Large N lattice gauge theory results and their interpretation

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1. The large N limit

2. Lattice calculation of large N observables
   - Confinement and asymptotic freedom
   - Glueballs
   - Mesons

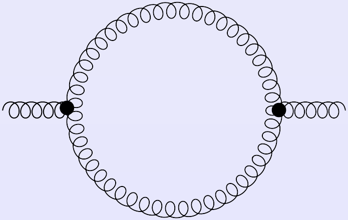
3. Conclusions

The 't Hooft large N limit – I

- In QCD some phenomena (confinement, chiral symmetry breaking) are non-perturbative
- Lattice calculations are successful in this regime, but more effective if some analytic guidance is available
- If we embed QCD in a larger context (SU(N) gauge theory with N_f quark flavours), the theory simplify in the limit $N \to \infty$, N_f and $g^2 N = \lambda$ fixed. Yet, it is non-trivial. Perhaps QCD is physically close to this limit?
- However large N QCD is still complicated enough that an analytic solution has not been found
- Lattice calculations can shed light on the existence of the limiting theory and on the proximity of QCD to it

The 't Hooft large N limit – II

SU(N) gauge theory (possibly enlarged with N_f fermions in the fundamental representation)

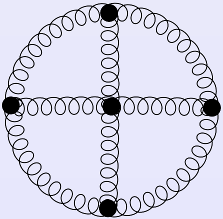


$$= f(g^2 N)$$

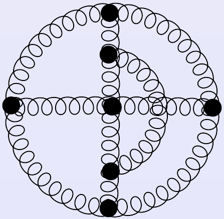
A sensible, non-trivial large N limit can be defined by keeping fixed $\lambda = g^2 N$, with N_f fixed ('t Hooft)

Planar graphs

Consider connected vacuum diagrams, e.g.



planar graph

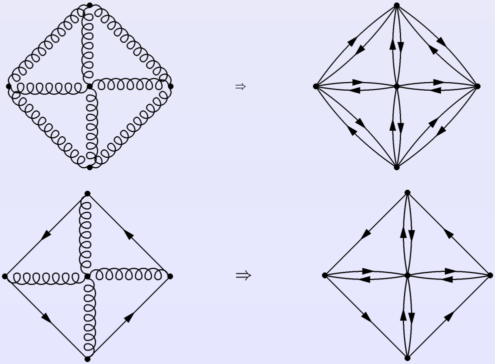


non-planar graph

In the large N limit only those diagrams survive that can be drawn in a plane without crossing of the lines (planar graphs)

Double line representation

Regarding the flow of the colour indices a gluon propagator is equivalent to a quark and an antiquark propagators



Diagrammatic at large N

For each vertex: N
For each propagator: 1/N $\Rightarrow$ $\langle \ \rangle \propto N^{F-E+V}$
For each loop: N

Euler characteristic $\chi = F - E + V = 2 - 2H - B$

1. The leading connected vacuum-to-vacuum graphs are of order N^2 (planar graphs made of gluons only)
2. The leading connected vacuum-to-vacuum graphs with quark lines are of order N (planar graphs with just one quark loop at the boundary)
3. Corrections down by factors of $1/N^2$ in the gauge theory and by factors of $1/N$ in the theory with fermions

Phenomenology and large N

- Quark loop effects $\propto 1/N \Rightarrow$ The $N = \infty$ limit is quenched
- Mixing glueballs-mesons $\propto 1/\sqrt{N} \Rightarrow$ No mixing between glueballs and mesons at $N = \infty$
- Meson decay widths $\propto 1/N \Rightarrow$ mesons do not decay at $N = \infty$
- OZI rule $\propto 1/N \Rightarrow$ OZI rule exact at $N = \infty$

$\hookrightarrow$ The simpler large N phenomenology can explain features of SU(3) phenomenology

Large N limit on the lattice

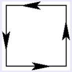
The lattice approach allows us to go beyond perturbative and diagrammatic arguments. For a given observable

1. Continuum extrapolation
   - Determine its value at fixed a and N
   - Extrapolate to the continuum limit
   - Extrapolate to $N \to \infty$ using a power series in $1/N^2$
2. Fixed lattice spacing
   - Choose a in such a way that its value in physical units is common to the various N
   - Determine the value of the observable for that a at any N
   - Extrapolate to $N \to \infty$ using a power series in $1/N^2$

Study performed for various observables both at zero and finite temperature for $2 \leq N \leq 8$

SU(N) Lattice Gauge Theories

- Link variables $U_\mu(i) = e^{ig_0 a A_\mu(i)}$
  Plaquettes $U_{\mu\nu}(i) = \prod_{U_\mu \in P_{\mu\nu}} U_\mu$



- Wilson action $\beta = (2N)/g_0^2$

$$S = \beta \sum_{i,\mu,\nu} \left(1 - \frac{1}{2N} \mathrm{Tr} \left(U_{\mu\nu}(i) + U^\dagger_{\mu\nu}(i) \right) \right)$$

- Partition function $Z = \int (\mathcal{D}U) \, e^{-S}$
- Invariance under SU(N) gauge transformations

$$\tilde{U}_\mu(i) = G^\dagger(i) U_\mu(i) G(i + \hat{\mu})$$

- Bulk phase transition $\Rightarrow$ avoided if β is large enough
- 't Hooft's coupling

$$\lambda = g_0^2 N \qquad \Rightarrow \qquad \frac{\beta(N)}{\beta(N')} = \frac{N^2}{N'^2}$$

- Tadpole improvement

$$g_I^2 = \frac{g_0^2}{\langle U_P \rangle} \qquad \Rightarrow \qquad \frac{\beta_I(N)}{\beta_I(N')} = \frac{N^2}{N'^2}$$

Correlation matrix

Trial operators $\Phi_1(t), \dots, \Phi_n(t)$ with the quantum numbers of the state of interest

$$
\begin{aligned}
C_{ij}(t) &= \langle 0| \left(\Phi_i(0)\right)^\dagger \Phi_j(t)|0\rangle \\
&= \langle 0| \left(\Phi_i(0)\right)^\dagger e^{-Ht}\Phi_j(0)e^{Ht}|0\rangle \\
&= \sum_n \langle 0| \left(\Phi_i(0)\right)^\dagger |n\rangle\langle n|e^{-Ht}\Phi_j(0)e^{Ht}|0\rangle \\
&= \sum_n e^{-\Delta E_n t}\langle 0| \left(\Phi_i(0)\right)^\dagger |n\rangle\langle n|\Phi_j(0)|0\rangle \\
&= \sum_n c^*_{in} c_{jn} e^{-\Delta E_n t} = \delta_{ij} \sum_n |c_{in}|^2 e^{-am_n t} \underset{t\to\infty}{\to} \delta_{ij}|c_{i1}|^2 e^{-am_1 t}
\end{aligned}
$$

Variational principle

1. Find the eigenvector v that minimises

$$am_1(t_d) = -\frac{1}{t_d} \log \frac{v_i^* C_{ij}(t_d) v_j}{v_i^* C_{ij}(0) v_j}$$

   for some t_d
2. Fit $v(t)$ with the law $Ae^{-m_1 t}$ to extract m_1
3. Find the complement to the space generated by $v(t)$
4. Repeat 1-3 to extract $m_2, \ldots, m_n$

Need a good overlap with the state of interest

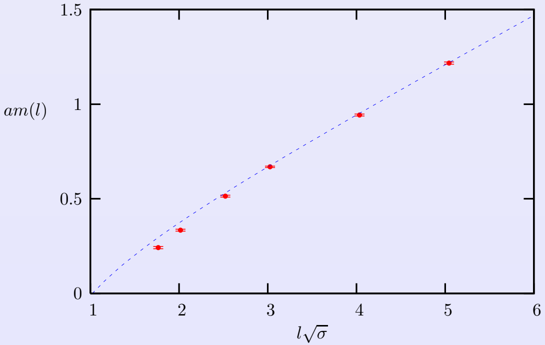
String tension

Confining potential: $V = \sigma R$

Polyakov loop $P_k(i) = \frac{1}{N} \mathrm{Tr} \prod_{j=0}^{L} U_k(i + j\hat{k})$

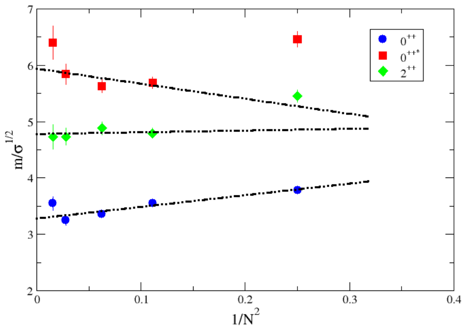
$$
\begin{aligned}
P(t) &= \sum_{\vec{n},k} P_k(\vec{n}, t) \\
C(t) &= \langle (P(0))^{\dagger} P(t) \rangle = \sum_j |c_j|^2 e^{-am_j t} \underset{t \to \infty}{\rightarrow} |c_l|^2 e^{-am_l t} \\
am_l &\simeq a^2 \sigma L - \frac{\pi (D-2)}{6L}
\end{aligned}
$$

Confining flux tubes



[H. Meyer and M. Teper, JHEP 0412 (2004) 031]

Glueball masses



[B. Lucini, M. Teper and U. Wenger, JHEP 0406 (2004) 012]

Masses at $N = \infty$

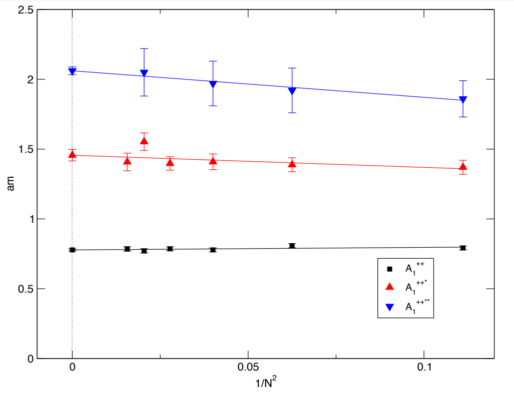
0^{++}
$$\frac{m}{\sqrt{\sigma}} = 3.28(8) + \frac{2.1(1.1)}{N^2}$$

0^{++*}
$$\frac{m}{\sqrt{\sigma}} = 5.93(17) - \frac{2.7(2.0)}{N^2}$$

2^{++}
$$\frac{m}{\sqrt{\sigma}} = 4.78(14) + \frac{0.3(1.7)}{N^2}$$

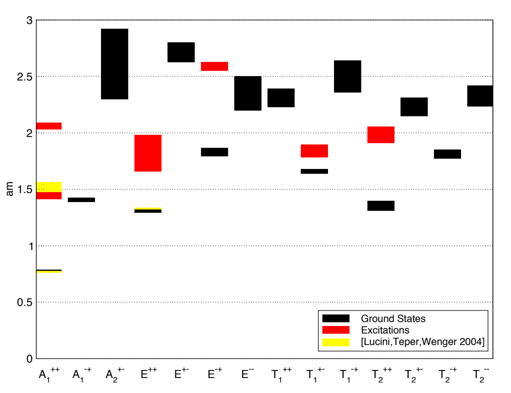
Accurate $N = \infty$ value, small $\mathcal{O}(1/N^2)$ correction

0^{++} excitations



Lattice spacing fixed by requiring $aT_c = 1/6$

Spectrum at $aT_c = 1/6$



[B. Lucini, A. Rago and E. Rinaldi, JHEP 1008 (2010) 119]

Lattice action for full QCD

Path integral

$$Z = \int (\mathcal{D}U_\mu(i)) \left(\det M(U_\mu)\right)^{N_f} e^{-S_g(U_{\mu\nu}(i))}$$

with

$$U_\mu(i) = Pexp\left(ig \int_i^{i+a\hat{\mu}} A_\mu(x) \mathrm{d}x \right)$$

and

$$U_{\mu\nu}(i) = U_\mu(i) U_\nu(i+\hat{\mu}) U^\dagger_\mu(i+\hat{\nu}) U^\dagger_\nu(i)$$

Gauge part

$$S_g = \beta \sum_{i,\mu} \left(1 - \frac{1}{N} \mathcal{R}\mathrm{e}\ \mathrm{Tr}(\mathrm{U}_{\mu\nu}(\mathrm{i})) \right) \qquad , \qquad with\ \beta = 2N/g_0^2$$

Wilson fermions

Take the naive Dirac fermions and add an irrelevant term that goes like the Laplacian

$$M_{\alpha\beta}(ij) = (m + 4r)\delta_{ij}\delta_{\alpha\beta} - \frac{1}{2}\left[(r - \gamma_\mu)_{\alpha\beta} U_\mu(i)\delta_{i,j+\mu} + (r + \gamma_\mu)_{\alpha\beta} U^\dagger_\mu(j)\delta_{i,i-\mu}\right]$$

This formulation breaks explicitly chiral symmetry

Define the hopping parameter

$$\kappa = \frac{1}{2(m + 4r)}$$

Chiral symmetry recovered in the limit $\kappa \to \kappa_c$ (κ_c to be determined numerically)

Quenched approximation

For an observable $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \frac{\int (\mathcal{D}U_\mu(i)) \left(\det M(U_\mu)\right)^{N_f} f(M) e^{-S_g(U_{\mu\nu}(i))}}{\int (\mathcal{D}U_\mu(i)) \left(\det M(U_\mu)\right)^{N_f} e^{-S_g(U_{\mu\nu}(i))}}$$

Assume $\det M(U_\mu) \simeq 1$ i.e. fermions loops are removed from the action

The approximation is exact in the $m \to \infty$ and $N \to \infty$ limit ($g^2 N$ is fixed)
$\hookrightarrow$ the large N spectrum is quenched for $m \neq 0$

As N increases, unquenching effects are expected for smaller quark masses

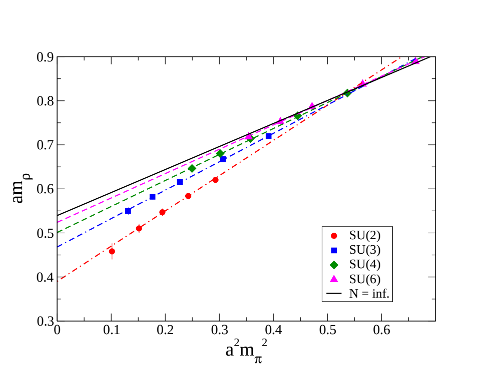
Fermionic operators

For isotriplet states (flavour index $\alpha \neq \beta$):

Particle	Bilinear	J^{PC}
a_0	$\bar{\psi}_\alpha \psi_\beta$	0^{++}
π	$\bar{\psi}_\alpha \gamma_5 \psi_\beta$, $\bar{\psi}_\alpha \gamma_0 \gamma_5 \psi_\beta$	0^{-+}
ρ	$\bar{\psi}_\alpha \gamma_i \psi_\beta$, $\bar{\psi}_\alpha \gamma_0 \gamma_i \psi_\beta$	1^{--}
a_1	$\bar{\psi}_\alpha \gamma_5 \gamma_i \psi_\beta$	1^{++}
b_1	$\bar{\psi}_\alpha \gamma_i \gamma_j \psi_\beta$	1^{+-}

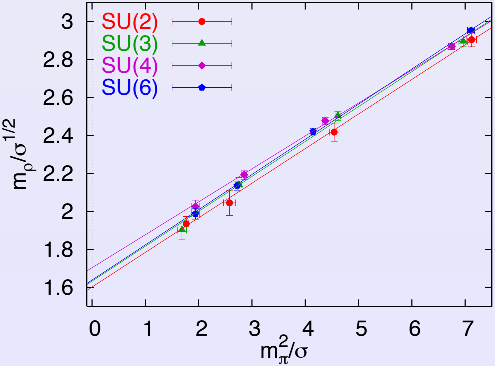
Flavour singlet states more difficult to study

m_ρ vs. m_π^2 at $N = \infty$



[L. Del Debbio, B. Lucini, A. Patella and C. Pica, JHEP 0803 (2008) 062]

m_π vs. m_ρ - fixing σ



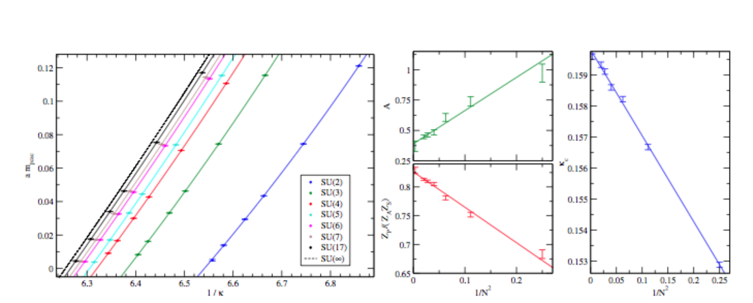
[G. Bali and F. Bursa, JHEP 0809 (2008) 110]

Numerical simulations

1. Aims: determining the mesonic spectrum (including some excitations) and decay constants
2. Calculations performed for $2 \leq N \leq 7$ and $N = 17$
3. β fixed across the various n by imposing $a\sqrt{\sigma} = 0.2093$, implying $a \simeq 0.093$fm (or $a^{-1} \simeq 2.1$ GeV)
4. Range of κ down to $m_\pi \simeq 0.5\sqrt{\sigma}$ for $N \geq 5$ and $m_\pi \simeq 0.75\sqrt{\sigma}$ for $N \leq 4$
5. Size $24^3 \times 48$ for $N \neq 17$, $12^3 \times 24$ for $N = 17$ (finite size effects negligible at large N)
6. 200 configurations (80 configurations for $N = 17$)

[G. Bali, F. Bursa, L. Castagnini, S. Collins, L. Del Debbio, B. Lucini and M. Panero, arXiv:1302.1502 and in preparation]

Numerical results for the PCAC mass



PCAC mass and chiral limit

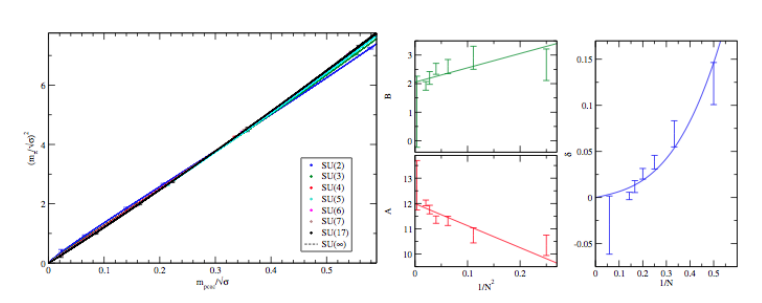
Putting together large-N and χPT predictions

$$a\, m_{\mathrm{PCAC}} = \frac{Z_P}{Z_A Z_S}\left(1 + A\, a\, m_{\mathrm{PCAC}}\right)\frac{1}{2}\left(\frac{1}{\kappa} - \frac{1}{\kappa_c}\right),$$

with A, K and $\frac{Z_P}{Z_A Z_S}$ expected to have $1/N^2$ corrections in the large-N limit:

$$\begin{aligned}
\frac{Z_P}{Z_A Z_S} &= 0.8291(20) - \frac{0.699(45)}{N^2},\\
A &= 0.390(13) + \frac{2.73(26)}{N^2},\\
\kappa_c &= 0.1598555(33)(447) - \frac{0.028242(68)(394)}{N^2}
\end{aligned}$$

The pseudoscalar



Fit results for the pseudoscalar

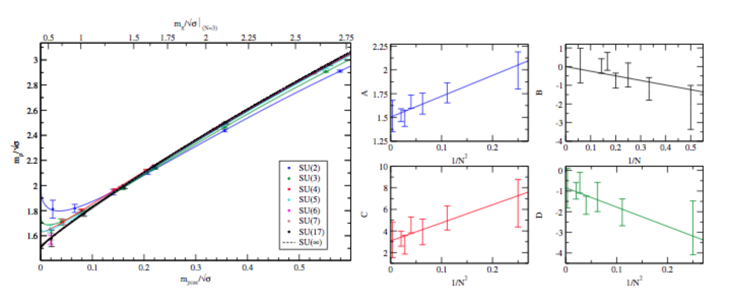
Ansatz:

$$(am_\pi)^2 = A\,(am_q)^{\frac{1}{1+\delta}} + B\,(am_q)^2$$

Results:

$$\begin{aligned} A &= 11.99(0.10) - \frac{8.7(1.6)}{N^2} \\ B &= 2.05(0.13) + \frac{5.0(2.2)}{N^2} \\ \delta &= \frac{0.056(19)}{N} + \frac{0.94(21)}{N^3} \end{aligned}$$

The vector


$m_\pi/\sqrt{\sigma}|_{(N=3)}$
$m_\rho/\sqrt{\sigma}$
$m_{pcac}/\sqrt{\sigma}$
SU(2)
SU(3)
SU(4)
SU(5)
SU(6)
SU(7)
SU(17)
SU(∞)
A
B
C
D
$1/N^2$
$1/N$

Fit results for the vector

Ansatz:

$$m_\rho = A + B m_q^{1/2} + C m_q + D m_q^{3/2}$$

Results:

$$\begin{aligned}
A &= 1.504(51) + \frac{2.19(75)}{N^2} \\
B &= -\frac{2.47(94)}{N} \\
C &= 3.08(53) + \frac{16.8(8.2)}{N^2} \\
D &= -0.84(31) - \frac{9.4(4.8)}{N^2}
\end{aligned}$$

Other states

$$\frac{m_{a_1}}{\sqrt{\sigma}} = \left(2.860(21) + \frac{0.84(36)}{N^2}\right) + \left(2.289(35) - \frac{2.02(61)}{N^2}\right) \frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

$$\frac{m_{b_1}}{\sqrt{\sigma}} = \left(2.901(23) + \frac{1.07(40)}{N^2}\right) + \left(2.273(38) - \frac{2.83(72)}{N^2}\right) \frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

$$\frac{m_{a_0}}{\sqrt{\sigma}} = \left(2.402(34) + \frac{4.25(62)}{N^2}\right) + \left(2.721(53) - \frac{6.84(96)}{N^2}\right) \frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

Excited states

$$\frac{m_{\pi^\star}}{\sqrt{\sigma}} = \left(3.392(57) + \frac{1.0(1.1)}{N^2}\right) + \left(2.044(80) - \frac{1.2(1.6)}{N^2}\right) \frac{m_{\rm PCAC}}{\sqrt{\sigma}}$$

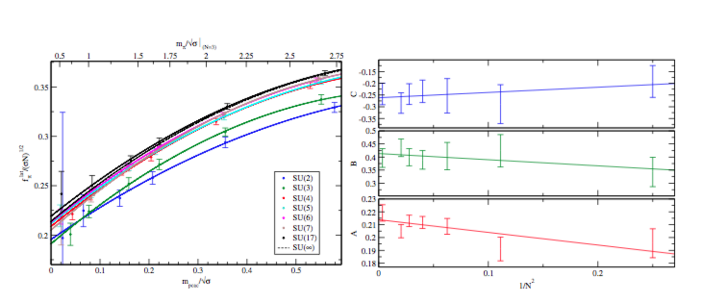
$$\frac{m_{\rho^\star}}{\sqrt{\sigma}} = \left(3.696(54) + \frac{0.23(55)}{N^2}\right) + \left(1.782(67) - \frac{1.30(54)}{N^2}\right) \frac{m_{\rm PCAC}}{\sqrt{\sigma}}$$

$$\frac{m_{a_0^\star}}{\sqrt{\sigma}} = \left(4.356(65) + \frac{1.8(1.4)}{N^2}\right) + \left(1.902(98) - \frac{2.9(2.1)}{N^2}\right) \frac{m_{\rm PCAC}}{\sqrt{\sigma}}$$

$$\frac{m_{a_1^\star}}{\sqrt{\sigma}} = \left(4.587(75) + \frac{1.2(1.2)}{N^2}\right) + \left(1.76(12) - \frac{2.1(19)}{N^2}\right) \frac{m_{\rm PCAC}}{\sqrt{\sigma}}$$

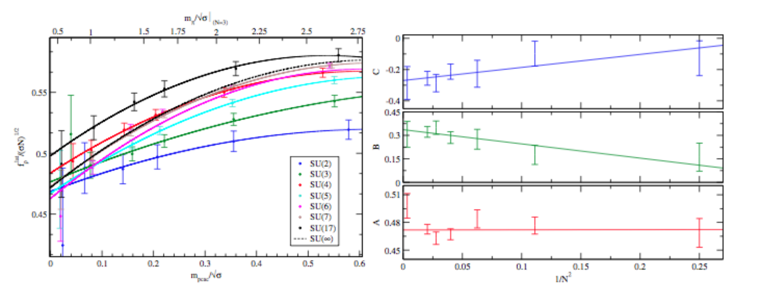
$$\frac{m_{b_1^\star}}{\sqrt{\sigma}} = \left(4.609(99) + \frac{1.7(1.5)}{N^2}\right) + \left(1.87(15) - \frac{2.5(2.2)}{N^2}\right) \frac{m_{\rm PCAC}}{\sqrt{\sigma}}$$

Pion decay constant



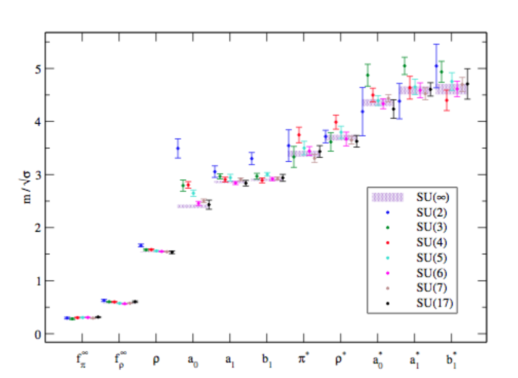
Fit $f_\pi^{\rm lat}/\sqrt{N\sigma} = A + B \cdot m_{\rm PCAC}/\sqrt{\sigma} + C \cdot m_{\rm PCAC}^2/\sigma$

ρ decay constant

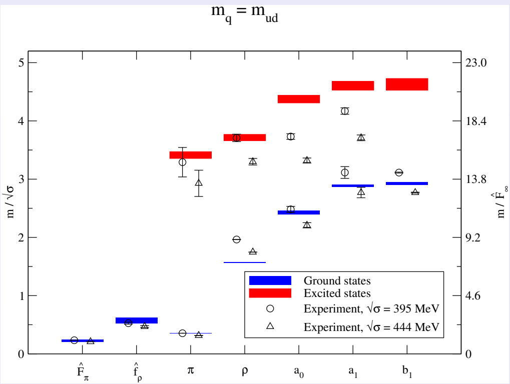


Fit $f_\rho^{\mathrm{lat}}/\sqrt{N\sigma} = A + B \cdot m_{\mathrm{PCAC}}/\sqrt{\sigma} + C \cdot m_{\mathrm{PCAC}}^2/\sigma$

Approaching $N = \infty$



Comparison with QCD



$\sqrt{\sigma}$ fixed from the condition $\hat{F}_\infty = 85.9\ \mathrm{MeV}$, m_{ud} from $m_\pi = 138\ \mathrm{MeV}$

Other topics

- Strings and k-strings
- Deconfinement phase transition
- Thermodynamics
- Topology
- Volume reduction
- Eguchi-Kawai
- 3D
- ...

For a comprehensive review, see B. Lucini and M. Panero, arXiv:1210.4997, Phys. Rept. to appear

Conclusions

- The lattice is a useful tool a non-perturbative investigations of the large N limit
- Several results for various observables available by know
- Many (all?) observables are well described by the expected leading correction for $2 \leq N \leq 8 \Rightarrow$ SU(3) differs from SU(∞) by small $\mathcal{O}(1/N^2)$ corrections
- Future work to include
  - Continuum spectrum
  - Isosinglet states
  - Effects of fermions
  - Other large N generalisations of QCD (e.g. Orientifold Planar Equivalence)