Hydrodynamics of quantum critical points

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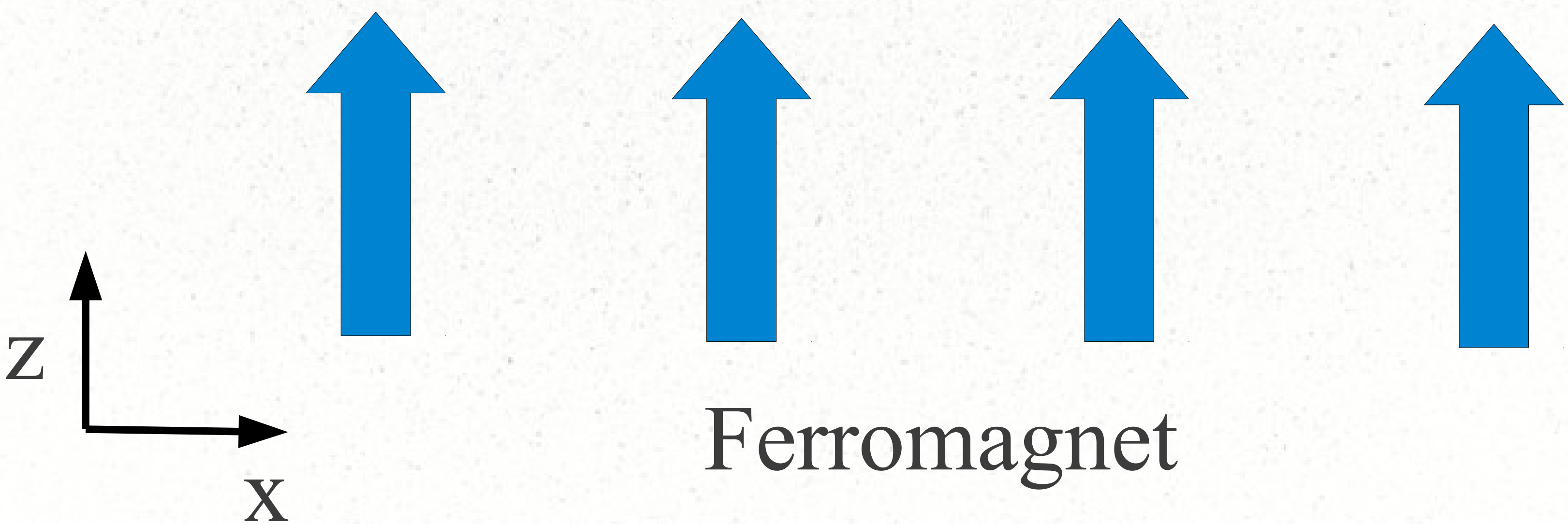
Quantum Critical Points

Example: Ising model

One-dimensional spin chain

$$H_I = -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$



Example: Ising model

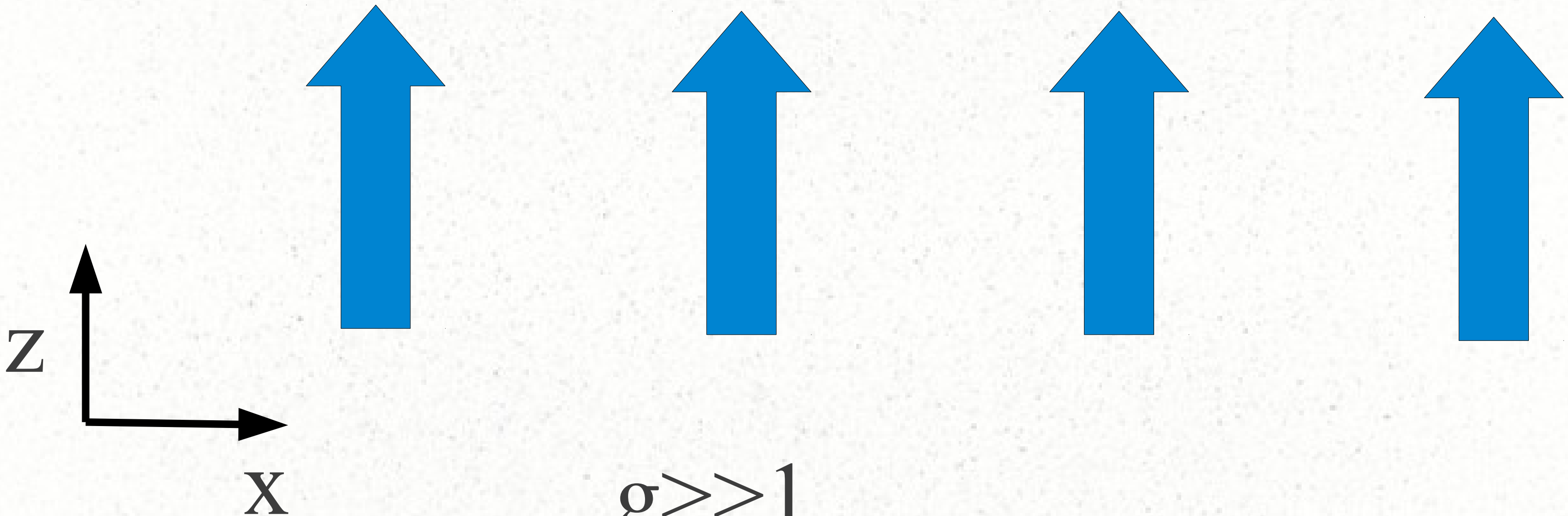
One-dimensional spin chain

$$H_I = -J \sum_i \left(g\hat{\sigma}_i^x + \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \right)$$

Magnetic field

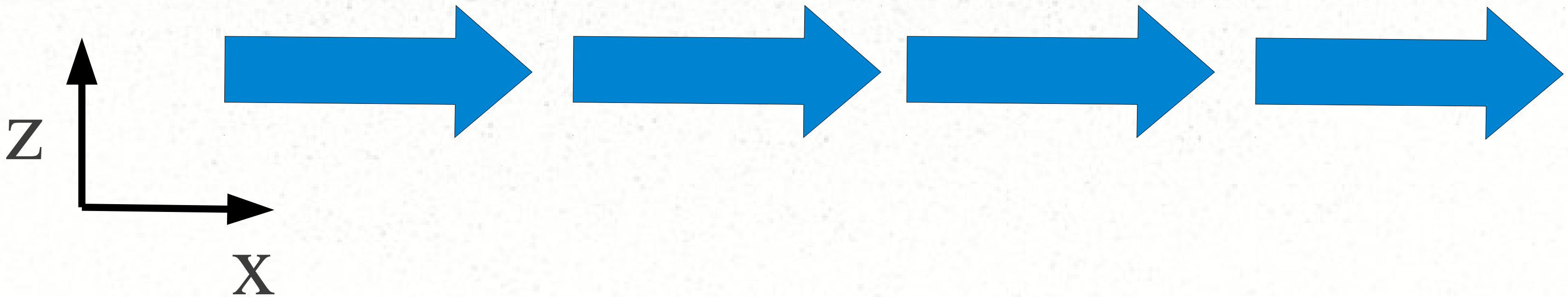
$g \ll 1$

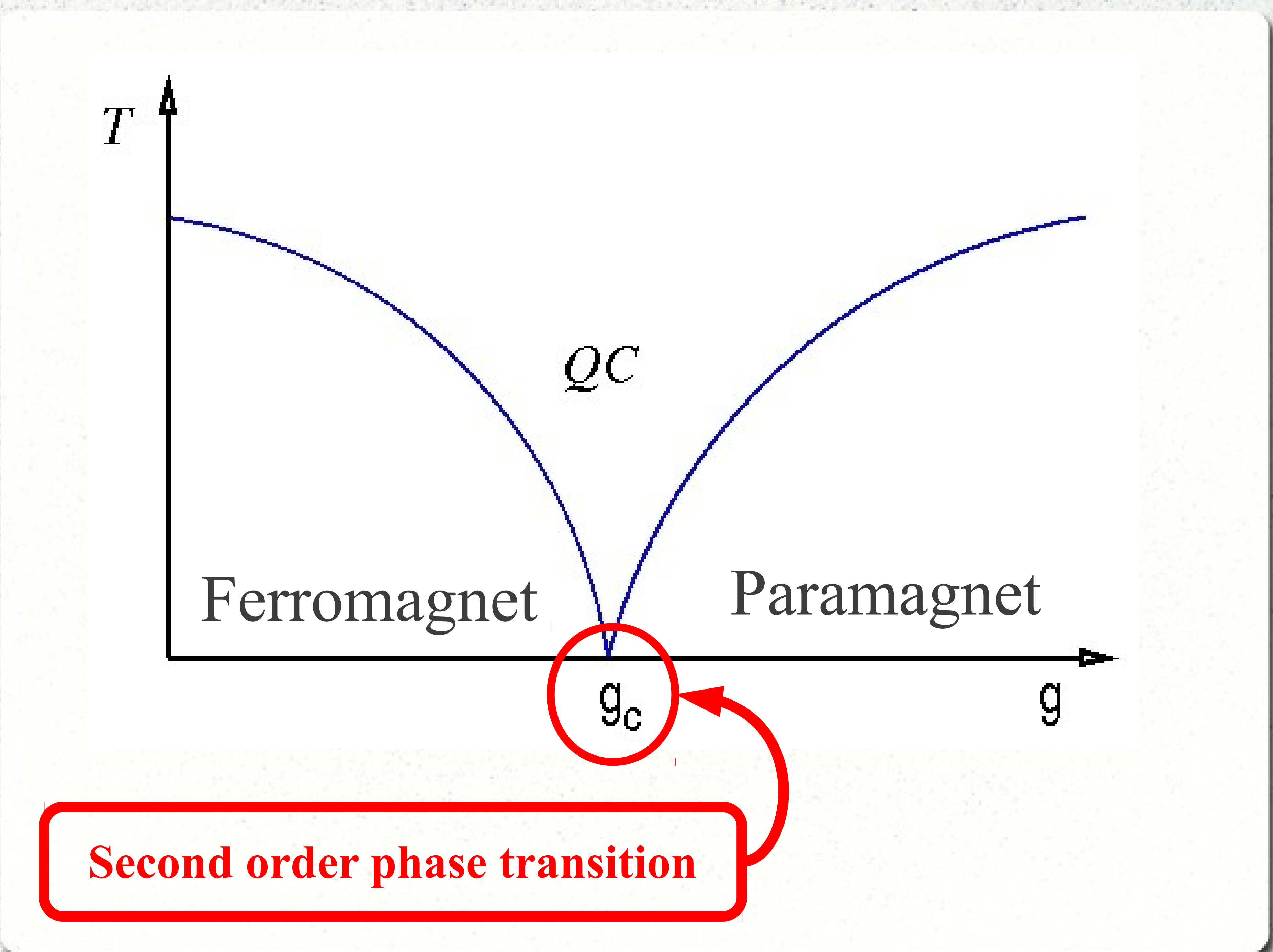
Ferromagnet



$g \gg 1$

Paramagnet





T
QC
Ferromagnet
Paramagnet
g_c
g
Second order phase transition

- **Energy gap:**

$$\Delta \sim J|g - g_c|^{z\nu}$$

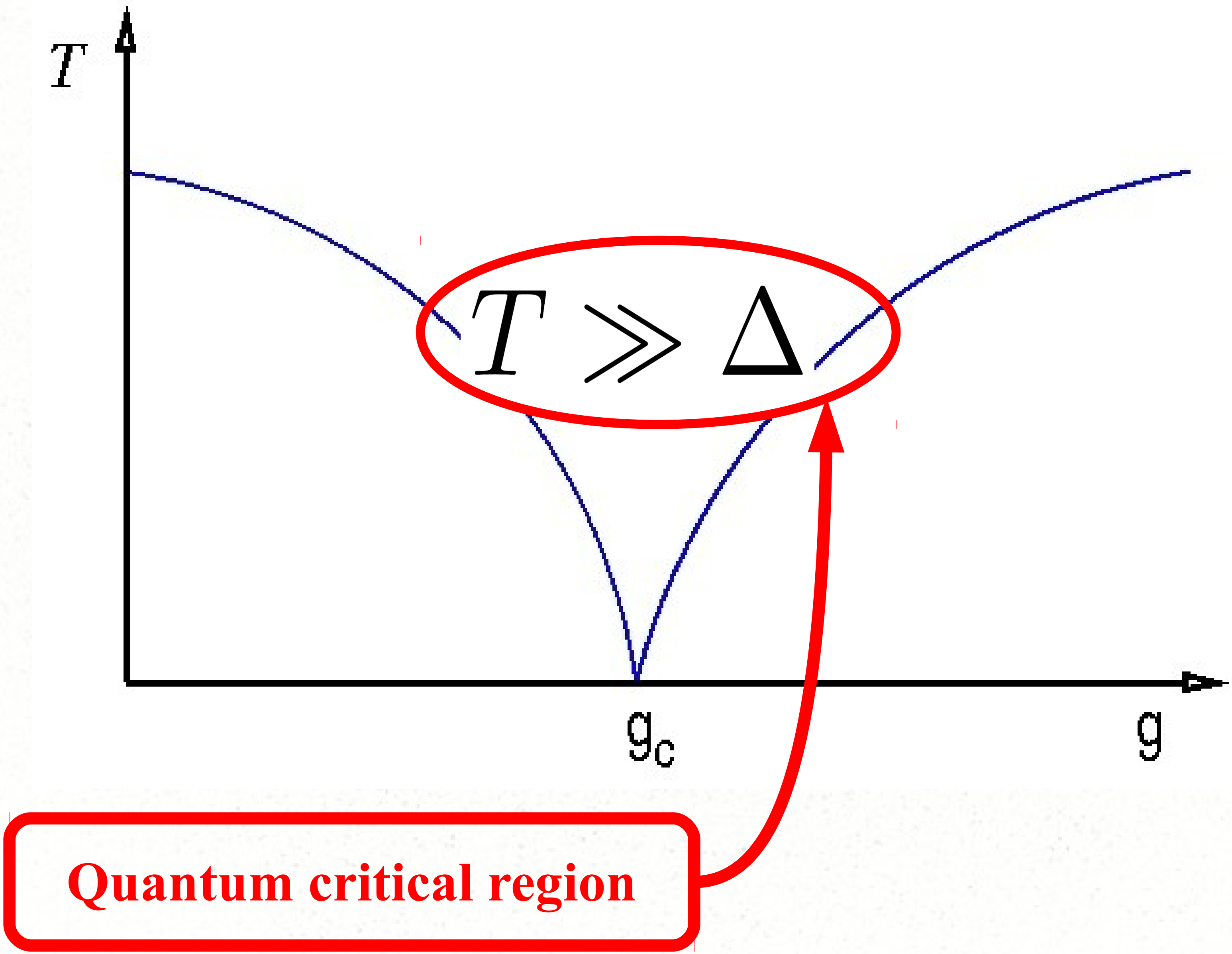
- **Correlation length:**

$$\xi^{-1} \sim \Lambda|g - g_c|^{\nu}$$

- **Critical point:**

$$\Delta \to 0 \quad \xi \to \infty$$

(Ising model: $z = \nu = 1$)


T
$T \gg \Delta$
g_c
g
Quantum critical region

Other Examples

- **Other spin chain models**
- **O(N) (quantum rotor) models**
- **Hubbard models**
- **Fermi liquids**
- **Bose-Einstein condensates**

Experimental relevance

- **Insulators with magnetic properties**

- **Strange metals:**

  - Heavy fermion compounds
  - Materials with high Tc superconductivity

Experiments can be made only at finite temperature

What is the signature of the quantum critical point?

Equilibration/mean free time

Quantum critical:

$$\tau \sim \hbar / k_B T$$

Non-critical:

$$\tau \sim (\hbar / k_B T) e^{|\Delta| / k_B T}$$

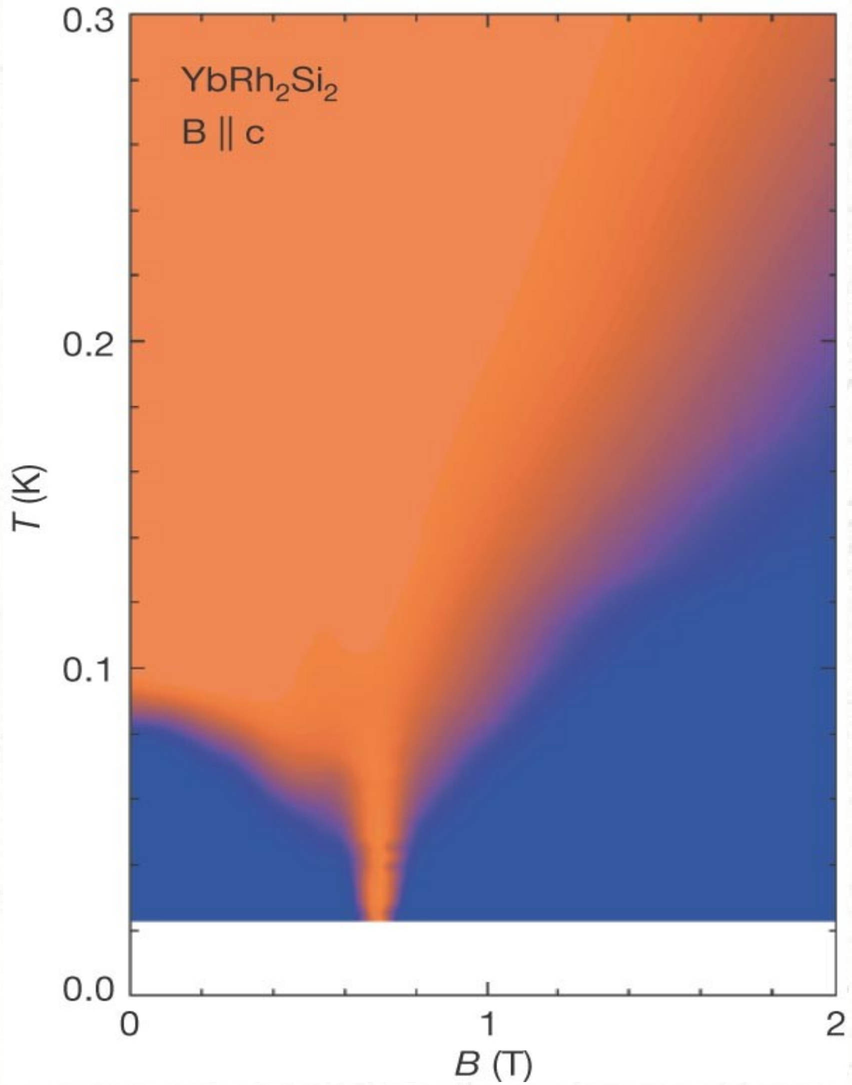
Large separation of time scales: classical relaxation (collisions of quasiparticles)

Equilibration/mean free time

Resistivity in critical region:

$$\boldsymbol{\rho}_{DC} = \frac{m_*}{4\pi e^2 n}\tau^{-1} \sim \frac{m_*}{4\pi e^2 n}\frac{k_B T}{\hbar}$$

Good agreement with experiments



Orange:
Resistivity Linear in Temperature

[Coleman, Schofield Nature 433, 226]

Quantum Critical Points

- **Continuum limit: quantum field theory**
- **Scaling symmetry at the critical point:**

$$t \to \lambda^z t, \quad x^i \to \lambda x^i$$

- **Free field example: "Lifshitz theory"**

$$S = \int dt\, d^d x \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{\kappa}{2z} ((\partial_i^2)^{z/2} \phi)^2 \right]$$

Connections to High Energy Physics

- There are now many examples of gravity duals with scale invariance
- **Canonical example: conformal field theories i.e. relativistic quantum critical points**
- Time scales in strongly coupled gauge theories with gravity duals

$$\tau \sim \frac{\eta}{Ts} \sim \hbar / k_B T$$

Hydrodynamic description very successful!

From gauge/gravity duality models, we expect a universal hydrodynamic description of scale-invariant theories

This was already argued by condensed matter physicists, but not developed in the same way as for relativistic theories

[Sachdev & Ye]

We initiate the formulation of hydrodynamics of quantum critical points at finite temperature

Hydrodynamics in QCP

- **Graphene:** [Fritz, Schmalian, Muller, Sachdev]

  z=1 (relativistic fermions)

  Hydrodynamic Drude model

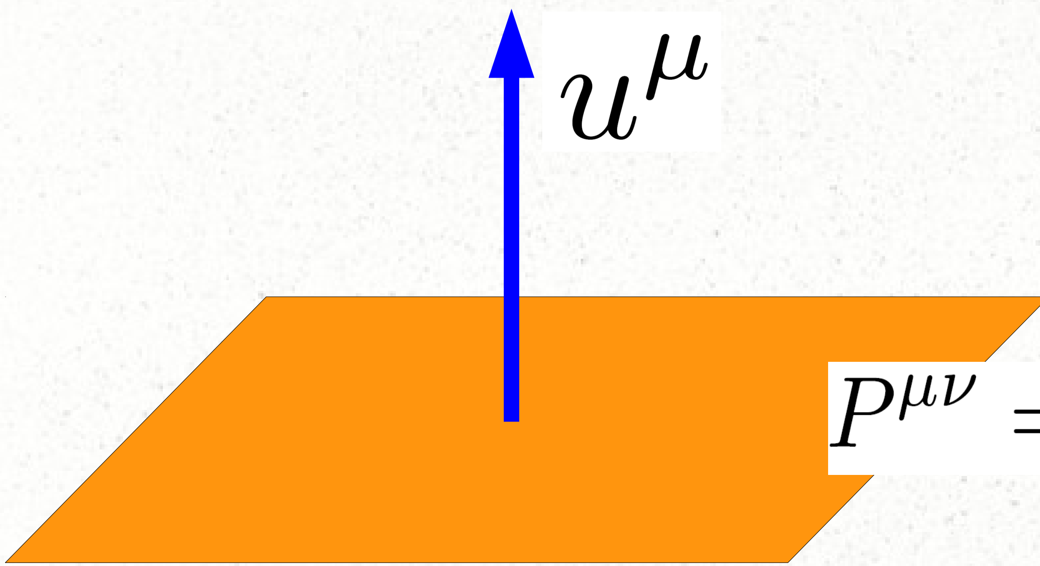
- **Fermions at unitarity:** [Cao et al.; Son & Wingate]

  z=2 (non-relativistic conformal invariance)

  Elliptic flow with low viscosity

Hydrodynamics

Time-like Killing vector: defines rest frame of the fluid



$$P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$$

$$\eta_{\mu\nu} u^\mu u^\nu = -1$$

$$u^\mu = (1, \beta^i)/\sqrt{1-\beta^2} \qquad \beta^i = \frac{v^i}{c}$$

Time derivatives:

$$\partial_t \phi = u^\mu \partial_\mu \phi$$

Space derivatives:

$$\nabla^2 \phi = P^{\mu\nu} \partial_\mu \partial_\nu \phi$$

Example: z=2 Lifshitz scalar

$$\mathcal{L} = \frac{1}{2}(u^\mu \partial_\mu \phi)^2 - \frac{\kappa}{4}(P^{\mu\nu}\partial_\mu \partial_\nu \phi)^2$$

Symmetry generators

Time and space translations:

$$P^{\|} = u^{\mu}\partial_{\mu}, \quad P_{\mu}^{\perp} = P_{\mu}{}^{\nu}\partial_{\nu}$$

Anisotropic dilatations:

$$D = z x^{\mu} u_{\mu} P^{\|} - x^{\mu} P_{\mu}^{\perp}$$

$$[D, P^{\|}] = z P^{\|} \;, \quad [D, P_{\mu}^{\perp}] = P_{\mu}^{\perp}$$

Ward identities

Conservation equations: $\partial_\mu T^{\mu\nu} = 0$

"Trace" of energy-momentum tensor:

$$zT_{\mu\nu}u^\mu u^\nu - T_{\mu\nu}P^{\mu\nu} = 0$$

Lorentz symmetry is broken:

$$T^{\mu\nu} \neq T^{\nu\mu}$$

Constitutive relations

Ideal energy-momentum tensor:

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu + p \eta^{\mu\nu}$$

Scale symmetry and equation of state:

$$z T_{\mu\nu} u^\mu u^\nu - T_{\mu\nu} P^{\mu\nu} = 0$$

$$\boxed{z\varepsilon = dp}$$

Temperature dependence

Scale symmetry:

$$z\varepsilon = dp$$

Thermodynamic relations:

$$\varepsilon + p = Ts \qquad s = \frac{\partial p}{\partial T}$$

Temperature dependence:

$$\varepsilon \sim p \sim T^{\frac{z+d}{z}}$$

Constitutive relations

Energy-momentum tensor:

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p\eta^{\mu\nu} + \pi_S^{(\mu\nu)} + \pi_A^{[\mu\nu]} + (u^\mu \pi_A^{[\nu\sigma]} + u^\nu \pi_A^{[\mu\sigma]})u_\sigma$$

$$T^{\mu\nu} \neq T^{\nu\mu}$$

Landau frame condition:

$$T^{\mu\nu}u_\nu = -\varepsilon u^\mu$$

Second law of thermodynamics

$$\partial_t S = \int d^d x \, \partial_t s \geq 0$$

Local form: $$\boxed{\partial_\mu s^\mu \geq 0}$$

$$\partial_\mu T^{\mu\nu} u_\nu = 0$$

Entropy current: $s^\mu = s u^\mu + \cdots$

Dissipative terms

Symmetric terms:

$$\pi_S^{(\mu\nu)} = -\eta P^{\mu\alpha} P^{\nu\beta} \Delta_{\alpha\beta} - \frac{\zeta}{d} P^{\mu\nu} \partial_\alpha u^\alpha$$

$$\Delta_{\alpha\beta} = 2\partial_{(\alpha} u_{\beta)} - \frac{2}{d} P_{\alpha\beta} (\partial_\sigma u^\sigma)$$

$$\partial_\mu s^\mu \geq 0 \quad \Rightarrow \eta \geq 0 \quad \zeta \geq 0$$

Dissipative terms

Anti-symmetric terms: $\partial_\mu s^\mu \geq 0$

$$\pi_A^{[\mu\nu]} = -\alpha^{\mu\nu\alpha\beta}(\partial_{[\alpha}u_{\beta]} - u_{[\alpha}u^\rho\partial_\rho u_{\beta]})$$

$$\tau_{\mu\nu}\alpha^{\mu\nu\sigma\rho}\tau_{\sigma\rho} \geq 0$$

Rotational invariance: $\alpha^{0i0j} = \alpha\delta^{ij} \geq 0$

Dissipative terms

Temperature dependence fixed by scaling:

$$\eta \sim \zeta \sim \alpha \sim T^{\frac{d}{z}}$$

New transport coefficient:

- **Dissipation due to non-inertial motion of the fluid**
- **Distinguishes Lifshitz from relativistic theories**

Application:
Drude model of strange metal

Non-relativistic limit

$$c \to \infty$$

$$\varepsilon = \rho c^2 - \rho \frac{v^2}{2} + U$$

Ideal equations of motion:

$$zU = dp$$

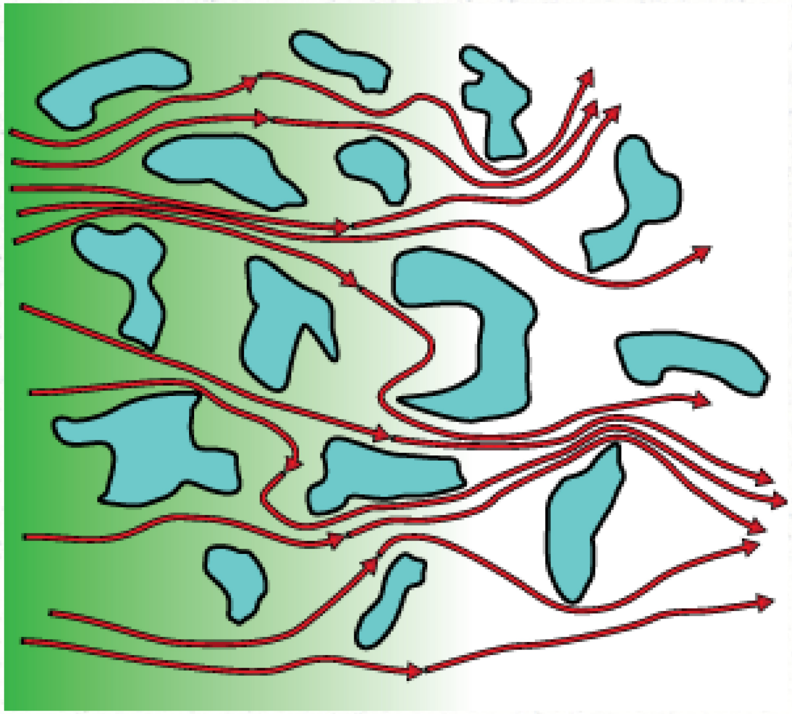
$$\partial_t \rho + \partial_i (\rho v^i) = 0$$

$$\partial_t U + \partial_i \left(U v^i \right) + p \partial_i v^i = 0$$

$$\partial_t (\rho v^i) + \partial_j (\rho v^j v^i) + \partial^i p = 0$$

Drude model: electron fluid moving though medium

- **Force by external electric field**
- **Drag force**



Ideal equations of motion:

$$\partial_t \rho + \partial_i(\rho v^i) = 0$$

$$\partial_t U + \partial_i \left(U v^i\right) + p\partial_i v^i = \lambda \rho v^2$$

$$\partial_t(\rho v^i) + \partial_j(\rho v^j v^i) + \partial^i p = \rho E^i - \lambda \rho v^i$$

Scaling dimensions:

$$[T] = z \qquad [v^i] = z - 1$$

$$[p] = [U] = z + d \qquad [\rho] = d + 2 - z$$

$$[\lambda] = z \longrightarrow \lambda \sim k_B T/\hbar$$

Conductivity:

$$J^i = \rho v^i \simeq \frac{\rho}{\lambda} E^i$$

Resistivity linear in T independent of z and d

Dissipative terms

$$\partial_t U + \partial_i \left(U v^i\right) + p \partial_i v^i$$

$$= \frac{\eta}{2} \sigma^{ij} \sigma_{ij} + \frac{\zeta}{d} (\partial_i v^i)^2 + \frac{\alpha}{2} (V_A^i)^2$$

Shear: $\sigma_{ij} = 2\partial_{(i} v_{j)} - (2/d)\delta_{ij} \partial_k v^k$

Linear acceleration:

$$D_t \equiv \partial_t + v^i \partial_i$$

$$V_A^i = 2 D_t v^i + \omega^{ij} v_j$$

Coriolis acceleration:

$$\omega_{ij} = 2\partial_{[i} v_{j]}$$

Corrections to the conductivity

$$\boldsymbol{\sigma}_{xx}(E_x) = \frac{\rho}{\lambda}\left[1 + \frac{1}{\rho\lambda E_x}\left(\eta\partial_y^2 E_x + \frac{\alpha}{6\lambda^2}\partial_y^2 E_x^3\right)\right]$$

Estimate for $E_x = E_0 y/L$

$$\frac{\boldsymbol{\sigma}_{xx} - \boldsymbol{\sigma}_{xx}^0}{\boldsymbol{\sigma}_{xx}^0} \sim 10^{-11}\left(\frac{m_e}{m_*}\right)^2\left(\frac{T}{\mathrm{K}}\right)^{-3}\frac{\alpha/\rho}{\mathrm{sec}}(\partial E_0)^2$$

$$(\partial E_0) = \frac{E_0/L}{\mathrm{N\,m^{-1}\,C^{-1}}}$$

Future directions

- **Transport coefficients beyond first order**
- **Additional conserved currents**
- **Superfluids with Lifshitz scaling**
- **Fluids in a curved space: Weyl anomaly?**
- **Anomalous currents**

Grazie!