The muon anomalous magnetic moment, a view from the lattice

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(C. Aubin, T. Blum, MG and S. Peris, Phys. Rev. D86 (2012) 054509
C. Aubin, T. Blum, MG and S. Peris, arXiv:1307.4701
MG, K. Maltman and S. Peris, arXiv:1309.2153)

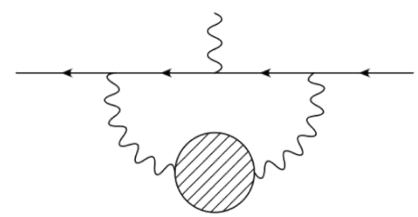
PhiPsi13, September 9-12, 2013

Current status

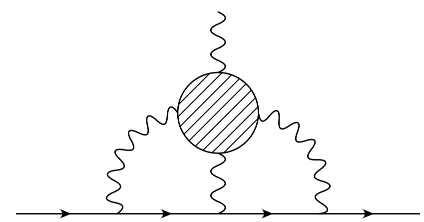
Experimental value: $a_\mu^{\rm exp} = 116592089(63) \times 10^{-11}$
error dominated by BNL E821, to be improved by factor 4 by Fermilab E989

Standard Model value: $a_\mu^{\rm SM} = 116591802(49) \times 10^{-11}$ (e.g. Hoecker, '11)

Hadronic Vacuum Polarization contribution: $6923(42) \times 10^{-11}$
error dominated by $\sigma(e^+e^- \to \text{hadrons})$



Hadronic Light by Light contribution: $105(26) \times 10^{-11}$
(model calculation!)



Discrepancy:

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 287(63)(49) \times 10^{-11} = 287(80) \times 10^{-11} \quad \to \quad 3.6\ \sigma$$

Reasons for using Lattice QCD

- Lattice QCD can provide a first-principle theoretical computation of hadronic contributions to a_μ , instead of experimental input plus models
  Need HVP to about 1%, HLxL to about 20%, hence focus on HVP

- Need check on current systematic errors
  Example: $a_\mu^{\rm HVP}$
  Obtain $I=1$ contribution in two different ways, from e^+e^- or from τ decays

$$a_\mu^{\rm HVP}(e^+e^-) = 6923(42)\times 10^{-11} \quad \text{vs.} \quad a_\mu^{\rm HVP}(\tau) = 7015(46)\times 10^{-11} \;:\; 1.8\,\sigma$$

(Hoecker '11; for (model-based) explanation, see Jegerlehner & Szafron '12)

Should we throw out $a_\mu^{\rm HVP}(\tau)$? Believe J&S model?

Note: using $a_\mu^{\rm HVP}(\tau)$ reduces discrepancy to $2.4\;\sigma$ instead of $3.6\;\sigma$!

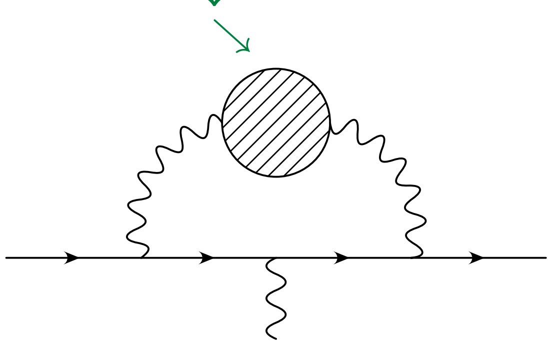
Methods using Lattice QCD

- Put quarks, gluons and photons on the lattice, probe with a muon, and compute the full hadronic contribution! Still in the future; "baby" version of this idea is used to go after HLxL (Blum et al. '08/'12)

- Treat QED (photons and muons) in perturbation theory, putting only quarks and gluons on the lattice

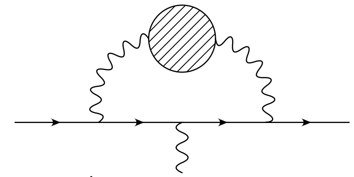
Example: $a_\mu^{\mathrm{HVP}} = 4\alpha^2 \int_0^\infty dQ^2\, f(Q^2, m_\mu^2)\, \underbrace{\left(\Pi(0) - \Pi(Q^2)\right)}$

(Lautrup & de Rafael '69, Blum '02)



Compute $\Pi(Q^2)$ using Lattice QCD

Hadronic **V**acuum **P**olarization on the lattice



Need to evaluate: $a_\mu^{\mathrm{HVP}} = 4\alpha^2 \int_0^\infty dQ^2\, f(Q^2, m_\mu^2)\, \underbrace{\left(\Pi(0) - \Pi(Q^2)\right)}$ $\downarrow$

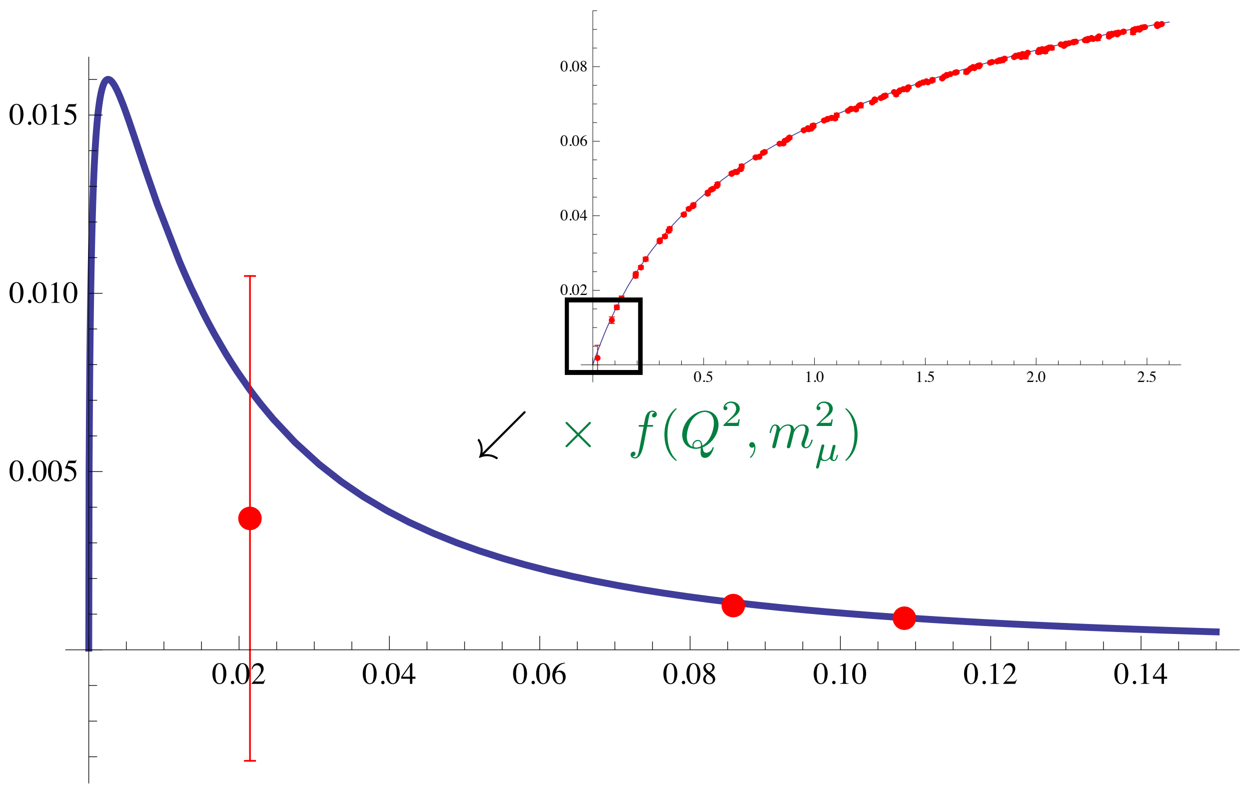
Integrand looks like:
peaks at $Q^2 \approx m_\mu^2/4$

red points:
typical lattice data!

$L^3 \times T = 64^3 \times 144$

$a \approx 0.06\ \mathrm{fm}$

$\swarrow \quad \times\ f(Q^2, m_\mu^2)$

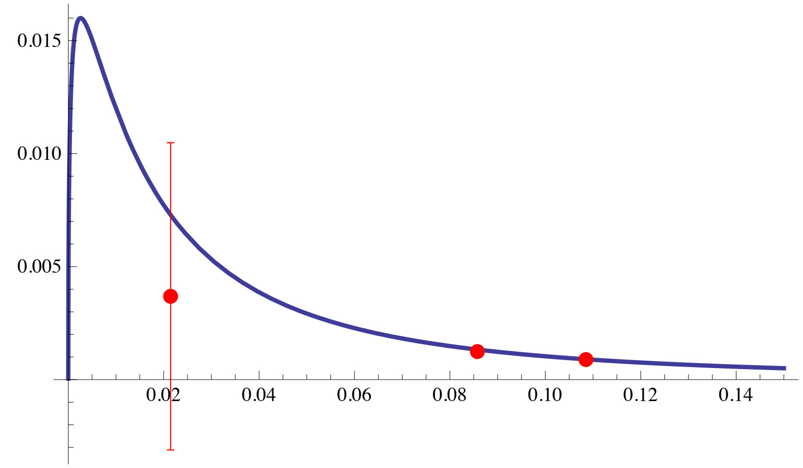


What is needed

- more data at small Q^2
  $\Rightarrow$ twisted boundary conditions

- theoretically reliable fit function
  of Q^2 behavior

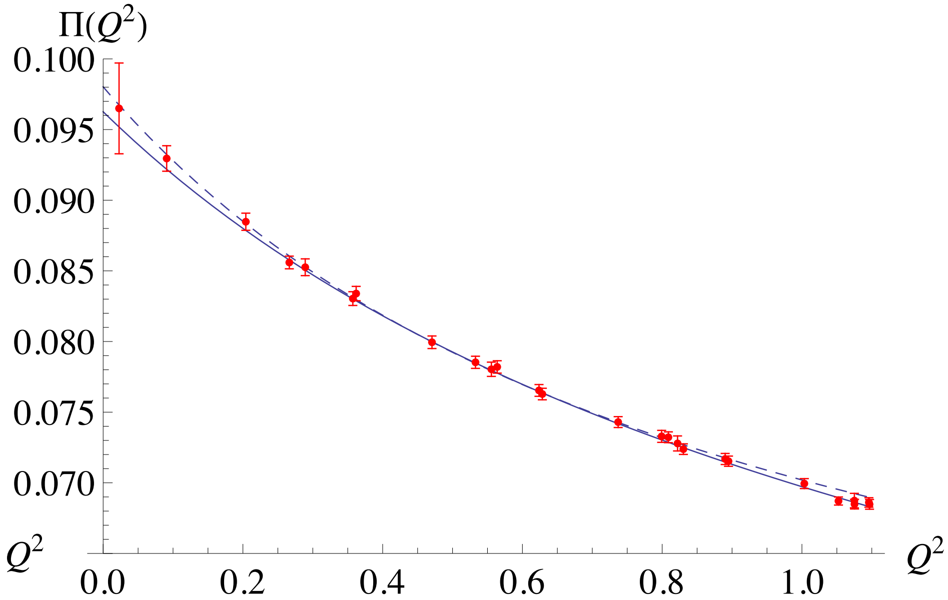
- (as almost always) smaller statistical errors



Smallest non-zero momentum on lattice: $Q = \left(0, 0, 0, \frac{2\pi}{aL}\right)$ (periodic bcs)

Standard fit function has been based on vector-meson dominance and tweaks:
assume essentially that $\Pi(Q^2) = A + B/(Q^2 + m_\rho^2) + \text{polynomial}$,
but cut starts at $Q^2 = -4m_\pi^2$, need to do much better!

Fitting the Q^2 behavior of $\Pi(Q^2)$



(Fig. from Aubin, Blum, MG & Peris, '12 , using MILC ensemble with $a = 0.09$ fm , $m_\pi \approx 450$ MeV)

Fit 1 (dashed): vector meson dominance

Fit 2 (solid): [1,1] Padé

Fit 1: $a_\mu^{\mathrm{HVP}} = 413(8)$

Fit 2: $a_\mu^{\mathrm{HVP}} = 350(8)$

Both good fits, but difference of about 15%!

At present, don't trust any numbers that claim to be more precise!

Padé approximants: model independent! (Aubin, Blum, MG & Peris ’12)

Using theorems by Baker ’69 and Barnsley ’73, can prove that the functions

$$\Pi(Q^2) = \Pi(0) - Q^2 \left(a_0 + \sum_{n=1}^{[P/2]} \frac{a_n}{b_n + Q^2} \right)$$

with $P = 2,\ 3,\ \ldots$ provide a converging sequence of PAs to the vacuum polarization, based on general analyticity properties of $\Pi(Q^2)$, and with

$$a_n > 0 \ ,$$
$$b_n \geq 4m_\pi^2 \ ,$$
$$a_0 = 0 \quad \text{for } P \text{ even}$$

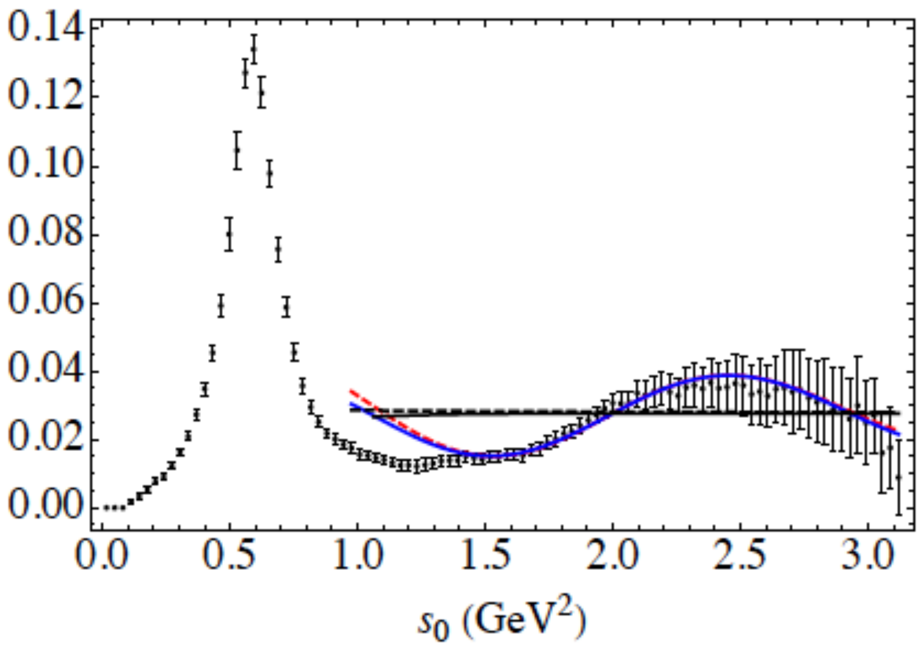
everywhere except near the cut $Q^2 \in (-\infty, -4m_\pi^2]$ (Minkowski axis).

Note: setting $b_1 = m_\rho^2$ (vector meson dominance) is not a valid PA!

Tests of $\Pi(Q^2)$ fits

(MG, Maltman & Peris'13)

Create a model $\Pi(Q^2)$ from non-strange vector tau spectral data extended beyond the tau mass using perturbation theory and a model for resonances:



(Fig. from Boito et al. '12; OPAL data)

- "Exact" (model) value:

$$10^7 \tilde{a}_\mu^{\mathrm{HVP},Q^2\leq 1\ \mathrm{GeV}^2} = 1.204$$

- Create fake data set at typical lattice momenta with typical lattice covariance matrix
- Try fits and see how they work!

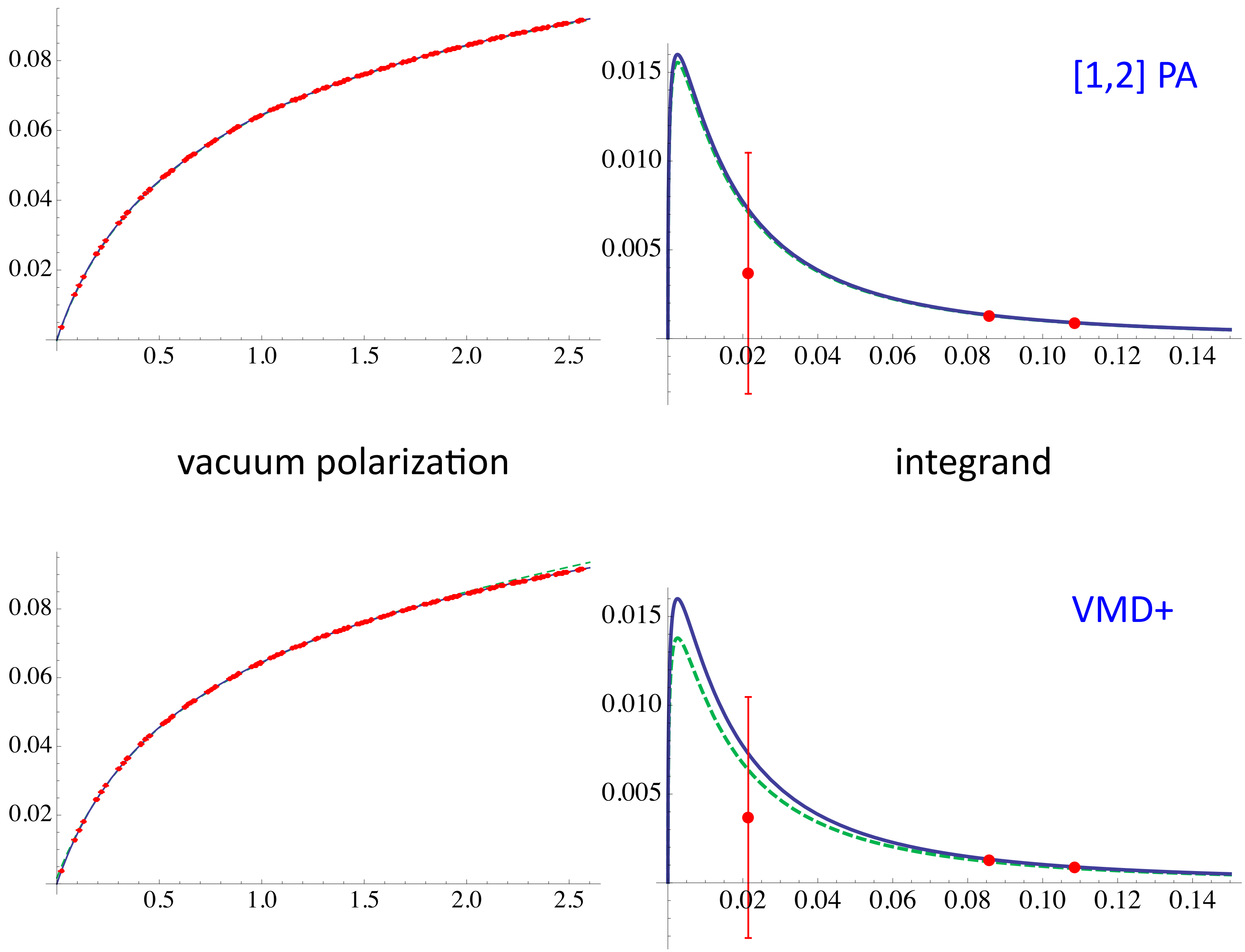
Using Q^2 values on an $a = 0.06$ fm , $L^3 \times T = 64 \times 144$ periodic lattice:

Fit	$\tilde{a}_\mu \times 10^7$	error$\times 10^7$	difference (σ)	χ^2/dof	
VMD	1.3201	0.0052	22	2189/47	
VMD+	1.0658	0.0076	18	67.4/46	$\leftarrow$
PA [0, 1]	0.8703	0.0095	35	285/46	
PA [1, 1]	1.116	0.022	4	61.4/45	
PA [1, 2]	1.182	0.043	0.5	55.0/44	$\leftarrow$
PA [2, 2]	1.177	0.058	0.5	54.6/43	

exact: $$10^7 \tilde{a}_\mu^{\mathrm{HVP}, Q^2 \le 1\ \mathrm{GeV}^2} = 1.204$$

difference: $$\sigma = \frac{|\mathrm{exact} - \mathrm{fitted\ value}|}{\mathrm{fit\ error}}$$

VMD fits may look good but fool you; PA reliable, but more data required



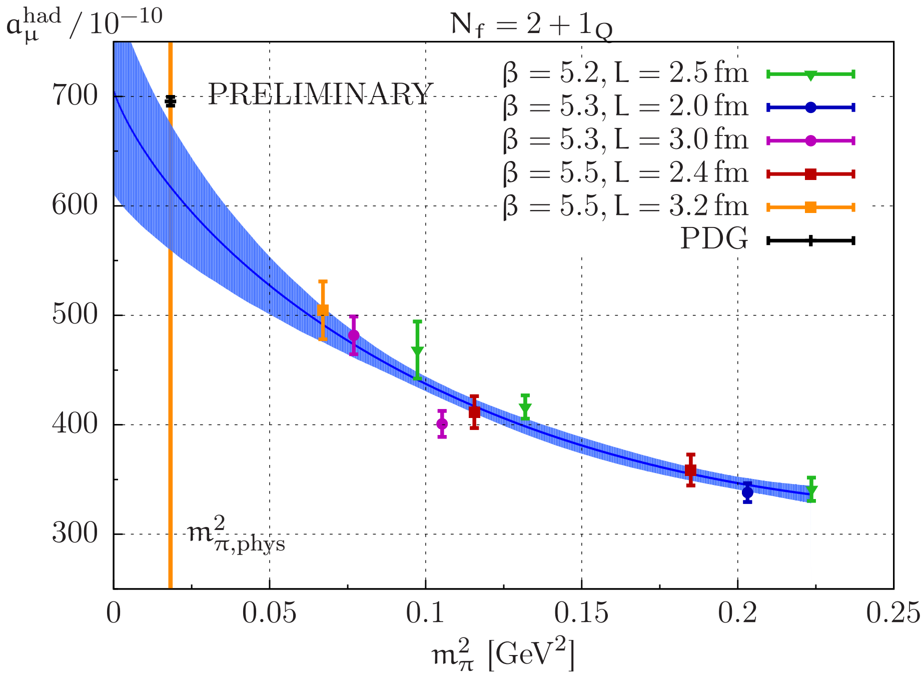
[1,2] PA
vacuum polarization
integrand
VMD+

Other improvements:

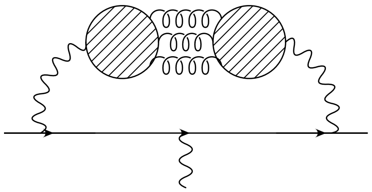
- Smaller errors (AMA = "all-mode averaging" (Blum, Izubuchi & Shintani '12))
- Extrapolation to physical pion masses non-trivial

  (Fig. from Della Morte et al. '12)

  Will need pion masses
  close to the physical value!



- Need "quark-disconnected" part



Concluding remarks

- There are issues with getting a_μ^{HVP} from $e^+e^- \to \mathrm{hadrons}$ and tau spectral data; a first-principle computation in QCD should be pursued: need Lattice QCD

- Computing a_μ^{HVP} with Lattice QCD is far from easy, but
  - we understand what is needed
  - it looks feasible in a time scale of 3-5 years

- Hadronic light-by-light is (even) harder, but not as much precision required – still, a tall order!

  Lattice is the only way to make progress (see Blum @ Lattice 2012)

BACK-UP SLIDES

Twisted boundary conditions

In a finite volume one needs to choose boundary conditions, even if they do not change the essential physics, which is the case if $m_\pi L \gg 1$

- Periodic boundary conditions:

$$f(x+L) = f(x)$$

$$e^{ip(x+L)} = e^{ipx} \qquad \rightarrow \qquad p = 2\pi n/L \ , \qquad n \in \mathbb{Z}$$

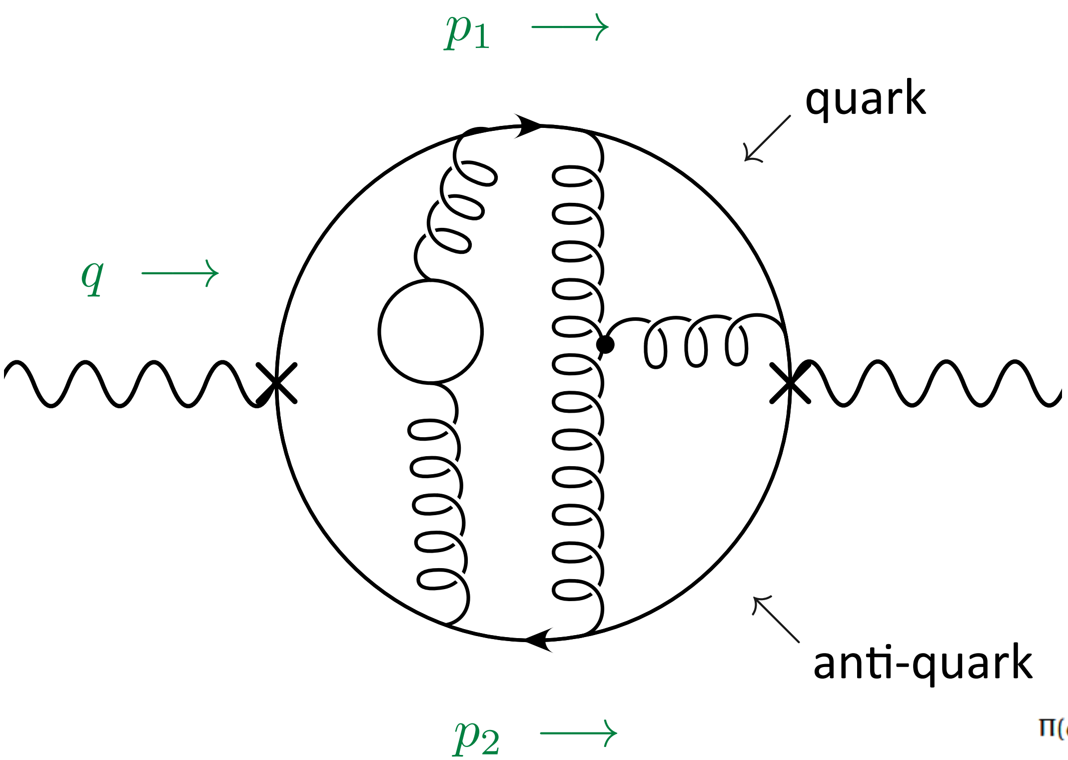
- Twisted boundary conditions:

$$f(x+L) = e^{i\theta} f(x)$$

$$e^{ip(x+L)} = e^{ipx+i\theta} \qquad \rightarrow \qquad p = (2\pi n + \theta)/L \ , \qquad n \in \mathbb{Z}$$

Taking $0 \leq \theta < 2\pi$ allows momentum to vary continuously!

(Bedaque '04, de Divitiis et al. '04, Sachrajda & Villadoro '05)

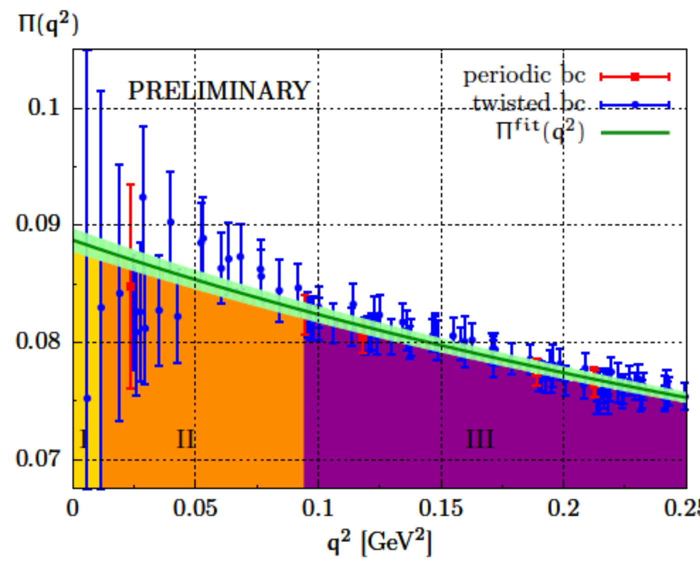


Choose quark with twist θ
anti-quark periodic, then

$$
\begin{aligned}
q &= p_1 + p_2 \\
&= \frac{2\pi n_1 + \theta}{L} + \frac{2\pi n_2}{L} \\
&= \frac{2\pi k + \theta}{L}\,, \qquad k \in \mathbb{Z}
\end{aligned}
$$

varies continuously

Fig. from Della Morte, Jäger, Jüttner & Wittig '11,'12; red: no twist; blue: various twists



Aubin, Blum, MG & Peris, in progress

Lattice errors -- general

- Lattice QCD evaluates infinite sums over Feynman diagrams through the path integral using Monte-Carlo techniques: statistical error
  reduce using smarter algorithms and larger computers

- Discretize space-time on a lattice with lattice spacing a in a finite volume $L^3 \times T$: systematic errors ($a = 0.06\ \mathrm{fm}$, $L = 4\ \mathrm{fm}$, $T = 8\ \mathrm{fm}$)
  reduce by extrapolating $a \to 0$ and $L,\ T \to \infty$

- Most computations: pions too heavy ($200 - 300\ \mathrm{MeV}$), in order to fit into finite box, need to extrapolate light quark masses $m_{u,d}$ to physical values

(Need good theoretical understanding of dependence on $a,\ L,\ T,\ m_{u,d}$!)