Unified dispersive approach to $\gamma^* \to \gamma\pi\pi$ and $\gamma\gamma \to \pi\pi$ at low energy

Bachir Moussallam
Project started with *Diogo Boito*
[arXiv:1305.3143]

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Introduction

- Goal: build representations for amplitudes

$$\gamma\gamma^*(q^2) \to \pi\pi \text{ or } \gamma^*(q^2) \to \gamma\pi\pi \qquad (|q^2| \lesssim 1 \text{ GeV}^2)$$

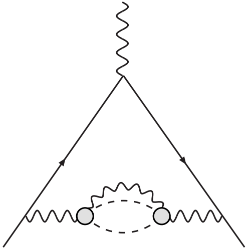
  → which combine nonperturbative QCD tools [Unitarity, analyticity, Chiral symmetry, soft photon theorems];

  → generalizes $\gamma\gamma \to \pi\pi$ [Morgan, Pennington Phys.Lett B192(1987), Phys.Lett B272 (1991)] [Donoghue, Holstein, Phys.Rev D48 (1991)]

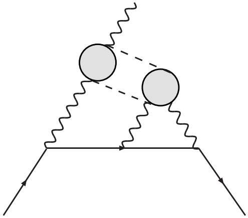
  Amplitude completely determined by matching with ChPT $O(p^4)$

- Applications: pion structure: generalized polarizabilities; sigma meson EM properties

- Relevance for muon $g-2$
  - → Evaluate separately:
    HVP (purely hadronic):
    and: $(\mathrm{H}+\gamma)$VP



  - → Use theoretical properties e.g. pion form factor v.s. $\pi\pi$ $J=1$ phase-shifts [de Troconiz, Yndurain Phys.Rev.D65 (2002)]
  - → Future: 2π contribution in light-by-light amplitude ([Martin Hoferichter et al.])



Theory of final-state interaction:

- Unitarity key ingredient to FSI
  $\pi\pi$ scattering elastic: $s \lesssim 1$ GeV^2

- Fermi-Watson phase theorem for $\gamma\gamma^*(q^2) \to \pi\pi$:

  → $s > 4m_\pi^2$, $q^2 \leqslant 4m_\pi^2$:

$$\mathrm{Im}\,(\gamma\gamma^* \to \pi\pi)_J = (\gamma\gamma^* \to \pi\pi)_J^*(\pi\pi \to \pi\pi)_J$$

  → $s > 4m_\pi^2$, $q^2 > 4m_\pi^2$:

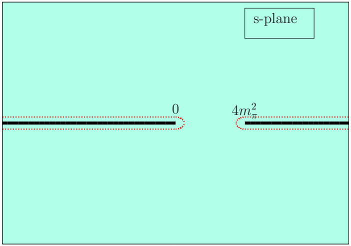
$$\mathrm{Im}\,(\gamma\gamma^* \to \pi\pi)_J = (\gamma\gamma^* \to \pi\pi)_J^*(\pi\pi \to \pi\pi)_J + (\gamma^* \to \pi\pi)(\gamma\pi\pi \to \pi\pi)_J$$

  Fermi-Watson does not apply here! [Creutz,Einhorn PR D1 (1970)2537.]

- Combine unitarity with analyticity [Omnès, NC 8 (1958) 316]. partial-wave amplitudes analytic s with two cuts

  → right-hand cut: $[4m_\pi^2, \infty]$

  → left-hand cut: $[-\infty, 0]$ (usual situation !)



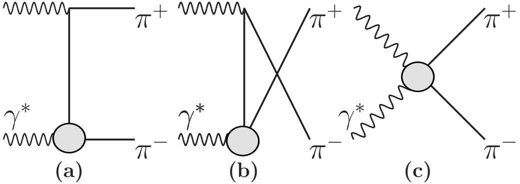
- Discontinuity on RHC: valid also when $q^2 > 4m_\pi^2$.

$$\mathrm{disc}(\gamma\gamma^* \to \pi\pi) = 2i(\gamma\gamma^* \to \pi\pi)_{s-i\epsilon}(\pi\pi \to \pi\pi)_{s+i\epsilon}$$

- Omnés-Muskhelishvili eqs.: Full amplitude in terms of $\pi\pi$ phase-shifts + left-hand cut piece

Phenomenology of left-hand cut

- Pion pole (Born Amplitude). Only $\pi^+\pi^-$



→ Vertex $\pi^+\pi^-\gamma^*$

$$\langle \pi^+(p)|j_\mu(0)|\pi^+(p')\rangle = (p+p')_\mu F_\pi^v((p-p')^2)$$

→ Diagram (c) needed to restore gauge invariance

$$W_{\mu\nu}^{(c)} = 2g_{\mu\nu}F_\pi^v(q^2)$$

- Partial-wave $J = 0$

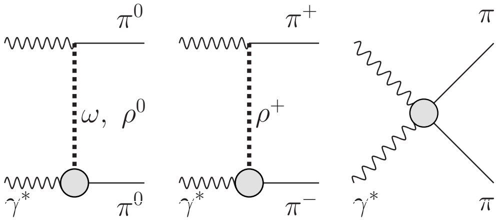
$$h^{Born}_{0,++}(s, q^2) = \frac{1}{s - q^2} \left[\frac{4m_\pi^2}{\sigma_\pi(s)} \log \frac{1 + \sigma_\pi(s)}{1 - \sigma_\pi(s)} - 2q^2 \right]$$

with $\sigma_\pi(s) = \sqrt{1 - 4m_\pi^2/s}$

- Singularities:
  - → left-hand cut $[-\infty, 0]$,
  - → also: pole at $s = q^2$ (soft photon, IR divergence)

Resonance exchange amplitudes

- Start with zero width resonances:



- Invariant amplitudes:

$$A^V(s,t,q^2) = \tilde{C}_V F_{V\pi}(q^2)\left[\frac{s-4m_\pi^2-4t+q^2}{t-M_V^2}+\frac{s-4m_\pi^2-4u+q^2}{u-M_V^2}\right]$$

$$B^V(s,t,q^2) = \tilde{C}_V F_{V\pi}(q^2)\left[\frac{1}{2(t-M_V^2)}+\frac{1}{2(u-M_V^2)}\right]$$

$$C^V(s,t,q^2) = \tilde{C}_V F_{V\pi}(q^2)\left[\frac{1}{t-M_V^2}-\frac{1}{u-M_V^2}\right]$$

- Must take width into account
- Usual propagators (BW, GS) not quite correct [first-sheet poles, left-hand cuts...]
- Analytically correct way: use Källen-Lehmann representation [E. Lomon,S. Pacetti,Phys.Rev.D85 (2012)]

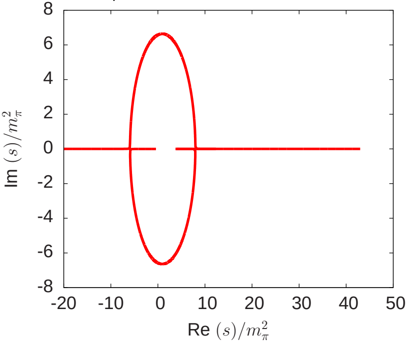
$$\frac{1}{M_V^2 - t} \longrightarrow \widetilde{BW}_V(t) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \, \frac{\sigma(t', M_V, \Gamma_V)}{(t' - t)}$$

with e.g. $\sigma(t', M_V, \Gamma_V) = \mathrm{Im}\, BW_V(t', M_V, \Gamma_V)$

- Also: use limiting prescription: $q^2 = \lim_{\epsilon \to 0} q^2 + i\epsilon$

- Left-hand cut extends in complex plane($q^2 > 4m_\pi^2$)

$\sqrt{q^2} = 0.95$ GeV



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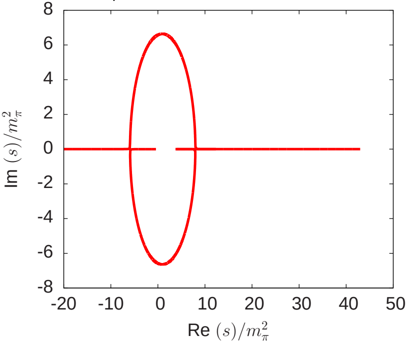
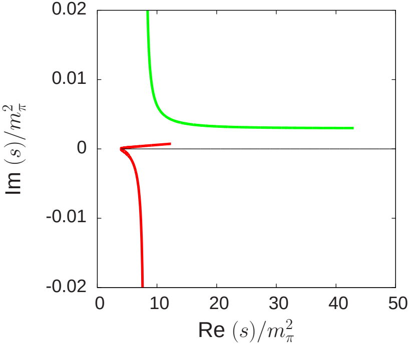


- Approaches the unitarity cut but does not touch it ($q^2 + i\epsilon$ prescription)



- Omnès method works as usual (no anomalous threshold)

Further theoretical constraints

1) Soft photon limit

$$\begin{array}{lcl} \pi^0\pi^0 & : & \lim_{s\to q^2} H^n_{\lambda\lambda'} = 0 \\ \pi^+\pi^- & : & \lim_{s\to q^2}[H^c_{\lambda\lambda'} - H^{Born}_{\lambda\lambda'}] = 0 \end{array}$$

2) Chiral symmetry: soft pion theorem (for $\pi^0\pi^0$)

$$\left[A(s,t,q^2) + 2q^2(B(s,t,q^2) - C(s,t,q^2))\right]_{s=0,t=0} = 0$$

for <u>any q^2</u>

<u>Physical</u> ++ helicity amplitude, $t = m_\pi^2$ has an Adler zero

$$H^n_{++}(s_A, q^2, \theta)\Big|_{t=m_\pi^2} = 0 \text{ with } s_A = O(m_\pi^2)$$

s_A depends on q^2

- LE expansion (ChPT), valid when $|q^2|,\ |s| << 1\ \mathrm{GeV}^2$
  - → NLO calculations:
    [Bijnens, Cornet NP B296 (1988) 557] $(q^2 = 0)$
    [Donoghue,Holstein, PR D48 (1993) 137] $(q^2 \neq 0)$
  - → NNLO calculations: $(q^2 = 0$ only$)$
    [Bellucci et al. (1994), Bürgi (1996), Gasser,Ivanov,Sainio (2005,2006)]

- NLO expression

$$H^{n}_{++}\big|_{NLO} = \frac{2(s-m_\pi^2)}{F_\pi^2}\,\bar{\mathcal{G}}(s,q^2)$$

$$H^{c}_{++}\big|_{NLO} = \frac{s}{F_\pi^2}\,\bar{\mathcal{G}}(s,q^2) + (\bar{l}_6 - \bar{l}_5)\,\frac{s-q^2}{48\pi^2 F_\pi^2} + H^{Born}_{++} F_\pi^{v,NLO}(q^2)$$

- Omnès repres. + constraints (based on 2-subtracted DR)

$$
\begin{aligned}
H^I_{++}(s,q^2,z) =& F^v_\pi(q^2)\,\bar{H}^{I,Born}_{++}(s,q^2,z) + \sum_{V=\rho,\omega} F_{V\pi}(q^2)\,\bar{H}^{I,V}_{++}(s,q^2,z) \\
&+ \Omega^I_0(s)\Bigg\{(s-q^2)\,b^I(q^2) \\
&\quad + s\,F^v_\pi(q^2)\left[\frac{s\left(J^{I,\pi}(s,q^2)-J^{I,\pi}(q^2,q^2)\right)}{s-q^2} - q^2\hat{J}^{I,\pi}(q^2)\right] \\
&\quad + s\sum_{V=\rho,\omega} F_{V\pi}(q^2)\left[s\,J^{I,V}(s,q^2) - q^2 J^{I,V}(q^2,q^2)\right] \Bigg\}
\end{aligned}
$$

→ First line: tree diagrams

→ $J^{I,\pi}$, $J^{I,V}$: rescattering integrals ($I = 0, 2$, J=0)

→ Omnès function: $\Omega^I_0(s) = \exp\left(\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'(s'-s)}\,\delta^I_0(s')\right)$

→ Matching with ChPT at $s = 0$

(Continued)

$$J^{I,\pi}(s,q^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{(s')^2(s'-s)} \frac{\sin\delta_0^I(s')}{|\Omega_0^I(s')|} \bar{h}_{0,++}^{I,\pi}(s',q^2)$$

$$J^{I,V}(s,q^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{(s')^2(s'-s)} \frac{\sin\delta_0^I(s')}{|\Omega_0^I(s')|} \bar{h}_{0,++}^{I,V}(s',q^2)$$

(integrals well defined)

→ Note: (from soft photon condition)

$$\hat{J}^{I,\pi}(q^2) = \left.\frac{\partial J^{I,\pi}(s,q^2)}{\partial s}\right|_{s=q^2}$$

→ $\hat{J}^{I,\pi}(q^2)$ diverges when phase-shift $\delta_0^I(q^2)$ has a cusp (inelasticity: self destructive formula)

- Parametrization of subtraction functions:

$$b^n(q^2) = b^n(0)\,\bar{F}(q^2) + \beta_\rho(GS_\rho(q^2) - 1) + \beta_\omega(BW_\omega(q^2) - 1)$$
$$b^c(q^2) = b^c(0) + \beta_\rho(GS_\rho(q^2) - 1) + \beta_\omega(BW_\omega(q^2) - 1)$$

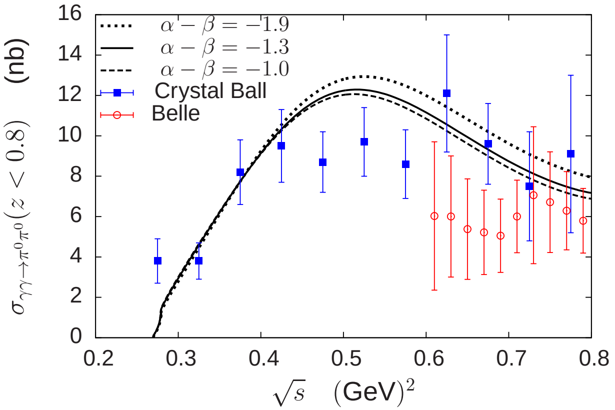
→ $\bar{F}(q^2)$ from chiral NLO

→ $b^n(0)$, $b^c(0)$: comp. with chiral amplitude
Note: relation to pion polarizabilities

$$\alpha_{\pi^0} - \beta_{\pi^0} = \frac{2\alpha}{m_\pi}\left[\lim_{s\to 0}\frac{1}{s}H_{++}^{n,V}(s,0,\theta) + b^n(0)\right]$$
$$\alpha_{\pi^+} - \beta_{\pi^+} = \frac{2\alpha}{m_\pi}\left[\lim_{s\to 0}\frac{1}{s}H_{++}^{c,V}(s,0,\theta) + b^c(0)\right]$$

→ β_ρ, β_ω: experimental inputs

$q^2 = 0$ comparison w. data $[\gamma\gamma \to \pi^0\pi^0]$



Crystal Ball: [PR D41 (1990) 3324]
Belle : [PR D78 (2008) 052004]
KLOE-2 [expected]

- $q^2 \neq 0$ need inputs for form-factors

  → F_π^v well known from many $e^+e^- \to \pi^+\pi^-$, $\tau^\pm \to \pi^\pm\pi^0\nu$ experiments

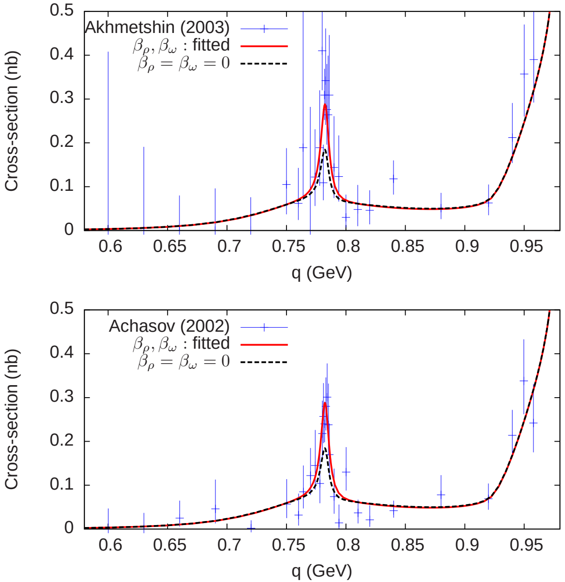
  → $F_{\omega\pi}$: measurements from $e^+e^- \to \omega\pi^0$
  [SND, Nucl.Phys.B569 (2000)]
  [CMD-2, Phys.Lett.B562(2003)]
  also from $\omega \to \pi^0 e^+ e^-$ [Landsberg, Phys.Repts.128(1985),NA60 Phys.Lett.B677(2009)]

  → $F_{\rho\pi}$: No experimental result !
  must be estimated from flavour symmetry arguments

- Two parameter fit of $\sigma_{e^+e^- \to \gamma\pi^0\pi^0}(q^2)$ data
  [Akhmetshin et al.[CMD2], Phys.Lett.B580(2004)119]
  [Achasov et al., [SND], Phys.Lett.B537 (2002) 201]

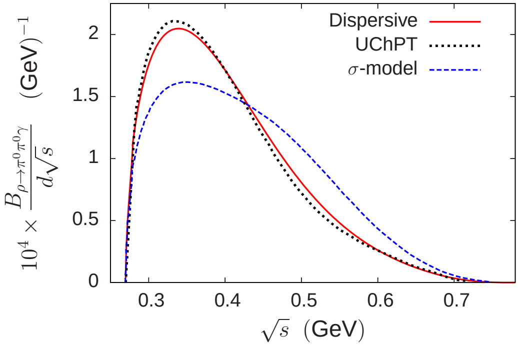
β_ρ	β_ω	χ^2/N_{dof}	ref.
0.14 ± 0.12	$(-0.39 \pm 0.12)\,10^{-1}$	20.2/27	SND (2002)
-0.13 ± 0.15	$(-0.31 \pm 0.15)\,10^{-1}$	15.0/21	CMD-2 (2003)
0.05 ± 0.09	$(-0.37 \pm 0.09)\,10^{-1}$	38.1/50	Combined

- (Two parameter) fit of $\sigma_{e^+e^- \to \gamma\pi^0\pi^0}(q^2)$ data



- Illustrative comp. w. other approaches

→ Consider $\rho^0 \to \pi^0\pi^0\gamma$

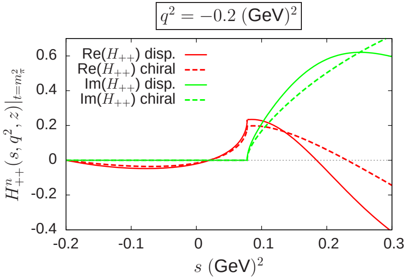


– Chiral Lagr.+V + unitarized pion loop [Palomar,Hirenzaki,Oset NP A707 (2002)161]

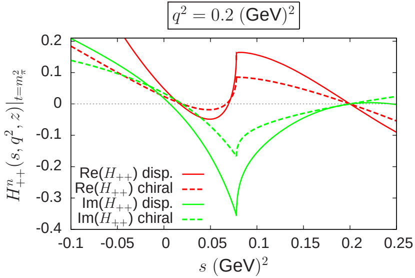
– Chiral Lagr.+V +pion loop +sigma meson [Bramon et al. PL B517 (2001) 345]

Omnès $\pi^0\pi^0$ amplitude compared with NLO ChPT

$q^2 = -0.2$ GeV^2



$q^2 = 0.2$ GeV^2



→ Form factor $=1$ at NLO

H$(+\gamma)$VP contrib. to muon $(g-2)/2$:

- Contrib. from Hadronic Vacuum Polarization

$$a_\mu = \frac{1}{4\pi^3}\sum_n \int_{m_\pi^2}^{\infty} dq^2\, K_\mu\left(q^2\right)\, [\sigma_{e^+e^-\to n} + \sigma_{e^+e^-\to n+\gamma} + \cdots](q^2)$$

with

$$n = \text{hadronic state}$$

and

$$K_\mu(q^2) = \int_0^1 dx\, \frac{x^2(1-x)}{x^2 + \frac{q^2}{m_\mu^2}(1-x)}$$

- $\pi^+\pi^-$ contribution

$$\sigma_{e^+e^-\to\pi^+\pi^-} = \frac{\pi\alpha^2}{3q^2}\sigma_\pi^3(q^2)|F_\pi^V(q^2)|^2$$

- $\gamma\pi\pi$ contribution in HVP

$$a_\mu^{[\gamma\pi\pi]} = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{q^2_{max}} dq^2\, K_\mu\left(q^2\right)\,\left(\sigma^c(q^2)+\sigma^n(q^2)\right)$$

σ^c, σ^n: charged, neutral pions

→ In terms of helicity amplitudes

$$\sigma^{c,n}(q^2) = \frac{\alpha^3}{12(q^2)^3}\int_{4m_\pi^2}^{q^2} ds(q^2-s)\sigma_\pi(s)\int_{-1}^{1} dz \sum |H^{c,n}_{\lambda\lambda'}(s,q^2,\theta)|^2$$

- Charged pions, σ^c:

  → Separate Born:

$$|H^c_{\lambda\lambda'}|^2 = |H^{Born}_{\lambda\lambda'}|^2 + 2\mathrm{Re}\,(H^{*\,Born}_{\lambda\lambda'} H^{V+resc}_{\lambda\lambda'}) + |H^{V+resc}_{\lambda\lambda'}|^2$$

  → σ^{Born}: s integral IR divergent
  defined: e.g. add rad. corr. part $\sigma(e^+e^- \to \pi^+\pi^-)$

  → σ^{Born} corresponds to scalar QED (sQED)

$$\sigma^{Born}(q^2) = \frac{\pi\alpha^2}{3\,q^2}\,\sigma^3_\pi(q^2)\,|F^v_\pi(q^2)|^2 \times \frac{\alpha}{\pi}\eta(q^2)$$

($\eta(q^2)$ known analytically[Jegerlehner, Nyffeler Phys.Repts. 477 (2009) 1])

- Numerical results ($q^{max} = 0.95$ GeV):

channel	cross-section	a_μ
$\gamma\pi^+\pi^-$	$\lvert H^{Born}\rvert^2$	41.9×10^{-11}
$\gamma\pi^+\pi^-$	$H^{*Born} H^{V+resc}$	$(1.31 \pm 0.30) \times 10^{-11}$
$\gamma\pi^+\pi^-$	$\lvert H^{V+resc}\rvert^2$	$(0.16 \pm 0.05) \times 10^{-11}$
$\gamma\pi^0\pi^0$	$\lvert H^{V+resc}\rvert^2$	$(0.33 \pm 0.05) \times 10^{-11}$

Remarks

→ $a_\mu[H^{*Born} H^{V+resc}] > 0$ unlike [Dubinsky et al. EPJ C40 (2005) 41]

→ $a_\mu[\lvert H^{V+resc}\rvert^2]$ compared w. σ-meson approx: $a_\mu^{[\gamma\sigma]} = 1.2 \times 10^{-11}$ [Narison (2003)], $= 1.5 \times 10^{-11}$ [Ahmadov, Kuraev, Volkov (2010)]

Conclusions

- Analyticity based treatment of FSI in $\gamma\gamma \to \pi\pi$ extended to $\gamma\gamma^*(q^2)$
- Main issue: left-hand cut [pion, resonances] becomes generalized one, but properly defined
- Good description of experimental data $e^+e^- \to \gamma\pi^0\pi^0$ (2 parameters)
- a_μ: contributions from $\gamma\pi^0\pi^0$, $\gamma\pi^+\pi^-$ $[q < 0.95$ GeV$]$
- Other applications: pion generalized polarizabilities, sigma meson (pole)-γ form factor
- Extensions possible [$q \gtrsim 1$ GeV] (coupled-channel MO). Double virtual scattering ?

Backup Slides

Loop function in NLO ChPT

$$\bar{\mathcal{G}}(s,q^2) = \frac{s\,\bar{G}_\pi(s) - q^2\,\bar{G}_\pi(q^2)}{s-q^2} - q^2\,\frac{\bar{J}_\pi(s) - \bar{J}_\pi(q^2)}{s-q^2}$$

with

$$\bar{J}_\pi(z) = \frac{1}{16\pi^2}\left(2 + \sigma_\pi(z)\log\frac{\sigma_\pi(z)-1}{\sigma_\pi(z)+1}\right)$$

$$\bar{G}_\pi(z) = -\frac{1}{16\pi^2}\left(1 + \frac{m_\pi^2}{z}\log^2\frac{\sigma_\pi(z)-1}{\sigma_\pi(z)+1}\right)$$

- Notes:
  - → $\mathrm{Im}\,\bar{G}_\pi(z), \bar{J}_\pi(z) \neq 0$ when $z > 4m_\pi^2$
  - → $\lim_{s\to q^2}\bar{\mathcal{G}}(s,q^2) = 0$ (soft photon theorem)

Omega-pion form factor $F_{\omega\pi}(q^2)$

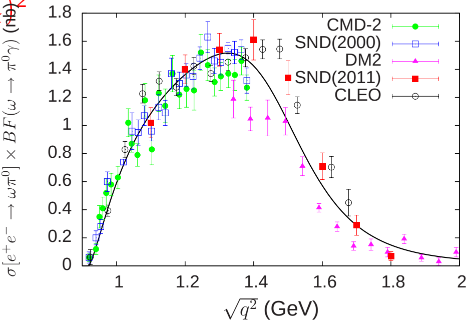
- We use similar parametrization to pion form factor:

$$F_{\omega\pi}(q^2) = \frac{1}{1+\beta'}\left[GS_\rho(q^2)\left(1+\delta\frac{q^2}{m_\omega^2}BW_\omega(q^2)\right)+\beta' GS_{\rho(1450)}(q^2)\right]$$

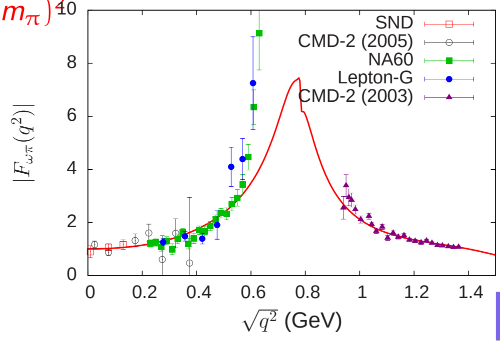
- Note: this is phenomenological. Better approach uses dispersion relations[Schneider,Kubis,Niecknig,PR D86 (2012) 054013]

- Parameters fitted to $e^+e^- \to \omega\pi$

- Result $q^2 > (m_\omega + m_\pi)^2$



- Result $q^2 < (m_\omega - m_\pi)^2$



- Puzzling behaviour in range $0.60 < q < 0.63$ GeV

- Rho-pion form factor $F_{\rho\pi}$:
  Parametrization:

$$F_{\rho\pi}(q^2) = \alpha_\omega \, BW_\omega(q^2) + \alpha_\phi \, BW_\phi(q^2) + \alpha_{\omega'} \, BW_{\omega'}(q^2)$$

- Unfortunately, no data in this case!
- Couplings α_ω, α_ϕ, $\alpha_{\omega'}$ estimated phenomenologically (relations to $g_{\omega\rho\pi}$, $g_{\phi\rho\pi}$, flavour symmetry...)