Final state interaction and asymmetry of pair production close to threshold in $e^+e^-$ annihilation

V.F. Dmitriev and A.I. Milstein,
Bulker Institute of Nuclear Physics,
Novosibirsk

From Φ to Ψ 2013, September 9-12, Rome
1 Coulomb final state interaction
   • A lepton pair production
Outline

1. Coulomb final state interaction
   - A lepton pair production

2. Strong Final State Interaction
   - $N\bar{N}$ pair production
Outline

1. Coulomb final state interaction
   - A lepton pair production

2. Strong Final State Interaction
   - $N\bar{N}$ pair production

3. Summary
A general form of the cross section is well known. The standard equation for the cross section

\[
\frac{d\sigma}{d\Omega} = \frac{\beta \alpha^2}{4Q^2} \frac{2\pi\eta}{1 - e^{-2\pi\eta}} \left[ |G_M(Q^2)|^2(1 + \cos^2 \theta) + \frac{4M^2}{Q^2} |G_E(Q^2)|^2 \sin^2 \theta \right].
\]

Near threshold, it is usually assumed \( G_E = G_M \), and the cross section is isotropic. At the same time, in the presence of long range Coulomb final state interaction (FSI), all partial waves give a nonzero contribution at the threshold, and \( G_E \neq G_M \), although the difference is small.
Let us consider the attractive Coulomb potential $U(r) = -e^2/r$ and the corresponding radial wave function $R_{k,l}^{(c)}(r)$, where $e$ is the electric charge, $k$ is the particle momentum, and $l$ is the angular momentum. Then the ratio $C_{k,l} = \left| R_{k,l}^{(c)}(r)/R_{k,l}^{(0)}(r) \right|^2$ at $kr \to 0$, where $R_{k,l}^{(0)}(r)$ is the radial wave function for plain wave, has the form,

$$C_{k,0} = \frac{2\pi\eta}{1 - e^{-2\pi\eta}}, \quad C_{k,l \neq 0} = C_{k,0} \prod_{s=1}^{l} \left( 1 + \frac{\eta^2}{s^2} \right), \quad \eta = \frac{\mu\alpha}{k}.$$

Here $\mu$ is the reduced mass of the system, $\alpha = e^2$ is the fine structure constant, units $\hbar = c = 1$ are used.

Note, $\eta \to \infty$ at threshold.
The Born amplitude of the process $e^+ e^- \rightarrow L \bar{L}$ near threshold in units $4\pi\alpha/Q^2$

$$T_{\lambda\mu}^B = \sqrt{2}\epsilon^*_\lambda \cdot \left[ G_s e_\mu + G_d \frac{k^2 e_\mu - 3(k \cdot e_\mu)k}{6M^2} \right].$$

Here $e_\mu$ is a virtual photon polarization vector, $\epsilon_\lambda$ is the spin-1 function of $L \bar{L}$ pair, $G_s = F_1(Q^2) + F_2(Q^2)$, $G_d = F_1(Q^2) - F_2(Q^2)$, $F_1(Q^2)$ and $F_2(Q^2)$ are the Dirac form factors of lepton. Two tensor structures in Eq.(1) correspond to the s-wave and d-wave production amplitudes. The total angular momentum of the $L \bar{L}$ pair is fixed by a production mechanism.
Exacte amplitude with Coulomb FSI

\[ T_{\lambda \mu} = \sqrt{2} \epsilon^{*}_{\lambda} \cdot \int \frac{d^3 p}{(2\pi)^3} \Phi_{k}^{(-)*}(p) \left[ G_{s} e_{\mu} + G_{d} \frac{p^2 e_{\mu} - 3(p \cdot e_{\mu})p}{6M^2} \right] , \]

where \( \Phi_{k}^{(-)}(p) \) is the Fourier transform of the function \( \psi_{k}^{(-)}(r) \), which is the wave function of the \( L\bar{L} \) pair in coordinate space:

\[ \psi_{k}^{(-)}(r) = \frac{1}{2k} \sum_{l=0}^{\infty} i^l e^{-i\delta_l} (2l + 1) R_{k l}(r) P_l(n \cdot \hat{k}) , \]

\[ \delta_l = \text{arg}\Gamma(l + 1 - \eta) . \]

Here \( n = r/r, \hat{k} = k/k, P_l \) are the Legendre polynomials

\[ R_{k l}(r) = \frac{2k l! \sqrt{C_{kl}}}{(2l + 1)!} (2kr)^l e^{-i\eta k r} F(\eta + l + 1, 2l + 2, 2i k r) , \]
Calculated exact amplitude

\[ T_{\lambda\mu} = \sqrt{2}e^{i\delta_0} \sqrt{C_{k0}} \epsilon_{\lambda}^* \cdot \left[ G_s e_{\mu} + \left( 1 - \frac{\eta^2}{2} - \frac{3i\eta}{2} \right) G_d \frac{k^2 e_{\mu} - 3(k \cdot e_{\mu})k}{6M^2} \right]. \]

The "dressed" form factor \( G_2 \)

\[ G_2 = \left( 1 - \frac{\eta^2}{2} - \frac{3i\eta}{2} \right) G_d, \]

and \( G_0 = G_s \).
Differential cross section

The cross section

\[
\frac{d\sigma}{d\Omega} = \frac{\beta\alpha^2}{2Q^2} C_k \left[ |G_0(Q^2)|^2 + \frac{\beta^2}{3} \text{Re}(G_0^*G_2)P_2(\cos \theta) \right.
\]

\[+ \left. \beta^4 |G_2(Q^2)|^2 [2 - P_2(\cos \theta)] \right]\]

where \( \beta = k/M \). Omitting the form factor \( F_2(Q^2) \) for lepton production, we obtain from Eq.(1)

\[
\frac{d\sigma}{d\Omega} = \frac{\beta\alpha^2}{2Q^2} C_k |F_1(Q^2)|^2 \left[ 1 + \frac{1}{3} \left( \beta^2 - \frac{\alpha^2}{8} \right) P_2(\cos \theta) \right] \]
The standard equation for the cross section

\[
\frac{d\sigma}{d\Omega} = \frac{\beta\alpha^2}{4Q^2} C_0 \left[ |G_M(Q^2)|^2 (1 + \cos^2 \theta) + \frac{4M^2}{Q^2} |G_E(Q^2)|^2 \sin^2 \theta \right].
\]

In terms of the form factor $G_0$ and $G_2$ the electromagnetic Sachs form factors have the form

\[
G_M = G_0 + \frac{\beta^2}{6} G_2, \quad \frac{2M}{Q} G_E = G_0 - \frac{\beta^2}{3} G_2.
\]

$\beta^2 G_2$ does not vanish in the limit $\beta \to 0$, we find that $G_E \neq G_M$ at threshold. Unlike the form factors $G_0$ and $G_2$, the electromagnetic Sachs form factors $G_E$ and $G_M$ are not singular in the limit $\beta \to 0$. 
For $N\bar{N}$ production near threshold there is a strong FSI between slowly moving nucleon and antinucleon. At small distances, where strong interaction dominates, we can forget about Coulomb interaction. The electromagnetic interaction is important only in the narrow region where $\beta \sim \pi\alpha$ and the nucleon energy is $E = M\beta^2/2 \sim 0.3$ MeV. Thus, we can work with the isospin amplitudes $T_{1\lambda\mu}^1$ and $T_{1\lambda\mu}^0$ related to $p\bar{p}$ and $n\bar{n}$ production amplitudes by

$$T_{\lambda\mu}^{(p)} = \frac{T_{1\lambda\mu}^1 + T_{1\lambda\mu}^0}{\sqrt{2}} \quad \text{and} \quad T_{\lambda\mu}^{(n)} = \frac{T_{1\lambda\mu}^1 - T_{1\lambda\mu}^0}{\sqrt{2}}.$$
The amplitude of $N\bar{N}$ pair production in a certain isospin channel $I = 0, 1$ near threshold in units $4\pi\alpha/Q^2$:

$$T^I_{\lambda\mu} = \sqrt{2}\epsilon^{*}_\lambda \left[ G^I_s e_\mu + G^I_d \frac{k^2 e_\mu - 3(k \cdot e_\mu)k}{6M^2} \right],$$

$$G^I_s = F^I_1(Q^2) + F^I_2(Q^2)$$

$$G^I_d = F^I_1(Q^2) - F^I_2(Q^2),$$

$e_\mu$ is a virtual photon polarization vector, and $\epsilon^{*}_\lambda$ is the spin-1 function of $N\bar{N}$ pair, $\lambda = \pm 1, 0$ is the projection of spin on the vector $k$. Here we omitted terms $\sim \beta^2$, where $\beta = k/M \ll 1$. 

V.F. Dmitriev and A.I. Milstein, Bulker Institute of Nuclear Physi
Full $N\bar{N}$ production amplitude

\[
T^I_{\lambda\mu} = \sqrt{2} \int \frac{d^3 p}{(2\pi)^3} \Phi_k^{I(-)*}(p) \cdot \left[ G_s^I e_\mu + G_d^I \frac{p^2 e_\mu - 3(p \cdot e_\mu) p}{6M^2} \right],
\]

where $\Phi_k^{I(-)}(p)$ is the Fourier transform of the function $\Psi_{k\lambda}(r)$, the wave function of the $N\bar{N}$ pair in coordinate space. This wave function is the solution of the Schrödinger equation

\[
\psi_{k\lambda}^{I(-)*}(r) \hat{H} = M\beta^2 \psi_{k\lambda}^{I(-)*}(r), \quad \hat{H} = \frac{p^2}{M} + V_{N\bar{N}},
\]

where $V_{N\bar{N}}$ is the optical potential.
Full $N\bar{N}$ production amplitude

\[
T_{I\mu}^I = \sqrt{2} \lim_{r \to 0} \left[ G_s^I e_\mu - G_d^I \frac{e_\mu \Delta - 3(\nabla \cdot e_\mu) \nabla}{6M^2} \right] \psi^{I*}_{k\lambda}(r),
\]

\[
\psi^{I}_{k\lambda}(r) = [u^{I*}_{11}(r)e_\lambda + w^{I*}_{11}(r)\sqrt{4\pi} Y^{2}_{1\lambda}(n)]
+ \sqrt{5} C^{1\lambda}_{20,1} [u^{I*}_{21}(r)e_\lambda + w^{I*}_{21}(r)\sqrt{4\pi} Y^{2}_{1\lambda}(n)].
\]

The radial wave functions $u^{I}_{n1}$ and $w^{I}_{n1}$, $n = 1, 2$, satisfy the system of equations

\[
\frac{p^2_r}{M} \chi + V \chi = 2E \chi,
\]

\[
V = \begin{pmatrix}
V^I_0 & -2\sqrt{2} V^I_3 \\
-2\sqrt{2} V^I_3 & V^I_2 - 2V^I_3
\end{pmatrix}, \quad \chi = \begin{pmatrix}
u^{I}_{n1} \\
w^{I}_{n1}
\end{pmatrix}.
\]
The asymptotic form of the solutions at large distances is,

\[ u_{11}^I(r) = \frac{1}{2ikr} \left[ S_{11}^{l_1} e^{ikr} - e^{-ikr} \right], \]
\[ w_{11}^I(r) = -\frac{1}{2ikr} S_{12}^{l_1} e^{ikr}, \]
\[ u_{21}^I(r) = \frac{1}{2ikr} S_{21}^{l_1} e^{ikr}, \]
\[ w_{21}^I(r) = \frac{1}{2ikr} \left[ -S_{22}^{l_1} e^{ikr} + e^{-ikr} \right]. \]

If \( V_3^I = 0 \), then \( w_{11}^I(r) = 0 \) and \( u_{21}^I(r) = 0 \).
Performing differentiation and putting $r = 0$ we obtain

$$T_{\lambda\mu}^I = \sqrt{2}\epsilon_\lambda^* \left\{ G_s^I e_\mu + G_d^I \frac{k^2 e_\mu - 3(k \cdot e_\mu)k}{6M^2} \right\},$$

where dressed $s$-wave and $d$-wave form factors are

$$G_s^I = G_s^I u_{11}^I(0) + \frac{5}{\sqrt{2}M^2} \lim_{r \to 0} \left( \frac{w_{11}^I(r)}{r^2} \right),$$

$$G_d^I = \frac{6G_s^I}{\sqrt{2} \beta^2} u_{21}^I(0) + 15 G_d^I \lim_{r \to 0} \left( \frac{w_{21}^I(r)}{k^2 r^2} \right).$$
$G_s^I = G_s^I u_{11}^I(0), \quad G_d^I = \frac{6G_s^I}{\sqrt{2}\beta^2} u_{21}^I(0)$

Final form of the amplitude

$T_{\lambda\mu}^I = G_s^I \left\{ \sqrt{2}u_{11}^I(0)(e_\mu \cdot e_\lambda^*) + u_{21}^I(0)[(e_\mu \cdot e_\lambda^*) - 3(\hat{k} \cdot e_\mu)(\hat{k} \cdot e_\lambda^*)] \right\}$,

where $\hat{k} = k/k$, and the only parameter $G_s^I = F_1^I + F_2^I$
Sachs form factors and cross section

\[ G_M^i = G_s^i + \frac{\beta^2}{6} G_d^i = G_s^i [u_{11}(0) + \frac{1}{\sqrt{2}} u_{21}(0)] , \]

\[ \frac{2M}{Q} G_E^i = G_s^i - \frac{\beta^2}{3} G_d^i = G_s^i [u_{11}(0) - \sqrt{2} u_{21}(0)] . \]

The ratio \( G_E^i / G_M^i \) is independent on the parameter \( G_s^i \)

\[ \frac{G_E^i}{G_M^i} = \frac{Q}{2M} \frac{u_{11}(0) - \sqrt{2} u_{21}(0)}{u_{11}(0) + \frac{1}{\sqrt{2}} u_{21}(0)} . \]

The cross section has a standard form

\[ \frac{d\sigma}{d\Omega} = \frac{\beta \alpha^2}{4Q^2} \left[ |G_M(Q^2)|^2 (1 + \cos^2 \theta) + \frac{4M^2}{Q^2} |G_E(Q^2)|^2 \sin^2 \theta \right] . \]
Cross section data

Figure: Cross sections for $e^+ e^- \rightarrow p\bar{p}$ (blue) and $e^+ e^- \rightarrow n\bar{n}$ (red)

$$T^{(p)}_{\lambda\mu} = \frac{T^{1}_{\lambda\mu} + T^{0}_{\lambda\mu}}{\sqrt{2}}, \quad T^{(n)}_{\lambda\mu} = \frac{T^{1}_{\lambda\mu} - T^{0}_{\lambda\mu}}{\sqrt{2}}.$$ 

Either $|T^0| \gg |T^1|$ or $|T^1| \gg |T^0|.$
Coulomb final state interaction
Strong Final State Interaction
Summary

$N\bar{N}$ pair production

Calculations with Paris $N\bar{N}$ potential

Cross section for $I = 0$

V.F. Dmitriev and A.I. Milstein, Bulker Institute of Nuclear Physics
Final state interaction and asymmetry of pair production close to
There is only one parameter $|G_s^0|$ in the fit of the cross section and $|G_E/G_M|$ is free of parameters. Here $|G_s^0|^2 = 50.9$
Isovector channel

Cross section for $l = 1$

Here $|G_s|^2 = 5092$; it does not look reasonable.
For $I = 1$ the wave function at the origin suppressed exponentially under the barrier, while for $I = 0$ it is enhanced due to attractive potential.
For lepton pair production the long range Coulomb final state interaction provides a nonzero contribution of all partial waves, $l \geq 0$, at the threshold. Interference of partial waves results in small asymmetry, $\sim \alpha^2$ in angular distribution of the produced lepton pair.
For lepton pair production the long range Coulomb final state interaction provides a nonzero contribution of all partial waves, \( l \geq 0 \), at the threshold. Interference of partial waves results in small asymmetry, \( \sim \alpha^2 \) in angular distribution of the produced lepton pair.

For \( N\bar{N} \)-pair production near threshold the asymmetry in angular distribution owes to the tensor component of the strong final state interaction. The ratio \( G_E/G_M \) is fully determined by the \( N\bar{N} \) optical potential without any free parameters.
Summary

1. For lepton pair production the long range Coulomb final state interaction provides a nonzero contribution of all partial waves, \( l \geq 0 \), at the threshold. Interference of partial waves results in small asymmetry, \( \sim \alpha^2 \) in angular distribution of the produced lepton pair.

2. For \( N\bar{N} \)-pair production near threshold the asymmetry in angular distribution owes to the tensor component of the strong final state interaction. The ratio \( G_E/G_M \) is fully determined by the \( N\bar{N} \) optical potential without any free parameters.

3. Due to strong repulsion at very short distances in the isovector part of the Paris \( N\bar{N} \)-potential, the isoscalar amplitude dominates near threshold in \( e^+e^- \rightarrow N\bar{N} \) process.