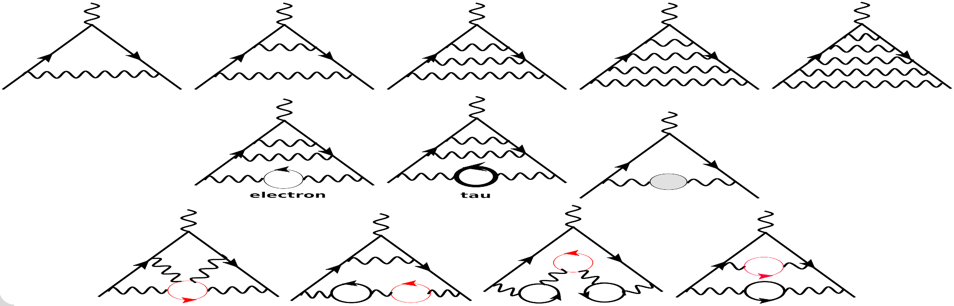
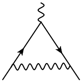
Towards analytic $(g-2)_\mu$ @ 4 loops

PhiPsi13, Rome, September 9-12, 2013

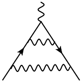
Matthias Steinhauser | TTP Karlsruhe



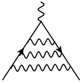
Lepton $g - 2$



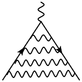
[Schwinger'48]



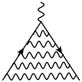
[Petermann'57;
Sommerfeld'58]



[Laporta,Remiddi'96]



[Laporta'93;
Kinoshita,Nio'06;
Aoyama,Hayakawa,
Kinoshita,Nio'07'08]



[Aoyama,Hayakawa,Kinoshita,Nio'12]

- $a_\mu(\mathrm{exp}) - a_\mu(\mathrm{SM}) \sim 240(87) \times 10^{-11}$
- $a_\mu(\text{4 loop QED}) \sim 380 \times 10^{-11}$

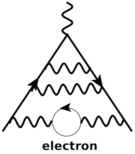
[Aoyama,Hayakawa,Kinoshita,Nio'12]

- aim: check 4-loop result
- independent calculation
- different method

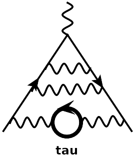
(Hadronic, electroweak, light-by-light: see talks by [Knecht, Stöckinger, Teubner, ...])

Outline

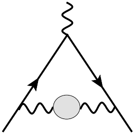
- 4-loop $(g-2)_\mu$: e^- loops



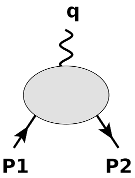
- 4-loop $(g-2)_\mu$: heavy lepton contribution



- 5-loop: polarization function insertions



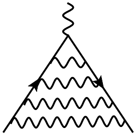
Form factor



$$= \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}}{2m} q_\nu F_2(q^2)$$

$$a = \frac{(g-2)}{2} = F_2(0) = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2)$$

$$\left.\begin{array}{c} p_1^2 = p_2^2 = M^2 \\ q \to 0 \end{array}\right\} \Rightarrow \text{on-shell integrals}$$



- 4-loop on-shell integrals
- not systematically studied in literature
- 3 loops: [Laporta,Remiddi'96; Melnikov,van Ritbergen'00]

n_l^2 and n_l^3 terms: 2 and 3 e^- loops

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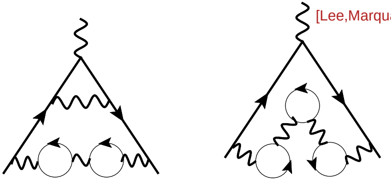


[Lee,Marquard,Smirnov,Smirnov,Steinhauser'13]

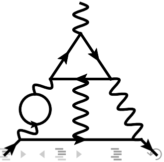
- gauge invariant subset
- results $\sim c_0 + c_1 \log(M_e/M_\mu) + c_2 \log^2(M_e/M_\mu) + \ldots + \frac{M_e}{M_\mu}[\ldots]$
- strategy:

$$\left.\begin{array}{lll} 1.)\ M_e = 0, M_\mu^{\rm OS}, \overline{\alpha}(\mu) & \Rightarrow & \log(\mu/M_\mu) \\ 2.)\ \overline{\alpha}(\mu) \to \alpha^{\rm OS} & \Rightarrow & \log(\mu/M_e) \end{array}\right\} \Rightarrow \log(M_e/M_\mu)$$

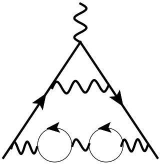
$$\overline{\alpha}(\mu) = \alpha^{\rm OS}\left[1 + \frac{2\alpha}{3\pi}\log\frac{\mu}{M_e}\right]$$

missing:

- $\mathcal{O}\left[M_e/M_\mu \log^n(M_e/M_\mu)\right]$ terms
- light-by-light-type contributions



Reduction to master integrals



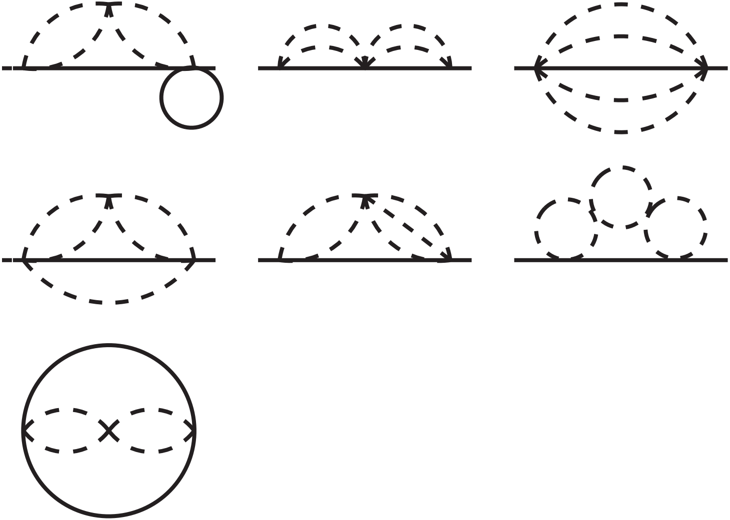
➪ $\sim 400\,000$ 4-loop integrals

reduce to minimal set of master integrals

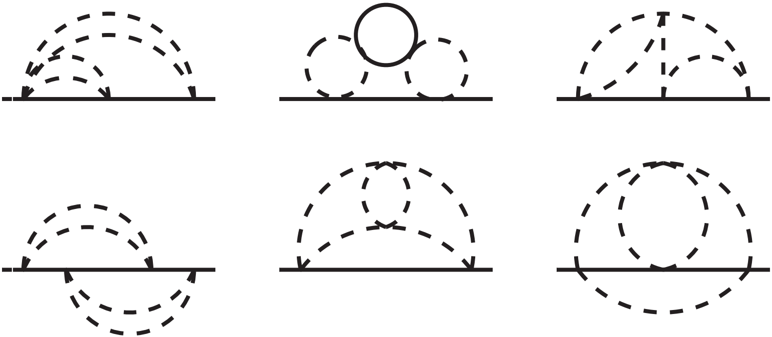
`FIRE` [Smirnov'08] (implementation of "Laporta" algorithm [Laporta'00])
`CRUSHER` [Seidel,Marquard]

➪ 13 master integrals

Master integrals I

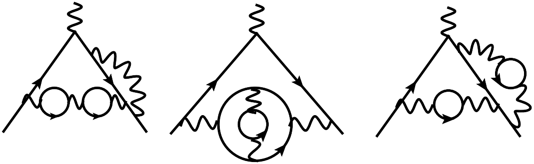


Master integrals II



Results

[Lee,Marquard,Smirnov,Smirnov,Steinhauser'13]



- new analytical results for n_l^2 terms

$a_1 = \log 2,\ a_n = \mathrm{Li}_n(1/2),\ L_{\mu e} = \log(M_\mu^2/M_e^2)$

$$
\begin{aligned}
a_\mu^{(42)a} \quad = \quad & L_{\mu e}^2 \left[\pi^2\left(\frac{5}{36} - \frac{a_1}{6}\right) + \frac{\zeta_3}{4} - \frac{13}{24}\right] + L_{\mu e}\left[-\frac{a_1^4}{9} + \pi^2\left(-\frac{2a_1^2}{9} + \frac{5a_1}{3} - \frac{79}{54}\right)\right. \\
& \left. - \frac{8a_4}{3} - 3\zeta_3 + \frac{11\pi^4}{216} + \frac{23}{6}\right] - \frac{2a_1^5}{45} + \frac{5a_1^4}{9} + \pi^2\left(-\frac{4a_1^3}{27} + \frac{10a_1^2}{9}\right. \\
& \left. - \frac{235a_1}{54} - \frac{\zeta_3}{8} + \frac{595}{162}\right) + \pi^4\left(-\frac{31a_1}{540} - \frac{403}{3240}\right) + \frac{40a_4}{3} + \frac{16a_5}{3} - \frac{37\zeta_5}{6} \\
& + \frac{11167\zeta_3}{1152} - \frac{6833}{864}
\end{aligned}
$$

$$\approx -3.624\,27$$

[note: $M_e \to 0$]

[Kinoshita,Nio'04; Aoyama,Hayakawa,Kinoshita,Nio'12]: $-3.642\,04(1\,12)$

Results (2)

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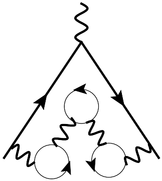
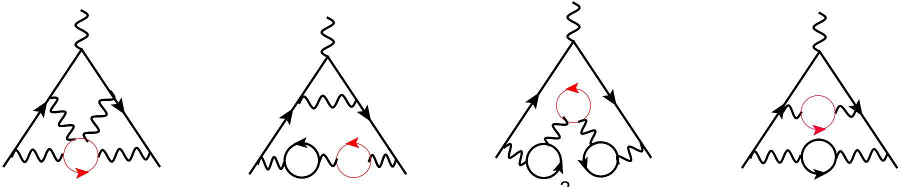
[Lee,Marquard,Smirnov,Smirnov,Steinhauser'13]



- n_l^3: 7.196 66
- $n_l^2 n_h$: 0.494 05
- agreement of analytic result with [Laporta'93; Aguilar,Greynat,De Rafael'08]

n_l^1 terms: 1 e^- loop

[Marquard,Smirnov,Smirnov,Steinhauser'13]

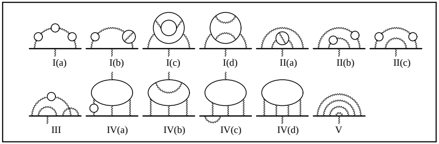


- 105 master integrals
- 30 master integrals known analytically
- rest: numerical evaluation FIESTA [Smirnov,Smirnov,Tentyukov'09'11]
- $n_l n_h$: -1.06 ± 0.05
  agreement with [Aoyama,Hayakawa,Kinoshita,Nio 2012]: -1.046
- $n_l n_h^2$: 0.0280 (analytic)
  agreement with [Laporta'93]
- $n_l n_h^0$ term: soon

PRELIMINARY

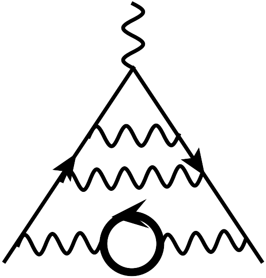
Heavy-lepton contribution

- $\left(\frac{\alpha}{\pi}\right)^4 \frac{M_\mu^2}{M_\tau^2} \approx \left(\frac{\alpha}{\pi}\right)^5$

- 

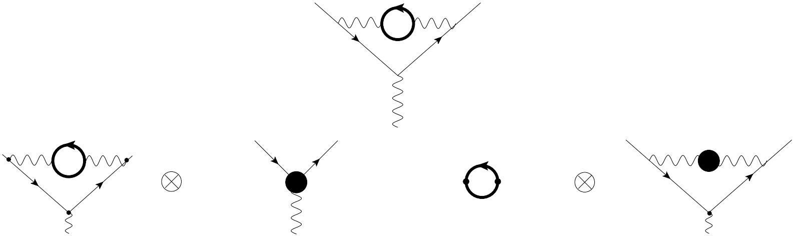
- [Aoyama,Hayakawa,Kinoshita,Nio'12]

group	$A_2^{(8)}(m_\mu/m_e)$	$A_2^{(8)}(m_\mu/m_\tau)$	$A_3^{(8)}$
I(a)	7.74547 (42)	0.000032 (0)	0.003209 (0)
I(b)	7.58201 (71)	0.000252 (0)	0.002611 (0)
I(c)	1.624307 (40)	0.000737 (0)	0.001807 (0)
I(d)	-0.22982 (37)	0.000368 (0)	0
II(a)	-2.77888 (38)	-0.007329 (1)	0
II(b)	-4.55277 (30)	-0.002036 (0)	-0.009008 (1)
II(c)	-9.34180 (83)	-0.005246 (1)	-0.019642 (2)
III	10.7934 (27)	0.04504 (14)	0
IV(a)	123.78551 (44)	0.038513 (11)	0.083739 (36)
IV(b)	-0.4170 (37)	0.006106 (31)	0
IV(c)	2.9072 (44)	-0.01823 (11)	0
IV(d)	-4.43243 (58)	-0.015868 (37)	0

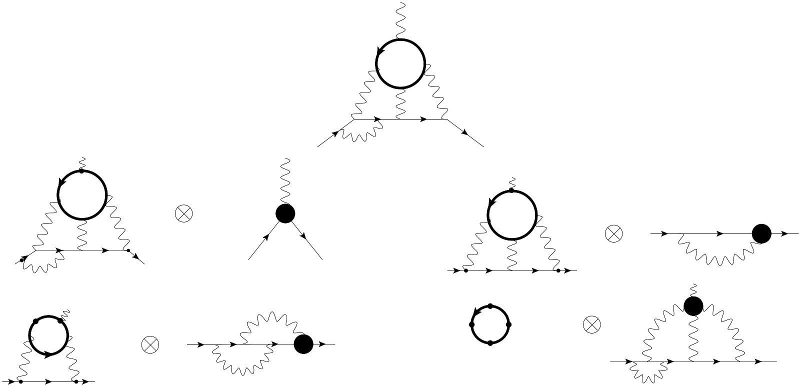


- Check with analytical asymptotic expansion for $M_\mu \ll M_\tau$

Graphical asymptotic expansion



Graphical asymptotic expansion (2)



Results

- analytic result [Kurz,Liu,Marquard,Steinhauser]

$$A_{2,\mu}^{(8)}(M_\mu/M_\tau) = \sum_n c_n \left(\frac{M_\mu}{M_\tau}\right)^n$$

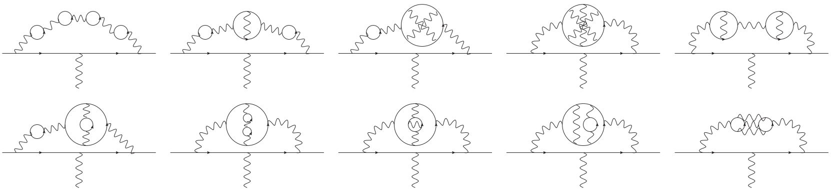
- $A_{2,\mu}^{(8)}(M_\mu/M_\tau)$
  $= 0.0421670 + 0.0003257 + 0.0000015 + \mathcal{O}\left(\frac{M_\mu}{M_\tau}\right)^8$
  $= 0.0424942 + \mathcal{O}\left(\frac{M_\mu}{M_\tau}\right)^8$
- $0.04234(12)$ [Aoyama,Hayakawa,Kinoshita,Nio'12]

PRELIMINARY

Polarization function insertions

4 loops: [Baikov,Broadhurst'95]

5 loops: [Baikov,Chetyrkin,Kühn,Sturm'13; Baikov,Marquard,Maier'13] (see also [Aguilar,Greynat,De Rafael'08])

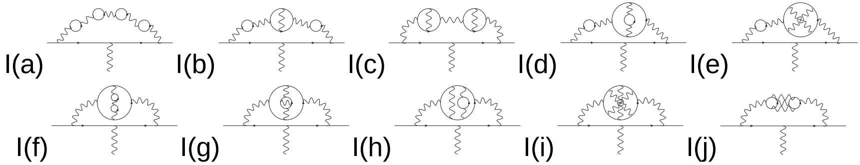


$$\boxed{a_\mu = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x(1-x)\left[-\Pi(s_x)\right]} \qquad s_x = -\frac{x^2}{1-x}m_\mu^2$$

- $\Pi(q^2)$: combine results for $q^2 \to 0, q^2 \to \infty$ and $q^2 \to 4m^2$ using Padé approximation
- 3 loops: [Baikov,Broadhurst'95; Chetyrkin,Kühn,Steinhauser'95]

  4 loops: [Hoang,Mateu,Zebarjad'08; Kiyo,Maier,Maierhöfer,Marquard'09; Greynat,Peris'10]

Results

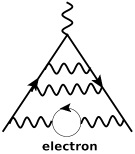


	[Baikov,Marquard,Maier'13]	[Baikov,Chetyrkin,Kühn,Sturm'13] $M_e \to 0$	[Aoyama,Hayakawa,Kinoshita,Nio]
I(a)	20.142 813	20.183 2	20.142 93(23)
I(b)	27.690 061	27.718 8	27.690 38(30)
I(c)	4.742 149	4.817 59	4.742 12(14)
I(d)+I(e)	6.241 470	6.117 77	6.243 32(101)(70)
I(e)	-1.211249	-1.33141	$-1.20841(70)$
I(f)+I(g)+I(h)	4.4469^{+6}_{-4}	4.391 31	4.446 68(9)(23)(59)
I(i)	0.0746^{+8}_{-19}	0.25237	0.087 1(59)
I(j)	-1.2469^{+4}_{-3}	-1.21429	$-1.24726(12)$

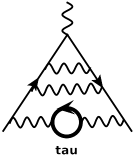
Conclusions

- First (analytic) results for $a_\mu^{4-\mathrm{loop}}$, e^- loops, $M_e \to 0$

  (also computed: $\overline{\mathrm{MS}}$–on-shell quark mass relation in QCD)



- Analytic results for 4-loop heavy lepton contribution



- 5-loop polarization function insertions