Numerical propagation of light beams in refracting/diffracting devices

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Summary

•Needs for optical simulations

•General principles of numerical propagation : several methods

- •Some examples :
  - •Fourier Transform
  - •Hankel Transform
  - •Modal
  - •Monte-Carlo

•Advantages/drawbacks

Needs for Optical simulations in GW interferometer design

1) Studies of different categories of mirror defects, bulk, polishing, coating

→Requirements for manufacturers

2) Commissioning : understanding the instrument
- thermal issues
- thermal compensation

4) Scattered light : mitigation strategies, baffles design

General principles of Propagation Methods

Expand optical field on a family of functions of known propagation

- Plane waves
- Bessel waves
- Gaussian modes (eg. HG or LG)
- Photons

Propagation by Fourier Transform :
General principles

Paraxial diffraction theory (vacuum)

Maxwell+single frequency → Helmholtz : $\left[\partial_x^2+\partial_y^2+\partial_z^2+\omega^2/c^2\right]E(\omega,x,y,z)=0$

Slowly varying envelope : $E(\omega,x,y,z)=\exp\left[ikz\right]F(\omega,x,y,z)$ and $\partial_z F << kF$

$$k\equiv\frac{\omega}{c}\equiv\frac{2\pi}{\lambda}$$

Paraxial diffraction equation (math. analogous to Heat-Fourier and to Schrödinger eq.) :

$$\left[\partial_x^2+\partial_y^2+2ik\partial_z\right]F(\omega,x,y,z)=0$$

2D Fourier Tr. : $\tilde{f}(p,q)\equiv\int_{R^2} f(x,y)e^{ipx}e^{iqy}\,dx\,dy$

$$\Longrightarrow \quad \left[2ik\partial_z-p^2-q^2\right]\tilde{F}(\omega,p,q,z)=0$$

$$\Longrightarrow \quad F(\omega,p,q,z_1)=\underbrace{\exp\left[-i\lambda\frac{p^2+q^2}{4\pi}(z_1-z_0)\right]}_{\text{propagator}}F(\omega,p,q,z_0)$$

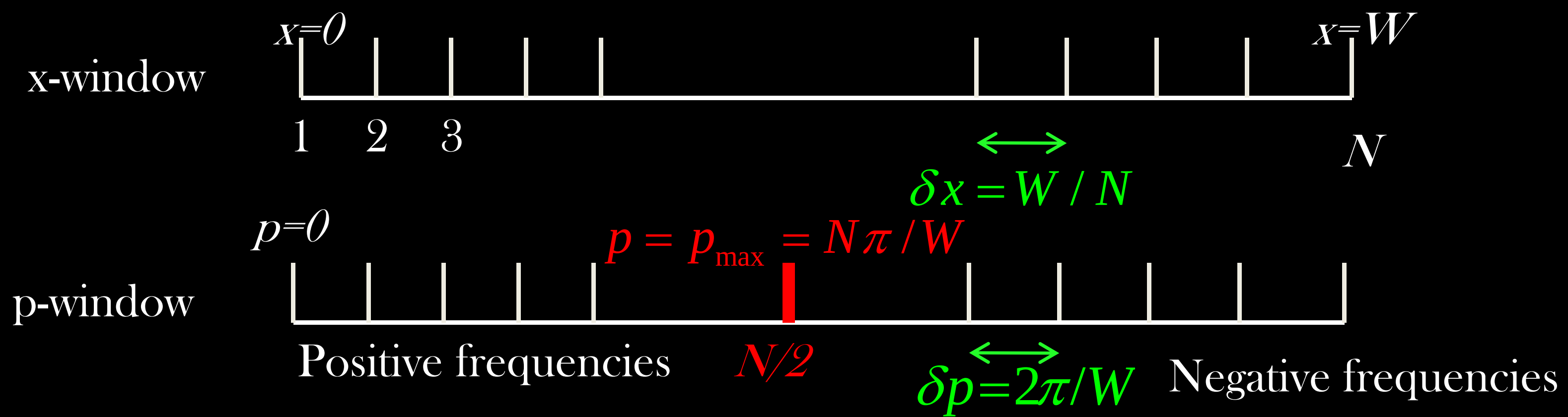
Propagation by Fourier Transform

$A(x, y, z_0)$ — Diffraction over Δz → $A(x, y, z_0 + \Delta z)$

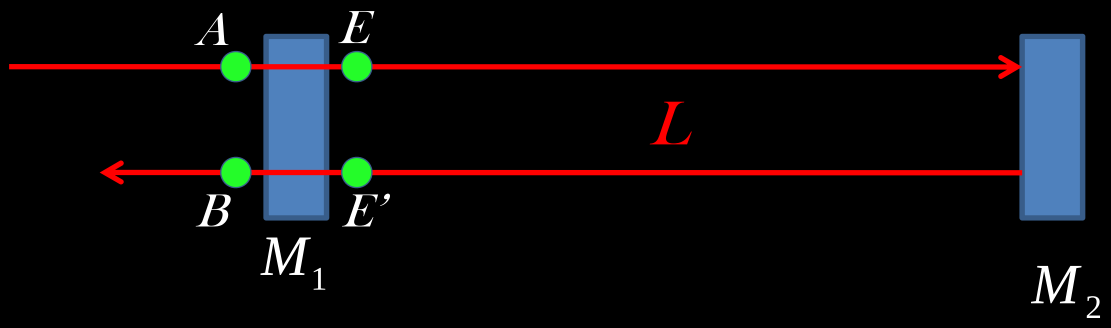
FT ↓   ↑ FT^{-1}

$$\tilde{A}(p, q, z_0) \times \exp\left[-i\frac{p^2+q^2}{2k}\Delta z\right] = \tilde{A}(p, q, z_0 + \Delta z)$$

Use of Discrete Fourier Transform (in practice : FFT)



Mode of a Fabry-Perot cavity



Implicit equation : $$E = T_1 * A + (R_1 * P_L * R_2 * P_L) * E$$

Mirror operators in xy plane

$$R_a = r_a \exp\left[-2ik\left(\frac{x^2+y^2}{2\rho_a} + f_a(x,y)\right)\right] \quad (a=1,2)$$

Curvature radius

Roughness as measured by the coating manufacturer

$$T_a = t_a \exp\left[ikh_a(x,y)\right]$$

Optical thickness

propagator

Propagation $$P_L * X \equiv \mathfrak{F}^{-1}\left[\mathcal{P}.\mathfrak{F}\left(X\right)\right]$$

Solution by simple relaxation scheme :

$$E_n = T_1 A + \mathcal{C} * E_{n-1} \qquad \mathcal{C} = M_1 * P * M_2 * P$$

With initial guess : $E_0 = \dfrac{t_1}{1 - r_1 r_2} TEM_{00}$

Large number of iterations if large finesse and/or large defects

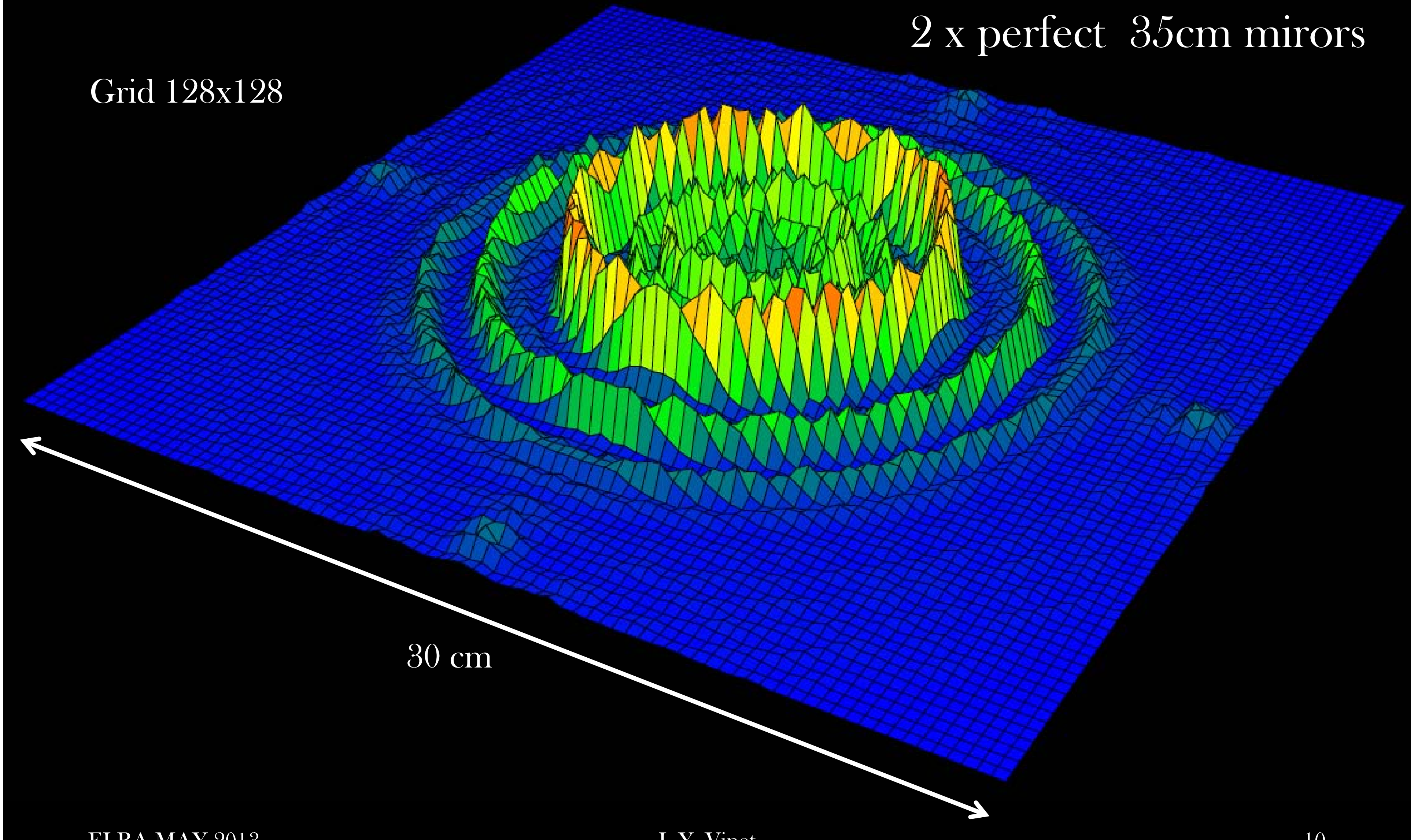
Accelerated convergence (a la Aitken):

$$E_n = \alpha_{n-1} E_{n-1} + \beta_{n-1} \underbrace{(T_1 A + \mathcal{C} * E_{n-1})}_{E_n^{'} \text{ of simple relaxation}}$$

With optimal choice of α_n, β_n at each iteration

See e.g. : Saha, JOSA A, Vol 14, No 9, 1997

Black fringe :(ideal TEM00) – (TEM00 reflected by a Virgo cavity) 10^{-8} W/W



Propagation by Hankel/Bessel Transform : General principles

Suitable for axisymmetrical problems

Fourier Transform :

$$\tilde{f}(p,q) = \frac{1}{2\pi}\int dxdy \exp\left[i(px+qy)\right] f(x,y)$$

Assume (axial symmetry) :

$$f(x,y) = f\left(r = \sqrt{x^2+y^2}\right)$$

then

$$x = r\cos\varphi, y = r\sin\varphi, \quad p = \rho\cos\theta, q = \rho\sin\theta$$

$$\tilde{f}(\rho) = \frac{1}{2\pi}\int rdrd\varphi \exp\left[i\rho r\cos(\varphi-\theta)\right] f(r) = \int_{R+} J_0(\rho r) f(r) rdr$$

Bessel Transform

Inverse B transform :

$$f(r) = \int_{R+} J_0(\rho r)\,\tilde{f}(\rho)\rho d\rho$$

Assume $f(r)$ negligible for $r > a$

Let $\{\varsigma_\alpha, \alpha = 1, \dots, \infty\}$ be the zeros of $J_1(r)$

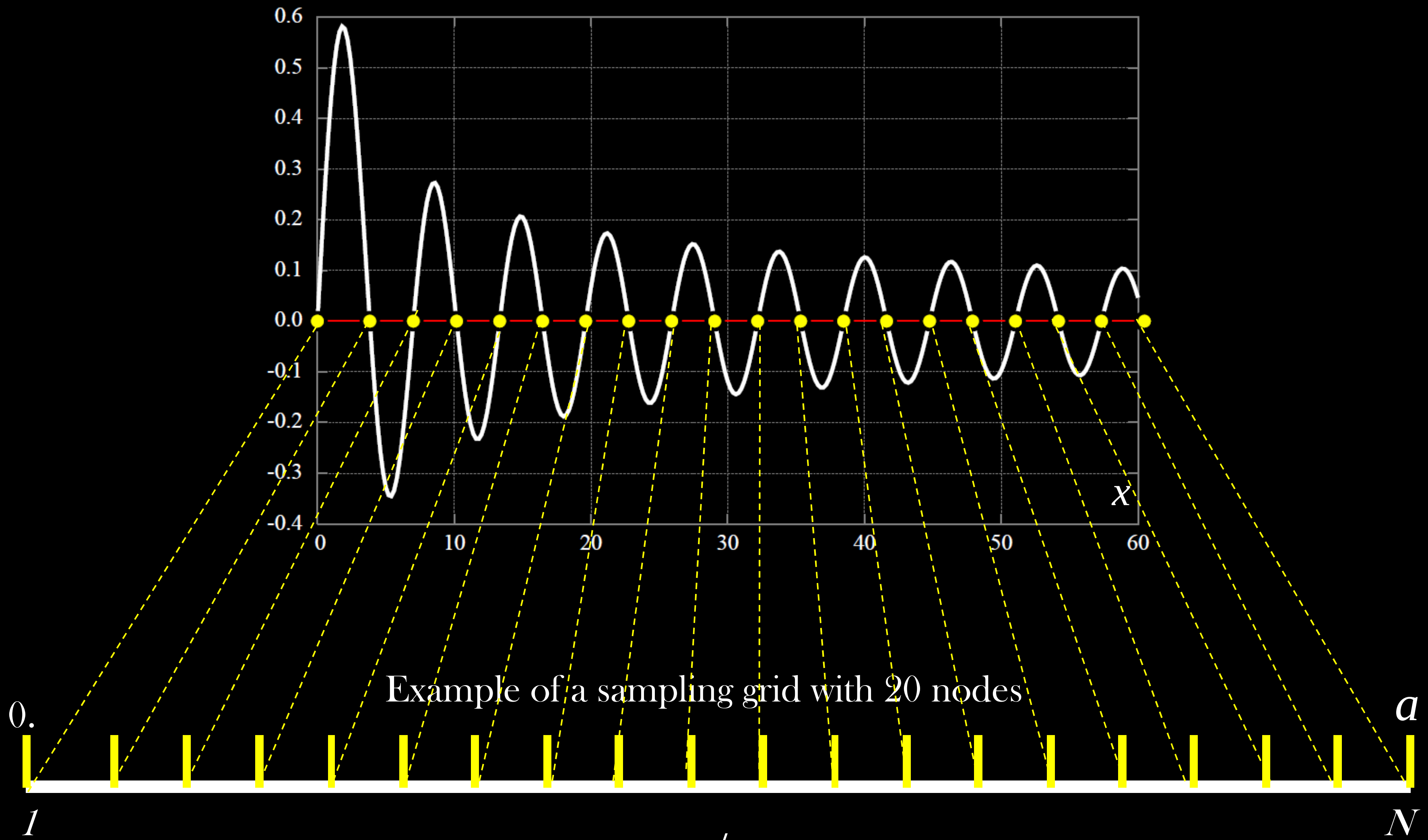
Sturm-Liouville theorem : the $\{\phi_\alpha(r) \equiv J_0(\zeta_\alpha r / a)\}$
are a complete, orthogonal family on $[0, a]$

$$\int_0^a \phi_\alpha(r)\phi_\beta(r) r dr = p_\alpha \delta_{\alpha\beta}, \quad p_\alpha \equiv \frac{a^2}{2} J_0^2(\varsigma_\alpha)$$

So that

$$f(r) = \sum_{\alpha=1}^{\infty} \frac{\tilde{f}_\alpha}{p_\alpha} \phi_\alpha(r) \quad \text{with} \quad \tilde{f}_\alpha \equiv \int_0^a f(r)\phi_\alpha(r) r dr$$

The first 20 zeros of $J_1(x)$



$$x_i = \varsigma_i a / \varsigma_N$$

$$f(r) = \sum_{\alpha=1}^{\infty} \frac{\tilde{f}_\alpha}{p_\alpha} \phi_\alpha(r) \qquad r_\alpha = \varsigma_\alpha a / \varsigma_N$$

$$\Rightarrow \quad f_\beta \equiv f(r_\beta) = \sum_{\alpha=1}^{\infty} \frac{\tilde{f}_\alpha}{p_\alpha} \phi_\alpha(r_\beta) = \sum_{\alpha=1}^{\infty} \frac{2}{a^2 J_0^2(\varsigma_\alpha)} J_0\left(\frac{\varsigma_\alpha \varsigma_\beta}{\varsigma_N}\right) \tilde{f}_\alpha$$

Reciprocal F transform :

$$f_\alpha = \sum_{\alpha=1}^{\infty} H_{\alpha\beta}^{(-)} \tilde{f}_\beta \quad \text{with} \quad H_{\alpha\beta}^{(-)} \equiv \frac{2 J_0\left(\frac{\varsigma_\alpha \varsigma_\beta}{\varsigma_N}\right)}{a^2 J_0^2(\varsigma_\beta)}$$

Direct F transform :

$$\tilde{f}_\alpha = \sum_{\alpha=1}^{\infty} H_{\alpha\beta}^{(+)} f_\beta \quad \text{with} \quad H_{\alpha\beta}^{(+)} \equiv \frac{2 a^2 J_0\left(\frac{\varsigma_\alpha \varsigma_\beta}{\varsigma_N}\right)}{\varsigma_N^2 J_0^2(\varsigma_\beta)}$$

$$\tilde{f}_\alpha = \tilde{f}(\rho_\alpha) \quad \text{with} \quad \rho_\alpha \equiv \frac{\varsigma_\alpha}{a}$$

Direct and inverse Bessel transforms are done with explicit matrices

Propagator in the Fourier space over distance Δz :

$$P(p,q,z) = \exp\left[-i\frac{\lambda\Delta z}{4\pi}\left(p^2+q^2\right)\right]$$

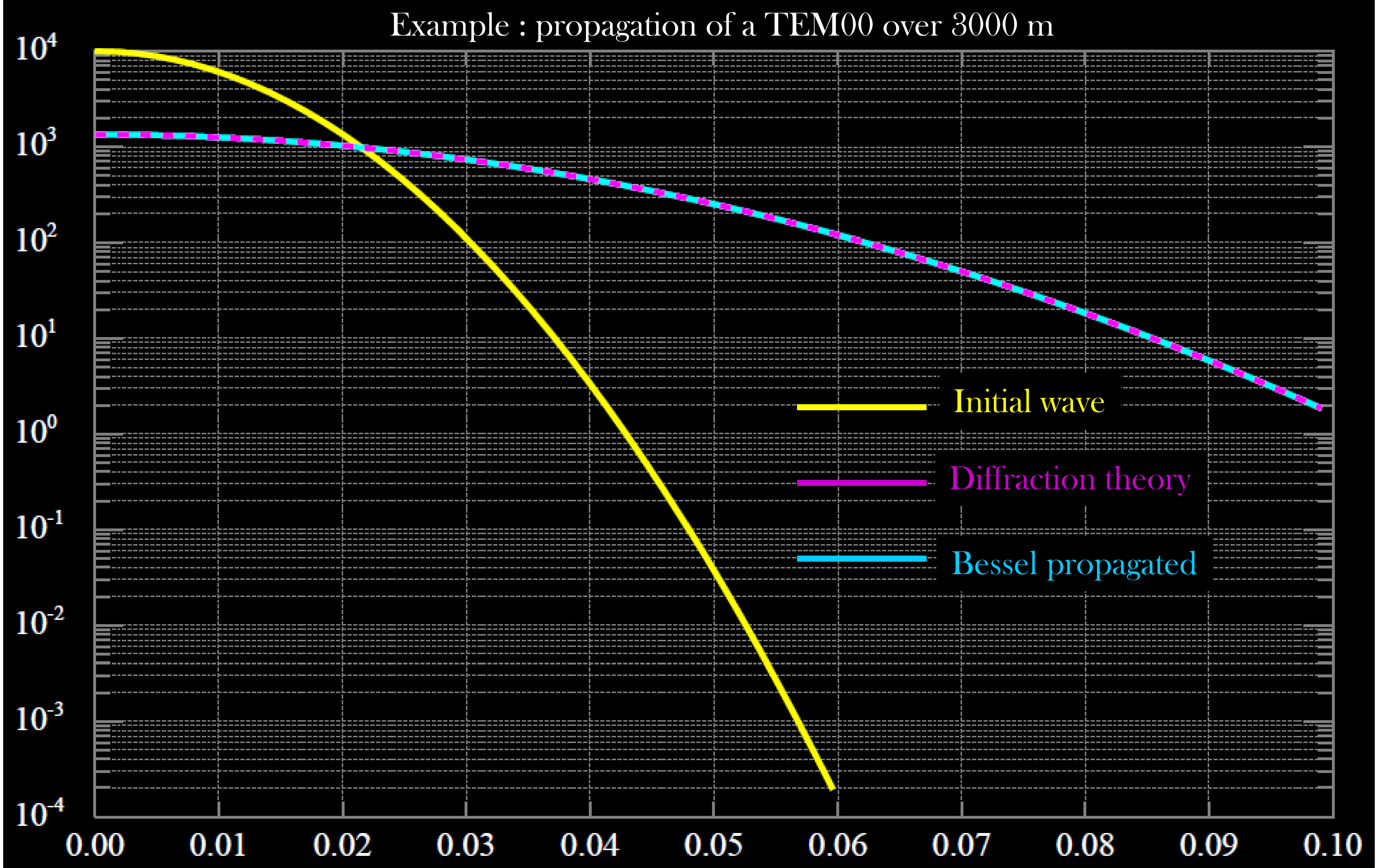
In the Fourier-Bessel space : $p^2+q^2=\rho^2$

After sampling : $\rho_\alpha \equiv \frac{\varsigma_\alpha}{a}$

$$P_\alpha(\Delta z) = \exp\left[-i\frac{\lambda\Delta z}{4\pi a^2}\varsigma_\alpha^2\right]$$

Diffraction step by a simple matrix product :

$$\Psi_\alpha(z+\Delta z) = \sum_{\beta=1}^{\infty} P_{\alpha\beta}\Psi_\beta(z) \quad \text{with} \quad P_{\alpha\beta}(\Delta z) \equiv \underbrace{\sum_{\sigma=1}^{\infty} H^{(-)}_{\alpha\sigma}P_\sigma(\Delta z)H^{(+)}_{\sigma\beta}}_{\text{To be computed once}}$$


Example : propagation of a TEM00 over 3000 m
10^4
10^3
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^-4
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.10
Initial wave
Diffraction theory
Bessel propagated

Representation of mirrors

Axially symmetrical defects : diagonal operator

$$M_\alpha = r \exp\left[\frac{4i\pi}{\lambda} \left(\frac{r_\alpha^2}{2R_c} + f(r_\alpha) \right) \right] \quad \text{with} \quad r_\alpha \equiv \frac{\varsigma_\alpha a}{\varsigma_N}$$

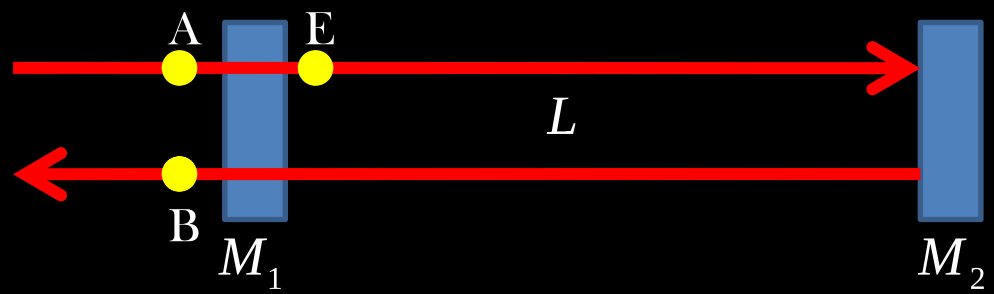
Pure parabolic contribution

defects

Reflected field :

$$\Psi'_\alpha = M_\alpha \Psi_\alpha$$

Example : reflectance of a Fabry-Perot cavity



Intracavity field : $$E = t_1 A + \mathrm{e}^{2ikL} \underbrace{M_1 P(L) M_2 P(L)}_{} E$$

$\mathcal{C}$ Matrix operator

Intracavity field by matrix inversion : $$E = \left[Id - \mathrm{e}^{2ikL} \mathcal{C} \right]^{-1} t_1 A$$

Reflected field by matrix product : $$B_\alpha = R_{\alpha\beta} A_\beta$$

With the reflectance operator $$R = \left(M_1^\dagger + t_1 P M_2 P \left[Id - \mathrm{e}^{2ikL} \mathcal{C} \right]^{-1} t_1 \right)$$

Modal propagation : general principles

The set of all complex functions $\Psi(x,y)$ of integrable square modulus has the structure of a Hilbert space, with a scalar product

$$\langle \Psi, \Phi \rangle = \int_{\mathbb{R}^2} dxdy\, \Psi^*(x,y)\Phi(x,y)$$

An example of a basis of such a HS is the Hermite-Gauss family of optical modes

$$\Psi_{nm}(x,y) = \gamma_{nm} H_n\left(\sqrt{2}\,\frac{x}{w}\right) H_m\left(\sqrt{2}\,\frac{y}{w}\right) \exp\left[-\frac{x^2+y^2}{w^2}\right] \exp\left[i\,\frac{2\pi}{\lambda}\,\frac{x^2+y^2}{R}\right]$$

So that any optical amplitude can be expanded in a series of HG modes

$$A(x,y) = \sum_{m,n} A_{nm} \Psi_{nm}(x,y)$$

Propagation of a HG mode of parameter (waist) w_0 :

$$\Psi_{lm}(x,y,z)=\sqrt{\frac{2P}{\pi w(z)^2}}\sqrt{\frac{1}{2^{m+n}m!n!}}\exp\left[-\frac{r^2}{w(z)^2}+ik\frac{r^2}{2R(z)}-iG_{lm}(z)\right]\times$$

$$\times H_l\left(\sqrt{2}\frac{x}{w}\right)H_m\left(\sqrt{2}\frac{y}{w}\right)$$

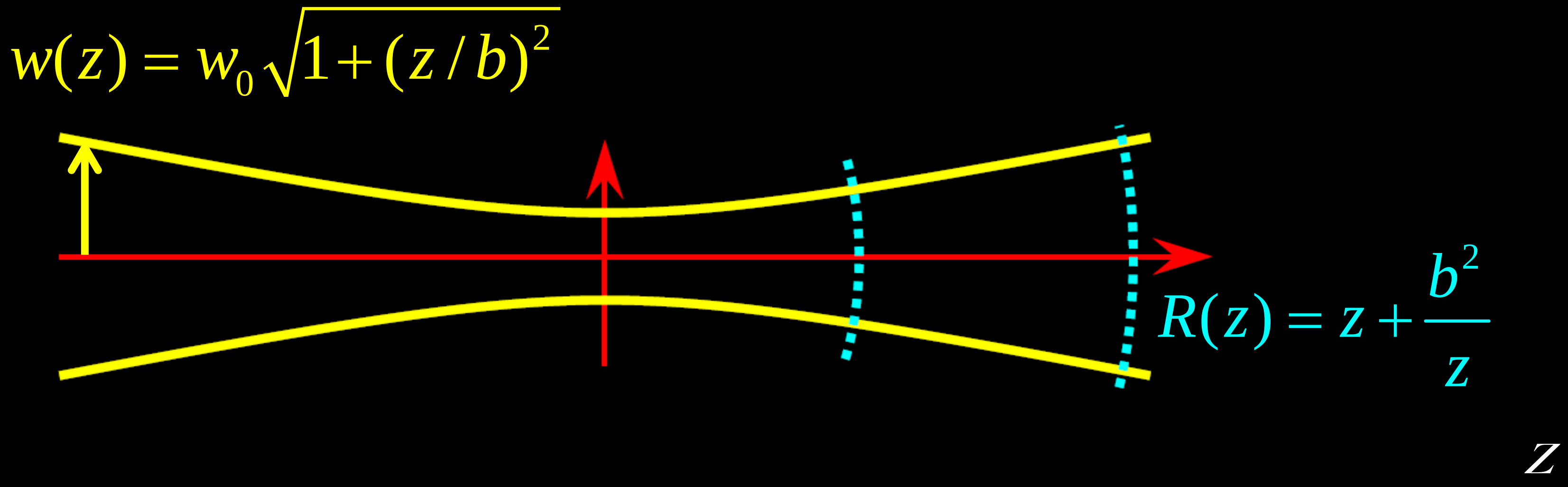
Rayleigh parameter : $b=\pi w_0^2/\lambda$

Beam width : $w(z)=w_0\sqrt{1+(z/b)^2}$

Curvature radius of the wavefront : $R(z)=z+\dfrac{b^2}{z}$

Gouy phase $G_{lm}(z)=(l+m+1)\arctan(z/b)$

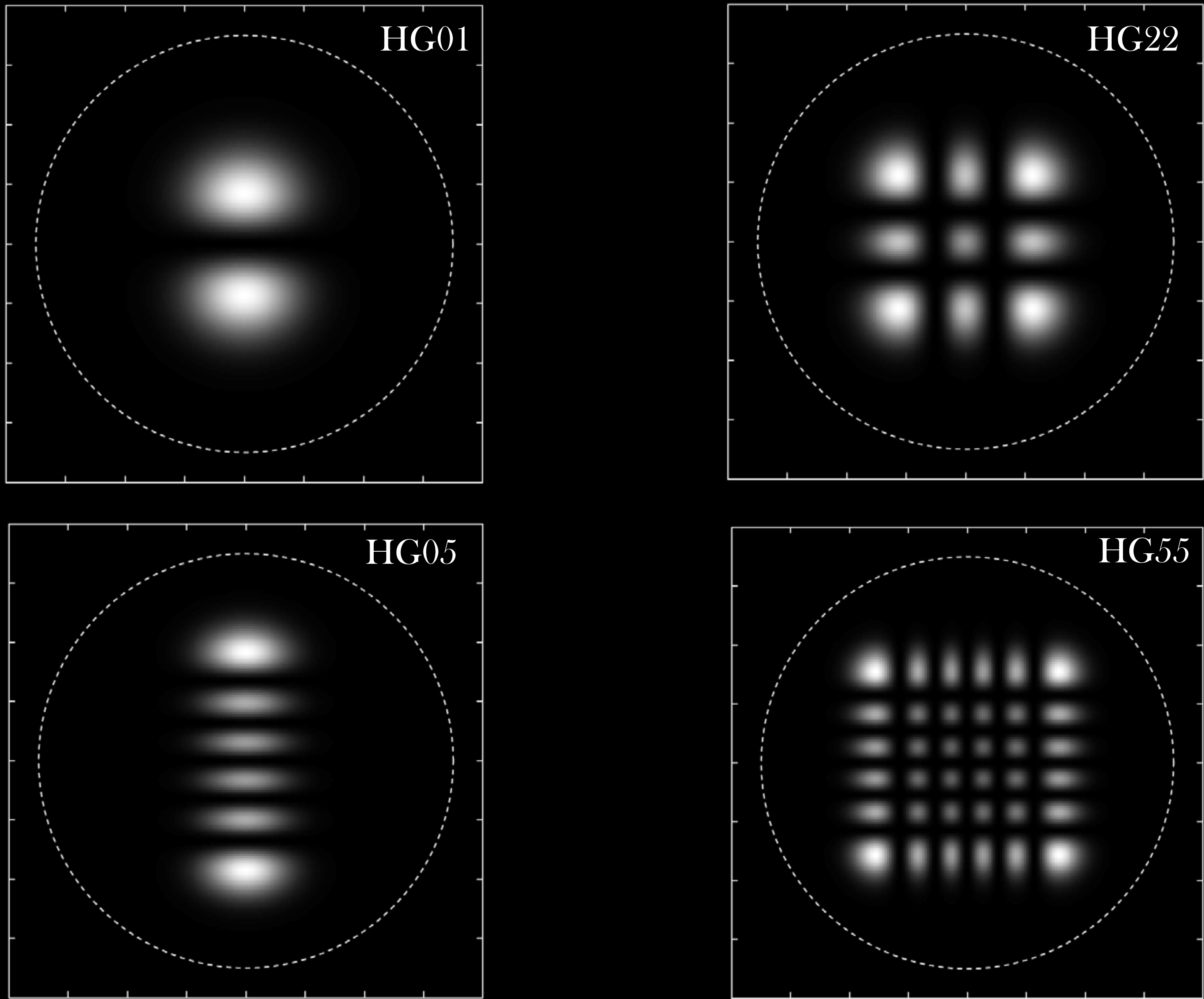
Diffraction of a Gaussian beam



Representation of mirrors by their matrix elements

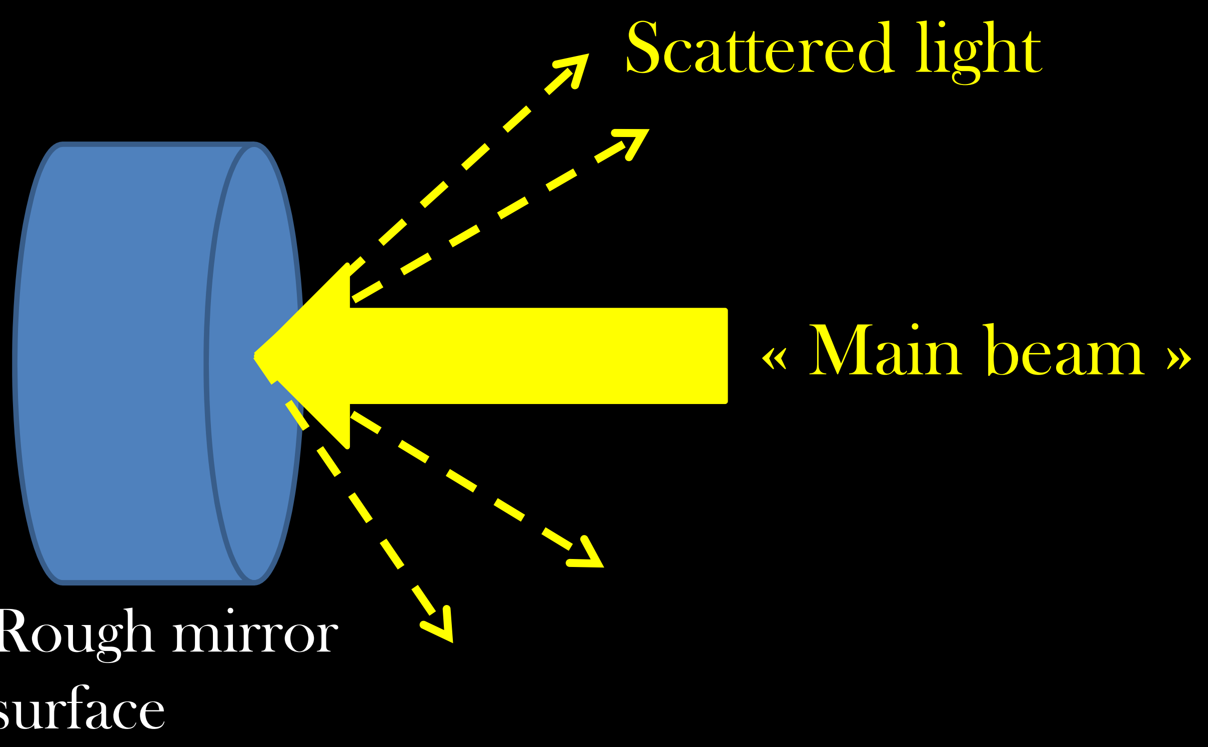
$$M_{abcd} = \langle \Psi_{ab}, M\Psi_{cd} \rangle$$

Propagation by simple linear algebra


HG01
HG22
HG05
HG55

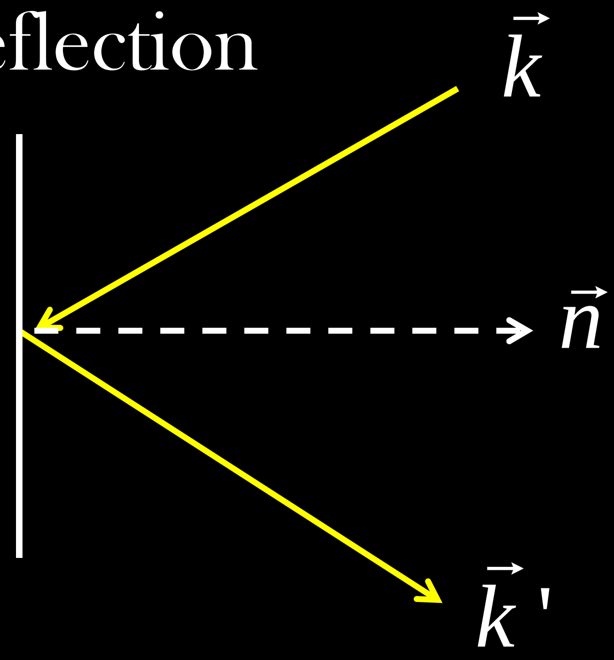
Propagation of light in complex structures by Monte-Carlo photons

Principle : send random pointlike particles (« photons ») from identified sources



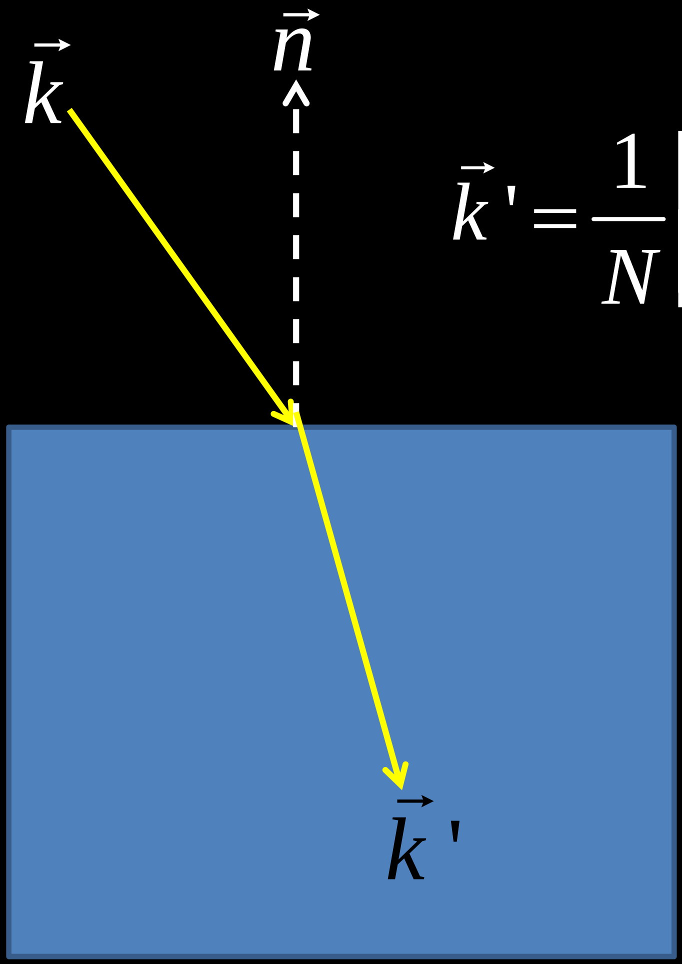
At each interface random decision :
absorption, reflection, refraction, scattering
according to the properties of the local material

1) Reflection



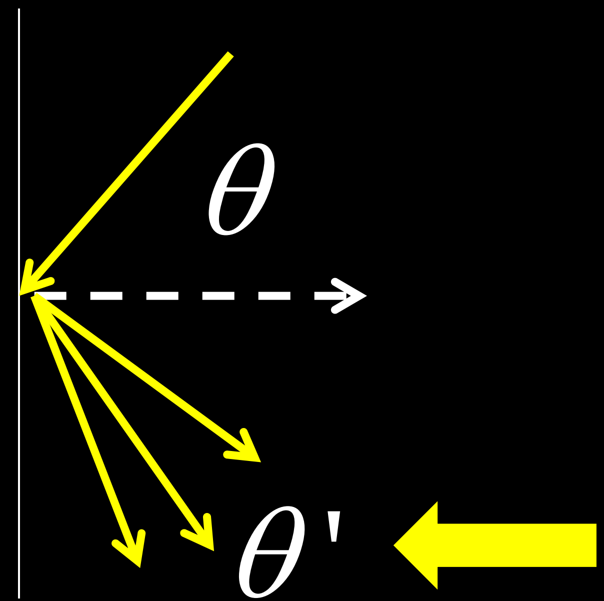
$$\vec{k}\,' = \vec{k} - 2(\vec{k}.\vec{n})\vec{n}$$

2) Refraction



$$\vec{k}\,' = \frac{1}{N}\left[\vec{k} - \left(\vec{k}.\vec{n} - \sqrt{N^2 - 1 + (\vec{k}.\vec{n})^2}\right)\vec{n}\right]$$

3) Diffusion



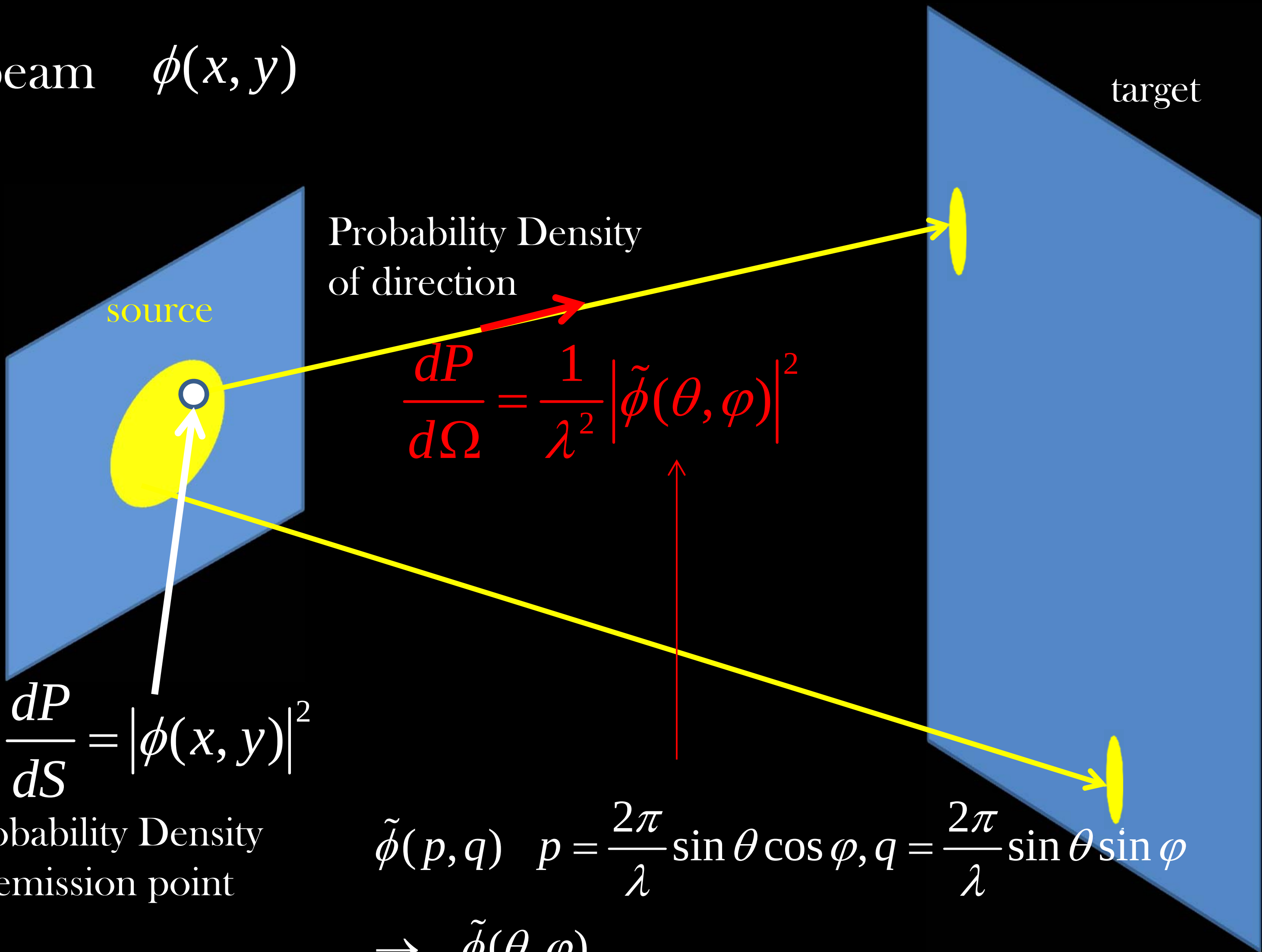
Rough surface

Random variable
with a PD that mimics
the BRDF of the material

Diffraction of photons ?

Example 1 : Propagation of a beam

Any beam $\phi(x, y)$



Probability Density of direction

$$\frac{dP}{d\Omega} = \frac{1}{\lambda^2}\left|\tilde{\phi}(\theta,\varphi)\right|^2$$

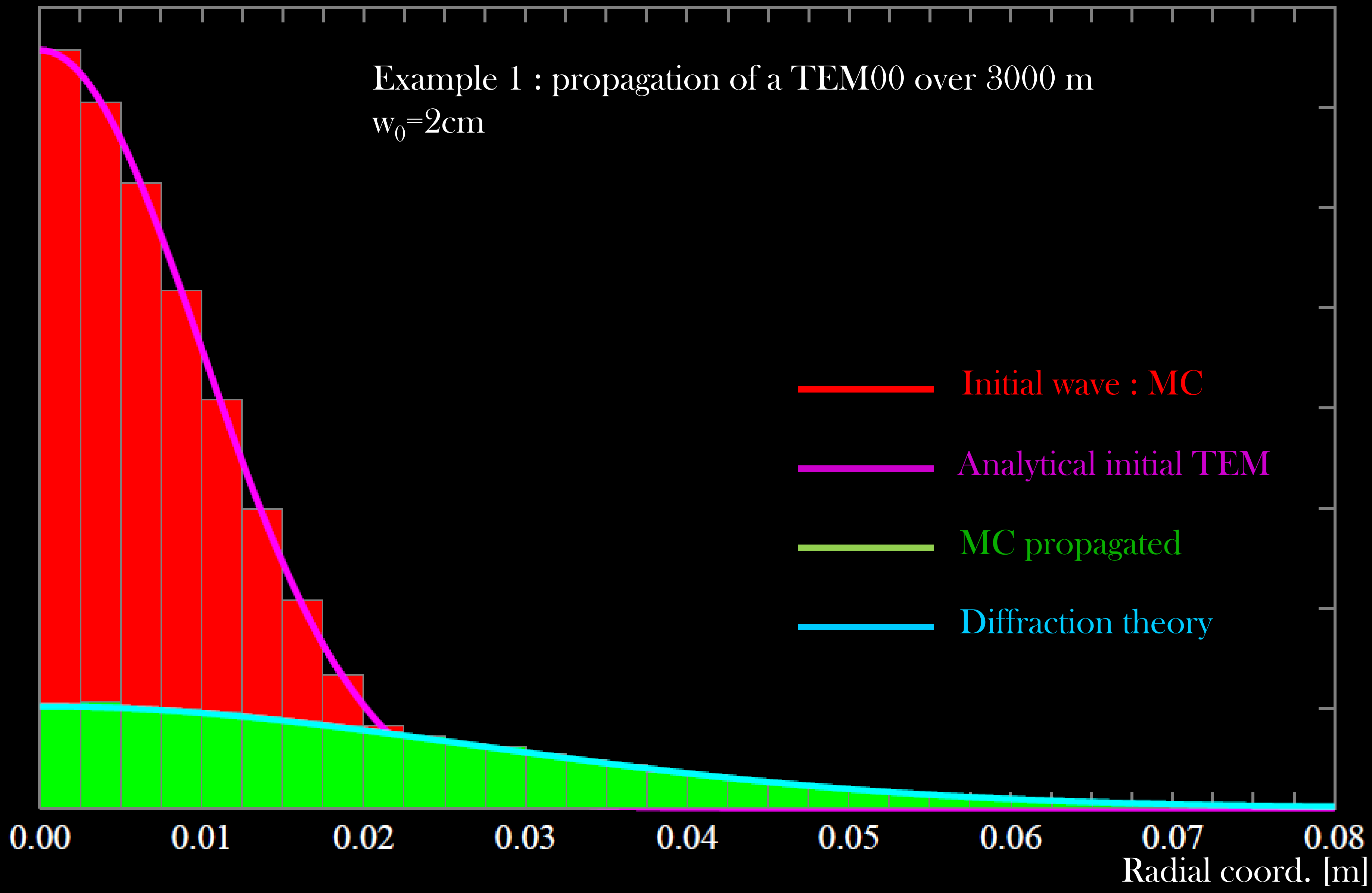
$$\frac{dP}{dS} = \left|\phi(x,y)\right|^2$$

Probability Density of emission point

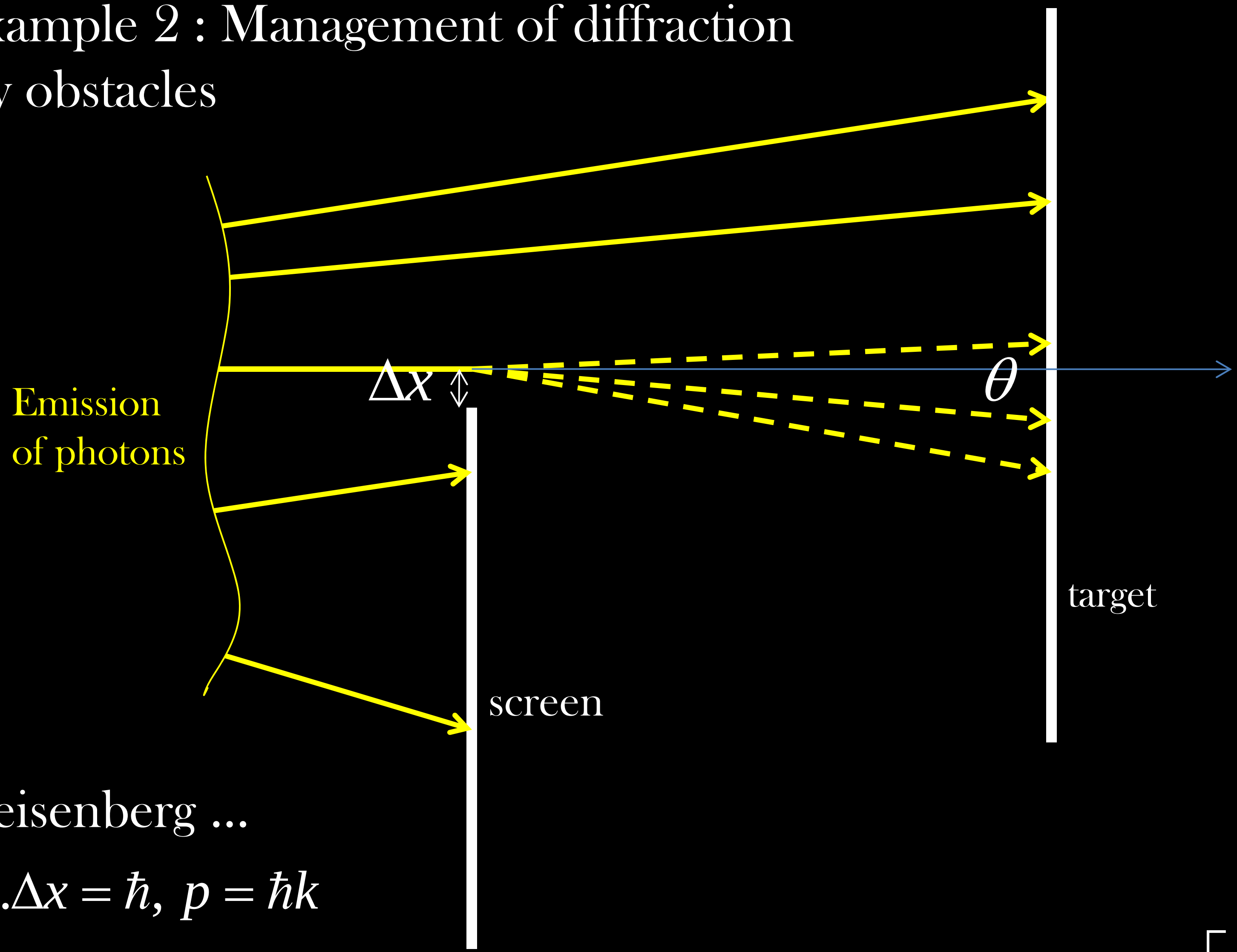
$$\tilde{\phi}(p,q) \quad p = \frac{2\pi}{\lambda}\sin\theta\cos\varphi, q = \frac{2\pi}{\lambda}\sin\theta\sin\varphi$$

$$\rightarrow \quad \tilde{\phi}(\theta,\varphi)$$

Monte-Carlo methods



Example 2 : Management of diffraction by obstacles

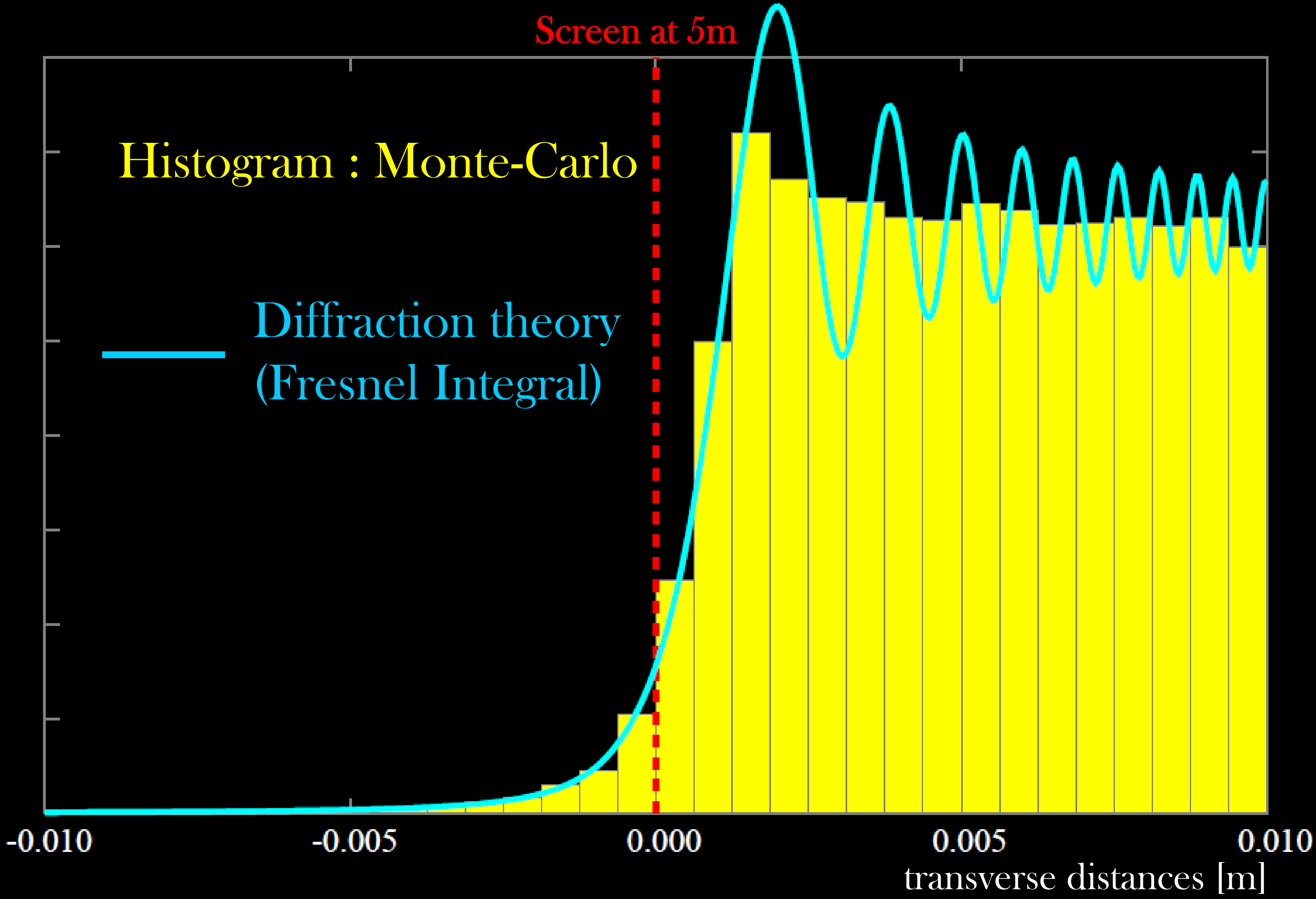


Heisenberg ...

$$\Delta p.\Delta x = \hbar,\ p = \hbar k$$

θ : Centered random deviate of standard deviation $\theta^* = \arctan\left[\dfrac{\lambda}{4\pi\Delta x}\right]$

Example 2 : diffraction by an edge



Conclusion

Discrete Wave optics

* FFT propagation : general purpose codes, suitable even for short spatial wavelength defects of mirrors : DarkF, SIS, FOG, OSCAR

• Propagation by Discrete Hankel transform : suitable for axisymmetrical problems (eg. heating by axisymmetrical beams)

• Propagation by modal expansion : very fast when a few modes are used → time dependent models for studying locking, servo loops, etc… SIESTA, FINESSE, MIST

Particle optics : mandatory for propagation of scattered light in complex structures (vacuum tanks, etc…)