Coulomb branches of 3d $\mathcal{N} = 4$ gauge theories

(Monopole operators and Hilbert series)

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Summary

Based on:

- SC, A. Hanany and A. Zaffaroni, arXiv:1309.2657 [hep-th].

What we do:

- Study the Coulomb branch of the moduli space of IR fixed points of 3d $\mathcal{N} = 4$ gauge theories of vector and hyper multiplets.
- Compute the Hilbert series: generating function of the chiral ring.

Why we do it:

- 3d mirror symmetry
- Instanton moduli spaces
- M-theory uplift of orientifold planes
- Non-Lagrangian Gaiotto theories

Moduli spaces of 3d $\mathcal{N} = 4$ gauge theories

HIGGS branch $\mathcal{M}_H$

- Hypermultiplets H_i, $i = 1, \ldots, n$
- $SU(2)_H$ R-symmetry
- $dim_{\mathbb{H}} \mathcal{M}_H = n - r$
- HyperKähler quotient

$$\{\vec{\mu}(H_i) = \vec{\xi}\}/G$$

  Triplet of D-term equations
- Non-renormalization Theorem

COULOMB branch $\mathcal{M}_C$

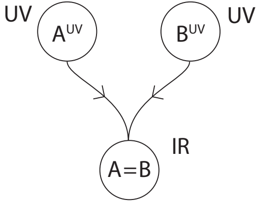
- Vector multipl's V_a, $a = 1, \ldots, r$
- $SU(2)_V$ R-symmetry
- $dim_{\mathbb{H}} \mathcal{M}_C = r$
- HyperKähler manifold:

$$(\vec{\sigma}_a, \varphi_a) , \quad d\varphi_a = *F_a$$

  at generic pts ($G \to U(1)^r$)
- Quantum corrections

- **Mirror Symmetry**: [Intriligator, Seiberg 1996]
  Infrared duality exchanging $\mathcal{M}_H$ and $\mathcal{M}_C$:

$$\mathcal{M}_H^A = \mathcal{M}_C^B$$
$$\mathcal{M}_C^A = \mathcal{M}_H^B$$



More on the Coulomb branch $\mathcal{M}_C$

- **Old problem**: How to dualise non-Abelian vector multiplet?
- **Modern answer**: Use 't Hooft monopole operators. ['t Hooft 1978]
  Can be defined in CFT. [Borokhov, Kapustin, Wu 2002]

Uses of monopole operators

- Chiral ring on the Coulomb branch (very simple theories) [BKW 2002] [Borokhov 2003]
- Global symmetry enhancement [Gaiotto, Witten 2008] [Bashkirov, Kapustin 2010]

We will go for the whole chiral ring of 3d $\mathcal{N} = 4$ SCFTs on $\mathcal{M}_C$.

Formula for the **Hilbert series** of $\mathcal{M}_C$, counting:

- Gauge invariant chiral operators modulo F-term relations
- Holomorphic functions on $\mathcal{M}_C$.

Plan

1. Hilbert series
2. Monopole operators
3. The Hilbert series of the Coulomb branch of a 3d $\mathcal{N} = 4$ SCFT
4. Examples
5. Outlook

Hilbert series

The Hilbert series of a graded ring

[Benvenuti, Feng, Hanany, He 2006]

Enumerates elements of a ring modulo relations, according to the grading.

- $\mathbb{C}[x]$: $$H_{\mathbb{C}}(t) = 1 + t + t^2 + t^3 + \cdots = \sum_{n=1}^{\infty} t^n = \frac{1}{1-t}$$
- $\mathbb{C}[x]/\langle x^N \rangle$: $$H(t) = 1 + t + t^2 + \cdots + t^{N-1} = \sum_{n=1}^{N} t^n = \frac{1-t^N}{1-t}$$
- $\mathbb{C}^2/\Gamma$ orbifold, $\Gamma \subset SU(2)$:

$$H_{\mathbb{C}^2/\Gamma}(t) = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} \frac{1}{\det(I_2 - g\, t)}$$

e.g. $\mathbb{C}^2/\mathbb{Z}_n$: $g \in \left\{\mathrm{diag}(\omega_n^k, \omega_n^{-k}),\ k = 0, \ldots, n-1\right\}$

$$H_{\mathbb{C}^2/\mathbb{Z}_n}(t) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{(1-\omega_n^k t)(1-\omega_n^{-k} t)} = \frac{1-t^{2n}}{(1-t^2)(1-t^n)^2}$$

Plethysm

- **Plethystic Exp PE** of $f(\vec{t})$ s.t. $f(\vec{0}) = 0$:

$$PE\left[f(t_1, t_2, \dots, t_n)\right] = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(t_1^k, \cdots, t_n^k)\right)$$

Generates symmetrization: $PE[t] = \frac{1}{1-t}$.

- **Plethystic Log PL** of $g(\vec{t})$ s.t. $g(\vec{0}) = 1$:

$$PL\left[g(t_1, t_2, \dots, t_n)\right] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(g(t_1^k, \cdots, t_n^k))$$

$$\mu(k) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \text{ has repeated prime factors} \\ (-1)^n & \text{if } k \text{ is the product of } n \text{ distinct primes} \end{cases}$$

PL=Inverse function of PE.

Useful to isolate generators and relations (and syzygies).

- $\mathbb{C}$: $$H_{\mathbb{C}}(t) = \sum_{n=1}^{\infty} t^n = \frac{1}{1-t} = PE[t]$$

Freely generated

- $\mathbb{C}[x]/\langle x^N \rangle$: $$H(t) = \sum_{n=1}^{N} t^n = \frac{1-t^N}{1-t} = PE[t - t^N]$$

Complete intersection

- $\mathbb{C}^2/\mathbb{Z}_n$: $$H_{\mathbb{C}^2/\mathbb{Z}_n}(t) = \frac{1-t^{2n}}{(1-t^2)(1-t^n)^2} = PE[t^2 + 2t^n - t^{2n}]$$

Complete intersection

- $\mathbb{C}^3/\mathbb{Z}_3$ $(1,1,-2)$: $$H_{\mathbb{C}^3/\mathbb{Z}_3}(t) = \frac{1+7t^3+t^6}{(1-t^3)^3}$$

$$PL[H_{\mathbb{C}^3/\mathbb{Z}_3}(t)] = 10t^3 - 27t^6 + 105t^9 - 540t^{12} + \ldots$$

Incomplete intersection

Casimir invariants

- Modding out by a continuous group G:

$$H(t) = \oint_G d\mu_G \, H(t,z) =$$

$$= \oint_{|z_1|=1} \frac{dz_1}{2\pi i z_1} \cdots \oint_{|z_r|=1} \frac{dz_r}{2\pi i z_r} \prod_{\alpha \in \Delta_+} \left(1 - \prod_{l=1}^{r} z_l^{\alpha_l}\right) H(t, z_1, \ldots, z_r)$$

e.g. Hilbert series of $\mathcal{M}_H$.

- Invariants of the adjoint representation of G:

$$H_G^{adj}(t) = \oint_G d\mu_G \, PE[\chi_G^{adj}(z)t] = \prod_{i=1}^{r} \frac{1}{1-t^{d_i}}$$

d_i: degrees of (independent) Casimir invariants.

The ring of Casimir invariants is freely generated.

Monopole operators

Monopole operators

['t Hooft 1978] [Borokhov, Kapustin, Wu 2002] [Kapustin 2005]

- Local disorder operator $\mathcal{O}(x)$: prescribed singularity in the Euclidean path integral at an insertion point x.
- Monopole operator $V_m(x)$: Dirac monopole singularity $U(1) \hookrightarrow G$.

Monopole operator $V_m(0)$

$$A_\pm = \frac{m}{2}(\pm 1 - \cos\theta)d\varphi + \mathcal{O}(1)\,, \qquad r \to 0$$

Gauge: $m \in \mathfrak{t}$ Cartan subalgebra of $\mathfrak{g}$, constant. Modulo Weyl group $\mathcal{W}_G$.

Generalised Dirac quantisation [Englert, Windey 1976]

$$\exp(2\pi i m) = \mathbb{1}_G$$

$\Rightarrow m \in \Gamma^*_{\widehat{G}}$ weight lattice of $\widehat{G}$, GNO dual group of G. [Goddard, Nuyts, Olive 1977]

Classification of monopole operators

- **Stronger**: Irreducible reps of the GNO dual group $\widehat{G}$, *i.e.* $\Gamma^*_{\widehat{G}}/\mathcal{W}_G$.
- **Weaker**: Topological symmetry group $\pi_1(G) = \Gamma^*_{\widehat{G}}/\Lambda_r(\widehat{\mathfrak{g}})$.

Examples:

G	$\widehat{G}$	$\pi_1(G)$
$U(N)$	$U(N)$	$U(1)$
$SU(N)$	$SU(N)/\mathbb{Z}_N$	1
$SU(N)/\mathbb{Z}_N$	$SU(N)$	$\mathbb{Z}_N$
$SO(2N+1)$	$USp(2N)$	$\mathbb{Z}_2$
$USp(2N)$	$SO(2N+1)$	1
$SO(2N)$	$SO(2N)$	$\mathbb{Z}_2$

Supersymmetric monopole operators

Parameterise the Coulomb branch $\mathcal{M}_C$ of a 3d susy gauge theory.

We study 3d $\mathcal{N} = 4$, but work in $\mathcal{N} = 2$ formalism ($\mathbb{C}^2$ instead of $\mathbb{H}$).

$\mathcal{N} = 4$ Vector Multiplet:

- $\mathcal{N} = 2$ Vector Multiplet: A_μ, $\sigma \in \mathbb{R}$, fermions, aux.
- $\mathcal{N} = 2$ Chiral Multiplet: $\phi \in \mathbb{C}$, fermions, aux.

$\mathcal{N} = 2$ supersymmetric monopole operator (Chiral Multiplet)

$$A_\pm = \frac{m}{2}(\pm 1 - \cos\theta)d\varphi + \mathcal{O}(1) \qquad r \to 0$$

$$\sigma = \frac{m}{2r} + \mathcal{O}(1) \qquad r \to 0$$

$\mathcal{N} = 2$ BPS equation: $(d - iA)\sigma = - * F$.

$\mathcal{N} = 4$ supersymmetric monopole operators

We have only used the $\mathcal{N} = 2$ Vector Multiplet. What about ϕ?

Adjoint breaking by monopole flux m

$$G \xrightarrow{m} H_m$$

whole gauge group	residual gauge group
	$[m, H_m] = 0$

$\mathcal{N} = 4$ BPS moduli: constant $\phi \in \mathfrak{h}_m$

$\phi = 0$: "Bare" monopole operators

$\phi \neq 0$: "Dressed" monopole operators

The Hilbert series of the Coulomb branch of a 3d $\mathcal{N} = 4$ SCFT

Quantum numbers of BPS (bare) monopole operators

- **Classical**: Topological symmetry $\pi_1(G) = \Gamma^*_{\widehat{G}}/\Lambda_r(\widehat{\mathfrak{g}})$.
- **Quantum**: "Canonical" $U(1)_R$ symmetry
  (dimension at free UV fixed point)

Canonical R-charge of bare monopole operator $V_m(x)$ [Bashkirov, Kapustin 2010]

$$\Delta(m) = -\sum_{\alpha \in \Delta_+} |\alpha(m)| + \frac{1}{2} \sum_{i=1}^{n} \sum_{\rho_i \in \mathcal{R}_i} |\rho_i(m)|$$

We will focus on *good/ugly* gauge theories: [Gaiotto, Witten 2008]
Δ is also the conformal dimension of BPS operators in the IR SCFT.

$\Delta(m) \geq \frac{1}{2}$ for all monopole fluxes m: Unitarity bound ✓

The HS of $\mathcal{M}_C$ of a 3d $\mathcal{N} = 4$ SCFT

1. $\mathcal{M}_C$ is parameterised by bare and dressed monopole operators.
2. Independent boundary counditions in the path integral.

Formula for the Hilbert series of $\mathcal{M}_C$

$$H_G(t,z) = \sum_{m \in \Gamma^*_{\widehat{G}}/\mathcal{W}_{\widehat{G}}} z^{J(m)} t^{\Delta(m)} P_G(t;m)$$

- $\sum_m$: sum over GNO sectors
- $z^{J(m)}$: topological charge in $\pi_1(G)$
- $t^{\Delta(m)}$: dimension of bare monopole operator V_m
- $P_G(t;m)$: dressing by adjoint $\phi \in \mathfrak{h}_m$

$$P_G(t;m) = \prod_{i=1}^{r} \frac{1}{1 - t^{d_i(m)}}$$

$d_i(m)$: degrees of independent Casimirs of H_m

The HS of $\mathcal{M}_C$ of a 3d $\mathcal{N} = 4$ SCFT

Formula for the Hilbert series of $\mathcal{M}_C$

$$H_G(t, z) = \sum_{m \in \Gamma^*_{\widehat{G}}/\mathcal{W}_{\widehat{G}}} z^{J(m)} t^{\Delta(m)} P_G(t; m)$$

- Exact formula for the generating function of the chiral ring on $\mathcal{M}_C$.
- Valid for the IR fixed point of any good/ugly 3d $\mathcal{N} = 4$ gauge theory.
- Encodes relations between monopole operators.
- Equivariant index of $\overline{\partial}$ on $\mathcal{M}_C$.

Examples

$U(1)$ with n electrons

$$H_{U(1),\,n}(t,z) = \frac{1}{1-t}\sum_{m\in\mathbb{Z}} z^m t^{\frac{n}{2}|m|} = \frac{1-t^n}{(1-t)(1-zt^{n/2})(1-z^{-1}t^{n/2})} =$$

$$= PE[t + (z+z^{-1})t^{n/2} - t^n]$$

Term	Interpretation
t	Φ adjoint chiral
$zt^{n/2}$	V_{+1} monopole operator
$z^{-1}t^{n/2}$	V_{-1} monopole operator
$-t^n$	$V_{+1}V_{-1} = \Phi^n$ relation

The result shows that $\mathcal{M}_C = \mathbb{C}^2/\mathbb{Z}_n$. [Intriligator, Seiberg 1996]

$U(k)$ with n fundamentals

Dimension formula:

$$\Delta(\vec{m}) = \frac{n}{2}\sum_{i=1}^{k}|m_i| - \sum_{i<j}|m_i - m_j|$$

Hilbert series:

$$H_{U(k),\,n}(t,z) = \sum_{m_1\geq m_2\geq\cdots\geq m_k>-\infty} t^{\Delta(\vec{m})} z^{\sum_{i=1}^{k} m_i} P_{U(k)}(t;\vec{m}) =$$

$$= \prod_{j=1}^{k} \frac{1-t^{n+1-j}}{(1-t^j)(1-zt^{n/2+1-j})(1-z^{-1}t^{n/2+1-j})} =$$

$$= PE\left[\sum_{j=1}^{k}\left(t^j + (z+z^{-1})t^{n/2+1-j} - t^{n+1-j}\right)\right]$$

- U, USp, SO with fundamentals: $\mathcal{M}_C$ are complete intersections
- SU, $Spin$, G_2, F_4 with fundamentals: $\mathcal{M}_C$ are incomplete intersections.

Affine ADE quivers and instantons

D2 $\to$ M2-branes probing an ADE singularity $\mathbb{C}^2/\Gamma_{\mathcal{G}}$.

$\mathcal{G}$: simple *ADE* group. $\Gamma_{\mathcal{G}}$: discrete subgroup of $SU(2)$ (McKay).

Quiver diagram: extended Dynkin diagram of ADE type.

Gauge group: $G = (\prod_{i=0}^{\mathrm{rk}(\mathcal{G})} U(n_i))/U(1)$

n_i: Dynkin indices of the extended Dynkin diagram.

Hilbert series of $\mathcal{M}_C$

$$H(t,z) = \sum_{k=0}^{\infty} [k, 0, \cdots, 0, k]_{\mathcal{G}}\, t^k$$

where $[1, 0, \cdots, 0, 1]_{\mathcal{G}}$ is the (Char of) adjoint rep of $\mathcal{G}$.

Agrees with conjectured HS of the reduced moduli space of 1 $\mathcal{G}$-instanton.

[Benvenuti, Hanany, Mekareeya 2010]

Outlook

Outlook

We found a simple formula for the Hilbert Series of the Coulomb branch of 3*d* $\mathcal{N} = 4$ SCFTs arising from good/ugly gauge theories:

- Predicts generators and relations in the chiral ring.
- Reduces computation of BPS correlators to the generators only.

Applications:

- Tests of mirror symmetry (proof for Abelian pairs).
- E-instanton moduli spaces without ADHM.
- Uplift of orientifolds to M-theory.
- Lagrangian mirrors of non-Lagrangian theories (*e.g.* 3d versions of Gaiotto's T_N) [SC, Hanany, Mekareeya, Zaffaroni, in progress]