4}$
3	$\mathbf{III}_{3,n}:\frac{SU(3,n)}{SU(3)\otimes SU(n)\otimes U(1)},\quad n\in\mathbb{N}$
4	$\mathbf{III}_{1,1}\otimes IV_{6,n}:\frac{SU(1,1)}{U(1)}\otimes\frac{SO(6,n)}{SO(6)\otimes SO(n)},\quad n\in\mathbb{N}\cup\{0\}\quad(\mathbb{R}\oplus\mathbf{\Gamma}_{n-1,5})$
5	$\mathbf{III}_{1,5}:\frac{SU(1,5)}{SU(5)\otimes U(1)}\quad(M_{1,2}(\mathbb{O}))$
6	$\mathbf{V}_6:\frac{SO^*(12)}{SU(6)\otimes U(1)}\quad(J_3^{\mathbb{H}})$
8	$\mathbf{5}:\frac{E_{7(7)}}{SU(8)}\quad\left(J_3^{\mathbb{O}_s}\right)$

❖ **charge orbits** of N>2-extended, D=4 sugras

Bellucci,Ferrara, Gunaydin,Kallosh,AM

Lu,Pope,Stelle

	$\frac{1}{\mathcal{N}}$-BPS orbits $\frac{G}{\mathcal{H}}$	non-BPS, $Z_{AB} \neq 0$ orbits $\frac{G}{\tilde{\mathcal{H}}}$	non-BPS, $Z_{AB} = 0$ orbits $\frac{G}{\hat{\mathcal{H}}}$
$\mathcal{N}=3$	$\frac{SU(3,n)}{SU(2,n)}$	–	$\frac{SU(3,n)}{SU(3,n-1)}$
$\mathcal{N}=4$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(4,n)}$	$\frac{SU(1,1)}{SO(1,1)} \otimes \frac{SO(6,n)}{SO(5,n-1)}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(6,n-2)}$
$\mathcal{N}=5$	$\frac{SU(1,5)}{SU(3)\otimes SU(2,1)}$	–	–
$\mathcal{N}=6$	$\frac{SO^*(12)}{SU(4,2)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(6)}$
$\mathcal{N}=8$	$\frac{E_{7(7)}}{E_{6(2)}}$	$\frac{E_{7(7)}}{E_{6(6)}}$	–

n=# *matter* (*vector*) *multiplets* (*matter coupling* possible only for N=3,4)

N=6 *pure* sugra is “twin” to **N=2** *matter coupled* magic sugra on quaternions H

❖ **attractor moduli spaces** of attractors in N>2-extended, D=4 sugras

Ferrara,AM

	$\frac{1}{\mathcal{N}}$-BPS moduli space $\frac{\mathcal{H}}{\mathfrak{h}}$	non-BPS, $Z_{AB} \neq 0$ moduli space $\frac{\widehat{\mathcal{H}}}{\widehat{\mathfrak{h}}}$	non-BPS, $Z_{AB} = 0$ moduli space $\frac{\widetilde{\mathcal{H}}}{\widetilde{\mathfrak{h}}}$
$\mathcal{N}=3$	$\frac{SU(2,n)}{SU(2)\otimes SU(n)\otimes U(1)}$	$-$	$\frac{SU(3,n-1)}{SU(3)\otimes SU(n-1)\otimes U(1)}$
$\mathcal{N}=4$	$\frac{SO(4,n)}{SO(4)\otimes SO(n)}$	$SO(1,1)\otimes\frac{SO(5,n-1)}{SO(5)\otimes SO(n-1)}$	$\frac{SO(6,n-2)}{SO(6)\otimes SO(n-2)}$
$\mathcal{N}=5$	$\frac{SU(2,1)}{SU(2)\otimes U(1)}$	$-$	$-$
$\mathcal{N}=6$	$\frac{SU(4,2)}{SU(4)\otimes SU(2)\otimes U(1)}$	$\frac{SU^*(6)}{USp(6)}$	$-$
$\mathcal{N}=8$	$\frac{E_{6(2)}}{SU(6)\otimes SU(2)}$	$\frac{E_{6(6)}}{USp(8)}$	$-$

$\mathfrak{h}$, $\widehat{\mathfrak{h}}$ and $\widetilde{\mathfrak{h}}$ are maximal compact subgroups of $\mathcal{H}$, $\widehat{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$, respectively, and n is the number of matter multiplets

Let’s reconsider the starting **Maxwell-Einstein-scalar Lagrangian density**

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}$$

…and introduce the following real 2n x 2n matrix :

$$\mathcal{M} = \begin{pmatrix} \mathbb{I} & -R \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -R & \mathbb{I} \end{pmatrix} = \begin{pmatrix} I + R I^{-1} R & -R I^{-1} \\ -I^{-1} R & I^{-1} \end{pmatrix}$$

$$\mathcal{M} = \mathcal{M}(R, I) = \mathcal{M}(\mathrm{Re}(\mathcal{N}), \mathrm{Im}(\mathcal{N})).$$

$$\mathcal{M}^T = \mathcal{M} \qquad \mathcal{M}\mathbb{C}\mathcal{M} = \mathbb{C}$$

$$\mathcal{M} = -\left(\mathbf{L}\mathbf{L}^T\right)^{-1} = -\mathbf{L}^{-T}\mathbf{L}^{-1},$$

L = element of the **Sp(2n,R)**-bundle over the scalar manifold
(= coset repr. for homogeneous spaces **G/H**)

…by virtue of this matrix, one can introduce a (scalar-dependent) **anti-involution** in *any* Maxwell-Einstein-scalar theory with symplectic structure :

$$\mathcal{S}(\varphi) \quad : \quad = \mathbb{C}\mathcal{M}(\varphi)$$

$$\mathcal{S}^2(\varphi) \;=\; \mathbb{C}\mathcal{M}(\varphi)\,\mathbb{C}\mathcal{M}(\varphi) = \mathbb{C}^2 = -\mathbb{I},$$

Ferrara,AM,Yeranyan; Borsten,Duff, Ferrara,AM

…in turn, this allows to define an **anti-involution** on the dyonic charge vector Q, which has been called (**scalar-dependent**) **Freudenthal duality**

$$\mathfrak{F}(\mathcal{Q}) := -\mathcal{S}(\varphi)(\mathcal{Q}).$$

$$\mathfrak{F}^2 \;=\; -Id.$$

By recalling

$$V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\,\mathcal{Q},$$

Freudenthal duality can be related to the **effective BH potential** :

$$\mathfrak{F} : \mathcal{Q} \rightarrow \mathfrak{F}(\mathcal{Q}) := \mathbb{C}\frac{\partial V_{BH}}{\partial \mathcal{Q}}.$$

All this enjoys a remarkable physical interpretation when evaluated **at the horizon** :

Attractor Mechanism $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau\rightarrow-\infty}\varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

Bekenstein-Hawking entropy $S = \frac{A_H}{4} = \pi \, V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T\mathcal{M}_H\mathcal{Q}$

…by evaluating the matrix M at the horizon $\lim_{\tau\rightarrow-\infty}\mathcal{M}(\varphi(\tau)) =: \mathcal{M}^H.$

one can define the **horizon Freudenthal duality** as:

$$\lim_{\tau\rightarrow-\infty}\mathfrak{F}(\mathcal{Q}) =: \mathfrak{F}_H(\mathcal{Q}) = -\mathbb{C}\mathcal{M}_H\mathcal{Q} = \frac{1}{\pi}\mathbb{C}\frac{\partial S_{BH}}{\partial \mathcal{Q}} =: \tilde{\mathcal{Q}},$$

$$\mathfrak{F}^2_H(\mathcal{Q}) = \mathfrak{F}_H(\tilde{\mathcal{Q}}) = -\mathcal{Q}$$

non-linear (scalar-independent) anti-involutive map on Q (hom of degree one)

The Bekenstein-Hawking BH entropy is **invariant** :

$$S(\mathcal{Q}) = S(\mathfrak{F}_H(\mathcal{Q})) = S\left(\frac{1}{\pi}\mathbb{C}\frac{\partial S}{\partial \mathcal{Q}}\right) = S(\tilde{\mathcal{Q}})$$

Ferrara,AM,Yeranyan

➢ the matrix M also allows for a universal expression of the symplectic vector of Abelian 2-form field strengths **at the horizon**…

$$H := \left(F^{\Lambda}, G_{\Lambda}\right)^{T};$$
$$^{*}G_{\Lambda|\mu\nu} := 2\frac{\delta\mathcal{L}}{\delta F^{\Lambda|\mu\nu}}.$$

$$\begin{aligned} H\left(\varphi, U, \mathcal{Q}\right) &= e^{2U}\mathbb{C}\mathcal{M}\left(\varphi\right)\mathcal{Q}dt\wedge d\tau + \mathcal{Q}\sin\theta d\theta\wedge d\psi \\ &= -e^{2U}\mathfrak{F}\left(\mathcal{Q}\right)dt\wedge d\tau + \mathcal{Q}\sin\theta d\theta\wedge d\psi, \end{aligned}$$

Denef

…in terms of the **horizon Freudenthal duality** :

$$\begin{aligned} H_H &= e^{2U_H}\mathbb{C}\mathcal{M}^H\mathcal{Q}dt\wedge d\tau + \mathcal{Q}\sin\theta d\theta\wedge d\psi \\ &= -e^{2U_H}\tilde{\mathcal{Q}}dt\wedge d\tau + \mathcal{Q}\sin\theta d\theta\wedge d\psi = -\mathfrak{F}_H\left(^{*}H_H\right) \end{aligned}$$

➢ The matrix M also occurs in the metric of the **D=3 enlarged scalar manifold**, obtained as **dimensional reduction** of the D=4 bosonic sector

$$\begin{aligned} ds^2_{D=3=(+,+,\mp)} &= \frac{1}{4}\left[4dU^2 + 2g_{ij}\left(\varphi\right)d\varphi^i d\varphi^j + e^{-4U}\left(d\mathbf{a} + \mathbf{Z}^T\mathbb{C}d\mathbf{Z}\right)^2 \mp 2e^{-2U}d\mathbf{Z}^T\mathcal{M}\left(\varphi\right)d\mathbf{Z}\right] \\ &= \frac{1}{4}\left[4dU^2 + 2ds^2_{D=4} + e^{-4U}\left(d\mathbf{a} + \langle\mathbf{Z}, d\mathbf{Z}\rangle\right)^2 \mp 4e^{-2U}V_{BH}\left(d\mathbf{Z}\right)\right] \end{aligned}$$

(+,+,-) : spacelike dim reduction : in sugra = **c-map** [Cecotti,Ferrara,Girardello]

(+,+,+) : timelike dim reduction : in sugra = **c*-map** [Breitenlohner,Gibbons,Maison]

What about dependence on **flat directions**? $\{\varphi\} = \{\tilde{\varphi}\} \cup \{\varphi_{flat}\}$

at least in symmetric spaces

Ferrara,AM

in the neighbourhood of *attractors* supported by the G-orbit $\mathcal{O} = \frac{G}{\mathcal{H}}$:

$$\frac{G}{H} \simeq [(G/H) \setminus (\mathcal{H}/mcs(\mathcal{H}))] \times \frac{\mathcal{H}}{mcs\,(\mathcal{H})}$$

The matrix M depends on flat dirs (also at the horizon) : $\frac{\partial \mathcal{M}}{\partial \varphi_{flat}} \neq 0; \quad \frac{\partial \mathcal{M}^H}{\partial \varphi_{flat}} \neq 0.$

…this is still true for its **contraction with Q** :

$$\frac{\partial\,(\mathcal{M}\mathcal{Q})}{\partial \varphi_{flat}} = \frac{\partial \mathcal{M}}{\partial \varphi_{flat}} \mathcal{Q} \neq 0. \quad \longrightarrow \quad \frac{\partial H}{\partial \varphi_{flat}} \neq 0. \quad \frac{\partial \mathfrak{F}\,(\mathcal{Q})}{\partial \varphi_{flat}} \neq 0;$$

…but **not** at the **horizon** :

$$\frac{\partial\,(\mathcal{M}^H\mathcal{Q})}{\partial \varphi_{flat}} = \frac{\partial \mathcal{M}^H}{\partial \varphi_{flat}} \mathcal{Q} = 0, \quad \longrightarrow \quad \frac{\partial H_H}{\partial \varphi_{flat}} = 0,$$

Attractor Mechanism for **2-form field strengths**

$$\frac{\partial \mathfrak{F}_H\,(\mathcal{Q})}{\partial \varphi_{flat}} = 0.$$

Attractor Mechanism for **Freudenthal duality**

Lie groups "of type E_7" : (G,**R**)

Brown;
Krutelevich;
Ferrara,Kallosh,AM;
AM,Orazi,Riccioni

❖ the (ir)repr. **R** is **symplectic** :

$$\exists! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}; \quad \langle Q_1, Q_2 \rangle \equiv Q_1^M Q_2^N \mathbb{C}_{MN} = -\langle Q_2, Q_1 \rangle ;$$

symplectic product

❖ the (ir)repr. admits a unique completely symmetric **invariant rank-4** tensor

$$\exists!\ K_{MNPQ} = K_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}]_s$$

(K-tensor)

G-invariant quartic polynomial

$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q =: \epsilon \, |I_4| \, , \quad \longrightarrow \quad S_{BH} = \pi \sqrt{|I_4|}$$

❖ defining a triple map in **R** as

$$T \colon \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R} \qquad \langle T(Q_1, Q_2, Q_3), Q_4 \rangle \equiv K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q$$

it holds

$$\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle \, K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q$$

All electric-magnetic duality groups of D=4 sugras with **symmetric** scalar manifolds and *at least* **8** supersymmetries are **"of type E_7"**

$N=2$

G	R
$U(1,n)$	$(\mathbf{1+n})$
$SL(2,\mathbb{R})\times SO(2,n)$	$(\mathbf{2},\mathbf{2+n})$
$SL(2,\mathbb{R})$	$\mathbf{4}$
$Sp(6,\mathbb{R})$	$\mathbf{14'}$
$SU(3,3)$	$\mathbf{20}$
$SO^*(12)$	$\mathbf{32}$
$E_{7(-25)}$	$\mathbf{56}$

N	G	R
3	$U(3,n)$	$(\mathbf{3+n})$
4	$SL(2,\mathbb{R})\times SO(6,n)$	$(\mathbf{2},\mathbf{6+n})$
5	$SU(1,5)$	$\mathbf{20}$
6	$SO^*(12)$	$\mathbf{32}$
8	$E_{7(7)}$	$\mathbf{56}$

"degenerate" groups "of type E_7"

$$I_4(p,q)=\left(I_2(p,q)\right)^2$$

$$S_{BH}=\pi\sqrt{|I_4(p,q)|}=\pi\,|I_2(p,q)|\,.$$

In sugras with electric-magnetic duality group **"of type E_7"**, the G-invariant **K-tensor** determining the extremal BH Bekenstein-Hawking entropy

$$S_{BH} = \pi\sqrt{|I_4|}$$

$$I_4 := K_{MNPQ} \mathcal{Q}^M \mathcal{Q}^N \mathcal{Q}^P \mathcal{Q}^Q =: \epsilon\, |I_4|\,,$$

can generally be expressed as adjoint-trace of the product of G-generators (dim **R** = 2n, and dim **Adj** = d)

$$K_{MNPQ} = -\frac{n\,(2n+1)}{6d}\left[t^{\alpha}_{MN} t_{\alpha|PQ} - \frac{d}{n(2n+1)}\mathbb{C}_{M(P}\mathbb{C}_{Q)N}\right]$$

The **horizon Freudenthal duality** can be expressed in terms of the **K-tensor**

$$\mathfrak{F}_H(\mathcal{Q})_M = \tilde{\mathcal{Q}}_M = \frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial \mathcal{Q}^M} = \epsilon\frac{2}{\sqrt{|I_4(\mathcal{Q})|}} K_{MNPQ} \mathcal{Q}^N \mathcal{Q}^P \mathcal{Q}^Q$$

Borsten,Dahanayake,Duff,Rubens

the **invariance** of the BH entropy under **horizon Freudenthal duality** reads as

$$I_4\,(\mathcal{Q}) = I_4(\mathbb{C}\tilde{\mathcal{Q}}) = I_4\left(\mathbb{C}\frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial \mathcal{Q}}\right)$$

In sugras with electric-magnetic duality group **"of type E_7"**, and **in absence of flat directions** at the attractor points :

$$\mathcal{M}^H_{MN}(\mathcal{Q}) = -\frac{1}{\sqrt{I_4}}\left(2\,\tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N - 6K_{MN} + \mathcal{Q}_M\mathcal{Q}_N\right)$$

(at $I_4>0$)

$$K_{MN} := K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q$$

M at the horizon :
expressed only in terms of **Q**, of the **Freudenthal-dual of Q**, and of **K-tensor**

M at the horizon satisfies

$$M^H_{MN}M^H_{PQ}\mathbb{C}^{NP} = \mathbb{C}_{MQ}\,;$$

$$M^H_{MN}\mathcal{Q}^M\mathcal{Q}^N = -2\left|I_4\right|^{1/2}$$

…and it is **invariant** under **horizon Freudenthal duality** :

$$\mathfrak{F}_H\left(\mathcal{M}^H_{MN}\right) := \mathcal{M}^H_{MN}(\tilde{\mathcal{Q}}) = \mathcal{M}^H_{MN}(\mathcal{Q}).$$

<u>What about dependence on **flat directions**?</u> $\{\varphi\} = \{\tilde{\varphi}\} \cup \{\varphi_{flat}\}$

general split (everywhere in the scalar manifold): Ferrara,AM,Orazi,Trigiante

$$\mathcal{M} = \mathcal{M}_1(\tilde{\varphi}, \varphi_{flat})\,\mathcal{M}_0(\varphi_{flat}) \qquad [\mathcal{M}_1, \mathcal{M}_0] \neq 0$$

at the horizon :

$$lim_{\tau \to -\infty}: \; \mathcal{M}^H = \mathcal{M}_1^H(\tilde{\varphi}_H(\mathcal{Q}))\,\mathcal{M}_0(\varphi_{flat}) = \mathcal{M}^H(\mathcal{Q}, \varphi_{flat})$$

Attractor Mechanism : $\dfrac{\partial \mathcal{M}_1^H}{\partial \varphi_{flat}} = 0$

…and the split gets **commutative** : $\lim_{\tau \to -\infty} [\mathcal{M}_1, \mathcal{M}_0] = 0$

it can also be proved that in general $\mathcal{M}^H \in G$

Are there other relevant symplectic matrices at the horizon ? YES!

general conditions :

$$M_{\pm}^{H}(\mathcal{Q})^{T}\mathbb{C}M_{\pm}^{H}(\mathcal{Q}) = \epsilon\mathbb{C} \; ; \qquad \left(M_{\pm}^{H}\right)^{T}(\mathcal{Q}) = M_{\pm}^{H}(\mathcal{Q})$$

$$\mathcal{Q}^{T}M_{\pm}^{H}(\mathcal{Q})\mathcal{Q} = -2\sqrt{|I_4|},$$

2 solutions exist :

1]

$$M_{+|MN}^{H} = -\frac{2}{\sqrt{|I_4|}}\tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N + \epsilon\frac{6}{\sqrt{|I_4|}}K_{MN} - \epsilon\frac{1}{\sqrt{|I_4|}}\mathcal{Q}_M\mathcal{Q}_N$$

$$\mathfrak{F}_H(M_{+}^{H}) = M_{+}^{H}(\tilde{\mathcal{Q}}) = \epsilon M_{+}^{H}(\mathcal{Q}).$$

$$\left(M_{+}^{H}\right)^{-1}\hat{R}_{\mathcal{Q}}M_{+}^{H} \subset \hat{R}_{\mathcal{Q}} \Leftrightarrow M_{+}^{H} \in \mathrm{Aut}(G),$$

symplectic representation of G-transformations in **R** $\hat{R}_{\mathcal{Q}}^{T}\mathbb{C}\hat{R}_{\mathcal{Q}} = \mathbb{C}.$

$$I_4 > 0: \quad M_{+}^{H}\mathbb{C}M_{+}^{H} = \mathbb{C} \Longrightarrow M_{+}^{H} \in \mathrm{Inn}(G) \subset \mathrm{Aut}(G)$$

symplectic (inner) automorphism

$$I_4 < 0: \quad M_{+}^{H}\mathbb{C}M_{+}^{H} = -\mathbb{C} \Longrightarrow M_{+}^{H} \in \frac{\mathrm{Aut}(G)}{\mathrm{Inn}(G)}$$

anti-symplectic (outer) automorphism

Relation with the matrix M defining the BH effective potential :

$$\mathcal{M}^{H}(\mathcal{Q},\varphi_{flat}) = M_{+}^{H}(\mathcal{Q})\mathcal{A}(\mathcal{Q},\varphi_{flat}).$$

this allows to introduce a (generally scalar-dependent) matrix

$$\mathcal{A}^{T}\, M_{+}^{H}(\mathcal{Q})\mathcal{A} = M_{+}^{H}(\mathcal{A}^{-1}\mathcal{Q}) = M_{+}^{H}(\mathcal{Q}) \Longrightarrow \mathcal{A} \in \mathrm{Stab}_{\mathcal{Q}}(GL\,(2n,\mathbb{R}))$$

$$\mathcal{M}^{H} \in G, \mathcal{M}_{+}^{H} \in \mathrm{Aut}(G) \Longrightarrow \mathcal{A} \in \mathrm{Aut}(G)$$

$$\mathcal{A} \in \mathrm{Aut}(G) \cap \mathrm{Stab}_{\mathcal{Q}}(GL\,(2n,\mathbb{R})), \quad \mathcal{A}^{T} = \mathcal{A},\ \mathcal{A}^{2} = Id,$$

$$I_4 > 0 : \quad \left.\begin{array}{l} \mathcal{M}^{H}\mathbb{C}\mathcal{M}^{H} = \mathbb{C} \\ \\ M_{+}^{H}\mathbb{C}M_{+}^{H} = \mathbb{C} \end{array}\right\} \Rightarrow \mathcal{A}\mathbb{C}\mathcal{A} = \mathbb{C} : \mathcal{A} \in \frac{\mathrm{Inn}(G)}{\mathcal{H}_0} \cap \mathrm{Stab}_{\mathcal{Q}}\left[Sp\,(2n,\mathbb{R})\right]$$

$$I_4 < 0 : \quad \left.\begin{array}{l} \mathcal{M}^{H}\mathbb{C}\mathcal{M}^{H} = \mathbb{C} \\ \\ M_{+}^{H}\mathbb{C}M_{+}^{H} = -\mathbb{C} \end{array}\right\} \Rightarrow \mathcal{A}\mathbb{C}\mathcal{A} = -\mathbb{C} : \mathcal{A} \in \frac{\mathrm{Aut}(G)}{H_5} \cap \mathrm{Stab}_{\mathcal{Q}}\left[GL\,(2n,\mathbb{R})\right]$$

Dependence on flat directions ? → *general* split :

$$\mathcal{A}\left(Q,\varphi_{flat}\right)=\mathcal{A}_1(\mathcal{Q})\mathcal{A}_0\left(\varphi_{flat}\right)=\mathcal{A}_1(\mathcal{Q})\mathcal{M}_0\left(\varphi_{flat}\right)$$

in absence of flat directions :

$$\mathcal{M}_0\left(\varphi_{flat}\right)=Id\Longrightarrow\frac{\partial\mathcal{A}}{\partial\varphi_{flat}}=0,\ \mathcal{A}=\mathcal{A}_1(\mathcal{Q})\Longrightarrow\mathcal{M}^H=M_+^H(\mathcal{Q})\mathcal{A}_1(\mathcal{Q})=\mathcal{M}^H(\mathcal{Q})$$

in presence of flat directions :

$$I_4>0:\mathcal{M}_0\left(\varphi_{flat}\right)=Id\Longrightarrow\mathcal{A}=\mathcal{A}_1(\mathcal{Q})=Id\Longrightarrow\mathcal{M}^H=M_+^H(\mathcal{Q})$$

$$I_4<0:\mathcal{M}_0\left(\varphi_{flat}\right)=Id\Rightarrow\mathcal{A}=\mathcal{A}_1(\mathcal{Q})\neq Id\Rightarrow\mathcal{M}^H=\mathcal{M}^H=M_+^H(\mathcal{Q})\mathcal{A}_1(\mathcal{Q})\neq M_+^H(\mathcal{Q})$$

2] second solution : $M^{H}_{-|MN} = \frac{1}{\sqrt{|I_4|}}\tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N - \epsilon\frac{6}{\sqrt{|I_4|}}K_{MN}$

By comparing with

$$M^{H}_{+|MN} = -\frac{2}{\sqrt{|I_4|}}\tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N + \epsilon\frac{6}{\sqrt{|I_4|}}K_{MN} - \epsilon\frac{1}{\sqrt{|I_4|}}\mathcal{Q}_M\mathcal{Q}_N$$

we notice the **absence of the Q_MQ_N term**

it generally holds : $M^{H}_{-|MN} = -\partial_M\partial_N\sqrt{|I_4|},$

(opposite of the) **Hessian of the BH entropy** $S_{BH} = \pi\sqrt{|I_4|}$

$$\left(M^{H}_{-}\right)^{-1}\hat{R}_{\mathcal{Q}}M^{H}_{-} \not\subset \hat{R}_{\mathcal{Q}} \Leftrightarrow M^{H}_{-} \notin \mathrm{Aut}(G).$$

This matrix is **not** an automorphism of G

Observation :

For $M^{H}_{-|MN} = -\partial_M \partial_N \sqrt{|I_4|},$ the property $M^{H}_{-|MN} \mathcal{Q}^M \mathcal{Q}^N = -2\sqrt{|I_4|}$

directly follows from the fact that the symplectic vector $\frac{\partial \sqrt{|I_4|}}{\partial \mathcal{Q}^M}$

is **homogeneous of degree 1** in the charges Q :

$$\frac{\partial^2 \sqrt{|I_4|}}{\partial \mathcal{Q}^N \partial \mathcal{Q}^M} \mathcal{Q}^N = \frac{\partial \sqrt{|I_4|}}{\partial \mathcal{Q}^M}$$

thus implying :

$$\frac{\partial^2 \sqrt{|I_4|}}{\partial \mathcal{Q}^N \partial \mathcal{Q}^M} \mathcal{Q}^M \mathcal{Q}^N = \frac{\partial \sqrt{|I_4|}}{\partial \mathcal{Q}^M} \mathcal{Q}^M = 2\sqrt{|I_4|} \Longleftrightarrow M^{H}_{-|MN} \mathcal{Q}^M \mathcal{Q}^N = -2\sqrt{|I_4|} \blacksquare$$

$$M^H_{-|MN} = \frac{1}{\sqrt{|I_4|}}\tilde{Q}_M\tilde{Q}_N - \epsilon\frac{6}{\sqrt{|I_4|}}K_{MN} = -\partial_M\partial_N\sqrt{|I_4|},$$

This matrix is the (opposite of) **pseudo-Euclidean metric** of a non-compact, non-Riemannian **rigid special Kaehler manifold** related to the **G-orbit** of BH em charges, which is an example of **pre-homogeneous vector space** (**PVS**)

Sato, Kimura

1st example : ***"large"*** **BPS** BH charge orbit in **maximal supergravity**

$$N=8, D=4: scalar\ manifold\ \mathrm{M}_{N=8} = \frac{E_{7(7)}}{SU(8)},\ dim_{\mathbb{R}} = 70,\ rank = 7$$

$$I_4 > 0: \frac{1}{8}\text{-}BPS\ E_{7(7)}\text{-}orbit\ in\ \mathbf{56}\ repr.space: \mathcal{O}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)}}$$

$$(quaternionic)\ moduli\ space\ \mathcal{M}_{I_4>0} = \frac{E_{6(2)}}{SU(6)\times SU(2)}\left(\subset\frac{E_{7(7)}}{SU(8)}\right),\ dim_{\mathbb{R}} = 40,\ rank = 4$$

$$M^H_- = -\partial^2\sqrt{I_4}: metric\ of\ \mathcal{O}_{I_4>0}\times\mathbb{R}^+ = \frac{E_{7(7)}}{E_{6(2)}}\times\mathbb{R}^+;\ (n_+, n_-) = (30, 26)$$

$$M^H_{-|MN} = \frac{1}{\sqrt{|I_4|}}\tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N - \epsilon\frac{6}{\sqrt{|I_4|}}K_{MN} = -\partial_M\partial_N\sqrt{|I_4|},$$

2nd example : ***"large"* non-BPS** BH charge orbit in **maximal supergravity**

$$I_4 < 0 : non-BPS\ E_{7(7)}-orbit\ in\ \mathbf{56}\ repr.space : \mathcal{O}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)}}$$

$$(real)\ m.s.\ \mathcal{M}_{I_4<0} = \frac{E_{6(6)}}{USp(8)} = \mathrm{M}_{N=8,D=5},\ dim_{\mathbb{R}} = 42,\ rank = 6$$

$$M^H_- = -\partial^2\sqrt{-I_4} : metric\ of\ \mathcal{O}_{I_4<0}\times\mathbb{R}^+ = \frac{E_{7(7)}}{E_{6(6)}}\times\mathbb{R}^+;\ \ (n_+,n_-) = (28,28)$$

Further results in generic N=2, D=4 sugras :

❖ **in any symplectic frame**, the matrix $\mathcal{M}(z,\bar{z}) = \mathcal{M}[\mathrm{Re}\mathcal{N}, \mathrm{Im}\mathcal{N}]$,
defining the BH effective potential
enjoys the general expression in terms of symplectic sections of SKG

$$\mathcal{M}(z,\bar{z}) = \mathbb{C}\left(V\bar{V}^T + \bar{V}V^T + U_i\, g^{i\bar{\jmath}}\bar{U}_{\bar{\jmath}}^T + \bar{U}_{\bar{\jmath}} g^{\bar{\jmath}i} U_i^T\right)\mathbb{C}\,.$$

❖ whenever an **holomorphic prepotential** F(X) exists
[and there are symplectic frame in which it does not!]

one can define the complex symmetric matrix

$$\mathcal{F}_{\Lambda\Sigma}(X) = \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma}\,,$$

and thus construct the real symmetric matrix

$$\mathcal{M} = \begin{pmatrix} \mathbb{I} & -R \\ 0 & \mathbb{I} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix}\begin{pmatrix} \mathbb{I} & 0 \\ -R & \mathbb{I} \end{pmatrix} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}$$

$$\mathcal{M}^{(F)} = \mathcal{M}^{(F)}(R,I) = \mathcal{M}^{(F)}\left(\mathrm{Re}\left(\mathcal{F}\right), \mathrm{Im}\left(\mathcal{F}\right)\right) = \mathcal{M}|_{\mathcal{N}_{\Lambda\Sigma}\longrightarrow\mathcal{F}_{\Lambda\Sigma}}$$

$$\mathcal{M}(z,\bar{z}) = -\mathcal{M}^{(F)}(z,\bar{z}) + 2\,\mathbb{C}\left(V\bar{V}^T + \bar{V}V^T + U_i\, g^{i\bar{\jmath}}\bar{U}_{\bar{\jmath}}^T\right)\mathbb{C}\,,$$

…and, *at least* at BPS attractors :

$$\mathcal{M}^H = M_+^H$$

$$\mathcal{M}^{(F)H} = M_-^H$$

Hints for Further Future Developments

❖ application to **homogeneous non-symmetric** special Kaehler manifolds
[classification available : deWit, Van Proeyen; Alekseevsky;]

❖ extension to **“small” G-orbits** : how to define Freudenthal duality ?

❖ extension to the **gauged sugras** :
how to incorporate the **embedding tensor formalism** within this framework ?

❖ extension to **multi-centered (extremal) BH solutions**:
first steps done in Yeranyan, arXiv :1205.5618
Ferrara,AM,Shcherbakov,Yeranyan, arXiv:1211.3262

❖ beyond the large charge (sugra) regime :
Freudenthal duality for integer, quantized charges ? Duff *et al.* arXiv:0903.5517

❖ application of the matrix M^{H}_{-} (Hessian of BH entropy) to the geometric approach to **BH thermodynamics** (*à la Weinhold-Ruppeiner*)

Thank You!