hadronic spectral function moments: perturbative expansions and $\alpha_s$

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Work done in collaboration with M. Beneke and M. Jamin

introduction
Spread in the results reflect (mainly) details of the theoretical input.
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There are still open questions (Renormalization Group Improvement, duality violations, ...)

see talk by Peris

\[ \alpha_s(m^2_{\tau}) = 0.36 \pm 0.04 \]

Braaten, Narison, and Pich ‘92
Sum rules for the spectral functions
(in tau decays) Braaten, Narison, and Pich, 1992

\[ \int_0^{s_0} ds \, w(s) \frac{1}{\pi} \text{Im} \tilde{\Pi}(s) = \frac{-1}{2\pi i} \int_{|z|=s_0} dz \, w(z) \tilde{\Pi}(z) \]

experiment (OPAL and ALEPH)

\[ J_\mu = \bar{u} \gamma_\mu (\gamma_5) q(x) \]

theory
Sum rules for the spectral functions
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experiment (OPAL and ALEPH)

\[ J_\mu = \bar{u} \gamma_\mu (\gamma_5) q(x) \]

Contributions to the sum rule (theory side)

\[ R_{V/A}^{w_i}(s_0) = \frac{N_c}{2} S_{EW} |V_{ud}|^2 \left[ \delta_{w_i}^{\text{tree}} + \delta_{w_i}^{(0)}(s_0) + \sum_{D \geq 2} \delta_{w_i, V/A}^{(D)}(s_0) + \delta_{w_i, V/A}^{\text{DV}}(s_0) \right] \]

OPE DVs

\[ \alpha_3^4 : \text{Baikov, Chetyrkin, Kühn 2008} \]

our focus is on \( \delta_{w_i}^{(0)} \) (moment dependence)
Sum rules for the spectral functions
(in tau decays) Braaten, Narison, and Pich, 1992

\[ \int_0^{s_0} ds \, w(s) \, \frac{1}{\pi} \text{Im} \tilde{\Pi}(s) = \frac{-1}{2\pi i} \int_{|z|=s_0} dz \, w(z) \tilde{\Pi}(z) \]

experiment
(OPAL and ALEPH)

\[ w_\tau(s) = (1 - s/s_0)^2 (1 + 2s/s_0) = (1 - x)^2 (1 + 2x) \]

Kinematical moment is a special case:

\[ s_0 = m_\tau^2 \implies R_\tau = \frac{\Gamma[\tau \to \text{hadrons} \nu_\tau]}{\Gamma[\tau \to e^- \bar{\nu}_e \nu_\tau]} = 3.6280 \pm 0.0094 \]

but the choice of \( w(s) \) is free
Open questions in the perturbative part

- In the literature several weight functions are used
  - DB et al '11, '12, Davier et al '08, Maltman and Yavin '08, ALEPH '98, ‘05, OPAL ‘99

- One often employs \( w_\tau(x) = (1 - x)^2(1 + 2x) \) (gives \( R_\tau = \frac{\Gamma[\tau \to \text{hadrons } \nu_\tau]}{\Gamma[\tau \to e^-\bar{\nu}_e \nu_\tau]} \))
  
  and many others:
  \[
  \begin{align*}
  w(x) &= 1, \quad w(x) = 1 - x^2, \quad w(x) = x(1 - x)^2, \quad w^{(k)}(x) = (1 - x)^3 x^k (1 + 2x), \\
  w^{(n)}(x) &= 1 - \frac{n}{n-1} x + \frac{n}{n-1} x^n \ldots
  \end{align*}
  \]

- Different emphasis on the experimental spectrum. Change the relative contributions on the theory side (pert., OPE, DVs)

- \( \alpha_s \) dependence comes mainly from \( \delta^{(0)}_{w_i} \)

Open questions in \( \delta^{(0)}_{w_i} \):

- Renormalization group improvement: what is the best prescription?
  - Contour Improved PT vs Fixed Order PT

- Moment dependence?

- Are there better moments to determine \( \alpha_s \)?
renormalization group
Description in terms of the Adler function (derivative of $\delta(0)$)

\[ D_{\text{pert}}^{(1+0)}(s) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_n^\mu \sum_{k=1}^{n+1} k c_{n,k} \left( \log \frac{s}{\mu^2} \right)^{k-1} \]

\[ a_\mu = \frac{\alpha(\mu)}{\pi} \]

- only $c_{n,1}$ are independent (known up to $c_{4,1}$). $c_{n,k}$ depend on $c_{n,1}$ and $\beta_m$.

Prescriptions for the RG improvement

**FOPT**

- $\mu = s_0$

\[ \delta_{\text{FO},w_i}^{(0)} = \sum_{n=1}^{\infty} a(s_0)^n k c_{n,k} J_{k-1}^{\text{FO},w_i} \]

**CIPT**

- $\mu = -s_0x$

\[ \delta_{\text{CI},w_i}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_{n}^{\text{CI},w_i}(s_0) \]

\[ J_n^{\text{FO},w_i}(s_0) \equiv \frac{1}{2\pi i} \int_{|x|=1} dx \ W_i(x) \log^n(-x) \]

\[ J_n^{\text{CI},w_i}(s_0) \equiv \frac{1}{2\pi i} \int_{|x|=1} dx \ W_i(x) a^n(-s_0x) \]

Le Diberder and Pich ‘92

\[ \alpha_s(m_\tau) = 0.3186 \]

\[ w(x) = 1 \]

\[ w_\tau(x) = (1-x)^2(1+2x) \]

\[ w(x) = (1-x)^3x^3(1+2x) \]

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higher orders
asymptotic series

\[ R \sim \sum_{n=0}^{\infty} r_n \alpha^{n+1}_s \] divergent but (hopefully) asymptotic

\[ \Downarrow \quad \Downarrow \]

\[ ? \quad \text{in QFT we only know the expansion} \]

- Define the Borel transformed series

\[ B[R](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!} \] which can be “summed” \[ \Rightarrow \tilde{R} \equiv \int_{0}^{\infty} dt \ e^{-t/\alpha} B[R](t) \]

- Divergent behaviour encoded in the singularities of \( B[R](t) \)

Strategy:

\[ \text{sing. of } B[R](t) \]

\[ \frac{c}{(1 + at)^{b}} + \cdots \]

(renormalons)

\[ B[R](t) \]

\[ \sum_{n} r_n \alpha^{n+1}_s \]

\[ D_{\text{pert}}^{(1+0)}(Q^2) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} \left( \frac{c_{n,1}}{\pi^n} \right) \alpha_Q^n \]
General structure of large-order behavior (believed to be) known

Borel transformed Adler function

\[ B[\hat{D}](t) \equiv \sum_{n=0}^{\infty} \frac{c_{n,1}}{\pi^n} \frac{t^n}{n!} \]

Borel sum:

\[ \hat{D}(\alpha) \equiv \int_0^\infty dt \, e^{-t/\alpha} \, B[\hat{D}](t) \]

Singularities in the \( t \) plane

- UV renormalons
  - sign alternating
  - leading sing. in the Adler function at \( u = -1 \)
  - no-sing alternation in known coeff.: small residue for the leading UV pole

- IR renormalons
  - fixed sign
  - sing. at \( u = 2, 3, 4... \) related to dim-4, dim-6, dim-8... contributions
  - \( u = 2 \) related to the gluon condensate

Structure of each singularity in principle calculable (up to \( c_p \))

\[ B[D_p] = \frac{c_p}{(p-u)^\gamma} \left[ 1 + \tilde{b}_1(p-u) + \cdots \right] \]
Reference model (RM)

Beneke & Jamin ‘08

\[ B[\hat{D}](u) = B[\hat{D}^{UV}_1](u) + B[\hat{D}^{IR}_2](u) + B[\hat{D}^{IR}_3](u) + d_0^{PO} + d_1^{PO} \, u \]

- Model with the leading UV and the first two IR singularities.
- Small polynomial terms to fix \( c_{1,1} \) and \( c_{2,1} \).
- Favors FOPT (related to the presence of \( u = 2 \) sing).

Alternative model (AM)

\[ B[\hat{D}](u) = B[\hat{D}^{UV}_1](u) + B[\hat{D}^{IR}_3](u) + B[\hat{D}^{IR}_4](u) + d_0^{PO} + d_1^{PO} \, u \]

- No IR singularity at \( u = 2 \) (related to the gluon condensate)
- Favors CIPT.

\[ \delta^{(0)} \]

\begin{align*}
\tau & \text{ spectral function moments: pt. exp. and } \alpha_s \\
\text{PhiPsi 2013, Roma, La Sapienza} & \\
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\end{align*}
First conclusions:

- Decision in favor of FOPT or CIPT depends on the higher order coefficients.
- Highly correlated with the first IR renormalon (gluon condensate).
moment analysis
Moments studied can have their behavior separated in classes

<table>
<thead>
<tr>
<th>$k$</th>
<th>$w_k(x)$</th>
<th>Reference</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>[DB et al]</td>
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<tr>
<td>2</td>
<td>$x$</td>
<td>[Maltman &amp; Yavin]</td>
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<tr>
<td>3</td>
<td>$x^2$</td>
<td>[Davier et al, ALEPH, OPAL]</td>
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<td>4</td>
<td>$x^3$</td>
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<td>5</td>
<td>$x^4$</td>
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<tr>
<td>6</td>
<td>$1 - x$</td>
<td>[DB et al]</td>
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<tr>
<td>7</td>
<td>$1 - x^2$</td>
<td>[DB et al]</td>
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<td>8</td>
<td>$1 - x^3$</td>
<td>[DB et al]</td>
</tr>
<tr>
<td>9</td>
<td>$1 - \frac{3x}{2} + \frac{x^3}{2}$</td>
<td>[Maltman &amp; Yavin]</td>
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<td>10</td>
<td>$(1 - x)^2$</td>
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<tr>
<td>11</td>
<td>$(1 - x)^3$</td>
<td>[DB et al]</td>
</tr>
<tr>
<td>12</td>
<td>$w_\tau(1 - x)^2(1 + 2x)$</td>
<td>[All recent works]</td>
</tr>
<tr>
<td>13</td>
<td>$(1 - x)^3(1 + 2x)$</td>
<td>[Davier et al, ALEPH, OPAL]</td>
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<td>14</td>
<td>$(1 - x)^2x$</td>
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</tbody>
</table>

Monomials

Pinched $[w(1) = 0]$ with a “1”

Pinched with no “1”

Note: moments with a term “$x$” form a separate class.
Pinched moments with a “1”

At least one of the methods approach the Borel result at relatively low orders

\[ w_\tau(x) = (1 - x)^2(1 + 2x) \]
\[ w(x) = 1 - x^2 \]
\[ w(x) = 1 - x^3 \]
Moments with the term $x$

- Very sensitive to $D = 4$. Unstable results if the $u = 2$ singularity is sizable.

\[ w(x) = 1 - x \]

\[ w(x) = (1 - x)^3 x (1 + 2x) \]
- Pinched moments starting at $x^2$ (or higher)
- Borel results are never well reproduced at low orders.

\[ w(x) = (1 - x)^3 x^2 (1 + 2x) \]

\[ w(x) = (1 - x)^3 x^3 (1 + 2x) \]
energy dependence
\[ w(x) = 1 - x^2 \]

\[ w(x) = (1 - x)^3 x^3 (1 + 2x) \]
consequences for $\alpha_s$

(exploratory study: power corrections and DVs as external inputs)
Data: Updated ALEPH [Davier et al (2008)]. Warning: Correlations due to unfolding missing in this data set. Experimental errors potentially underestimated!
conclusions
Decision in favor of FOPT or CIPT depends on the higher order coefficients.

Some moments are more suitable for the extraction of $\alpha_s$.

The pinched moments with a “1” and without an “x” are ideal:
- Good convergence of FOPT (RM) or CIPT (AM) at low orders

Moments composed only by powers of “x” should be avoided:
- problems in the convergence of both FOPT and CIPT,
- power corrections are too important.

Some of the recent extractions of $\alpha_s$ employed moments that are not optimal. similar conclusion also in Maltman & Yavin 2008

Conformal mapping: Promising strategy to deal with the RG improvement of pt. series Abbas, Ananthanarayan, Caprini, and Fischer, PRD 88 034026 (2013)
A new determination of $\alpha_s$ from $\tau$ decays

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A new determination of $\alpha_s$ from $\tau$ decays

extra
why FOPT is better in the reference model

Reference model

Separating the contributions in FOPT

\[
\delta_{\text{FO}, w_i}^{(0)}(s) = \sum_{n=1}^{\infty} \left[ c_{n,1} \delta^{\text{tree}}_{w_i} + g_n[w_i] \right] a(s_0)^n
\]

\[
g_n[w_i] = \sum_{k=2}^{n} k c_{n,k} J_{k-1}^{\text{FO}, w_i}
\]

Result at \(\alpha_s^n\). FOPT sums the first \(n\) rows. Important cancellations.

\[
\begin{array}{cccccccccc}
\alpha_s^n & c_{1,1} & c_{2,1} & c_{3,1} & c_{4,1} & c_{5,1} & c_{6,1} & c_{7,1} & c_{8,1} & g_n & \frac{c_n + g_n}{c_n} \\
1 & 1 & & & & & & & & & 1 \\
2 & g_2 & 3.56 & + & 1.64 & & & & & & 3.56 & 3.17 \\
3 & g_3 & 8.31 & + & 11.7 & + & 6.37 & & & & 20.0 & 4.14 \\
4 & g_4 & -20.6 & + & 30.5 & + & 68.1 & + & 49.1 & & 78 & 2.59 \\
& \vdots & & & & & & & & & \vdots & \vdots \\
6 & g_6 & -2924 & -2858 & -2280 & 2214 & 5041 & & 3275 & & -807 & 0.754 \\
& \vdots & & & & & & & & & \vdots & \vdots \\
8 & g_8 & 14652 & -29552 & -145846 & -502719 & -393887 & 260511 & 467787 & 388442 & -329054 & 0.153 \\
\end{array}
\]

CIPT sums the first \(n\) columns to all orders. Misses the cancellations.

\(\tau\) spectral function moments: pt. exp. and \(\alpha_s\) PhiPsi 2013, Roma, La Sapienza

Beneke & Jamin ‘08