Lepton mixing under the lepton charge nonconservation, neutrino masses and oscillations and the "forbidden" decay $\mu^- \rightarrow e^- + \gamma$

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1. Lepton charge nonconservation and mixing of leptons with different flavors

- The lepton charge nonconservation (nonconservation of the lepton numbers L_e, L_μ, L_τ) leads to the mixing of the electron, muon and tau neutrinos, which manifests itself in the spatial oscillations at the neutrino beam propagation in vacuum [B. Kayser, Phys. Lett. **B 667**, 163 (2008)]. At the same time, the lepton-charge non-conserving interaction should also weakly mix the ordinary leptons with the same electric charge (e^-, μ^-, τ^- , as well as e^+, μ^+, τ^+) and should be, in particular, the cause of the nonzero probabilities of the μ^- decay into the electron and γ quantum and the μ^+ decay into the positron and γ quantum, which are forbidden under the lepton charge conservation .

- Let us emphasize that, in the framework of the scheme under consideration, the total lepton number $L = L_e + L_\mu + L_\tau$ is conserved.

- It is accepted to take the lepton numbers of the electron and electron neutrino to be equal to $L_e = +1,\ L_\mu = 0,\ L_\tau = 0$, those of the negative muon and muon neutrino – to $L_e = 0,\ L_\mu = +1,\ L_\tau = 0$, and those of the τ^- lepton and tau neutrino – to $L_e = 0,\ L_\mu = 0,\ L_\tau = +1$. For antiparticles, the respective lepton numbers have the opposite sign : $L_e = -1,\ L_\mu = 0,\ L_\tau = 0$ for the positron and electron antineutrino, $L_e = 0,\ L_\mu = -1,\ L_\tau = 0$ for the positive muon and muon antineutrino, and $L_e = 0,\ L_\mu = 0,\ L_\tau = -1$ for the τ^+ lepton and tau antineutrino .

2. Mass matrix and neutrino states with the definite masses

- Taking into account the *CP* invariance (*T* invariance), the mass matrix for the neutrino family should be symmetric and, due to hermiticity, real. It has the general structure of the form :

$$\hat{m}^{(\nu)} = \begin{pmatrix} m_{ee}^{(\nu)} & m_{e\mu}^{(\nu)} & m_{e\tau}^{(\nu)} \\ m_{\mu e}^{(\nu)} & m_{\mu\mu}^{(\nu)} & m_{\mu\tau}^{(\nu)} \\ m_{\tau e}^{(\nu)} & m_{\tau\mu}^{(\nu)} & m_{\tau\tau}^{(\nu)} \end{pmatrix} \qquad (1)$$

- The diagonal elements of this mass matrix have the meaning of the masses of electron neutrino ($m_{ee}^{(\nu)} \equiv m_{e}^{(\nu)}$) , muon neutrino ($m_{\mu\mu}^{(\nu)} \equiv m_{\mu}^{(\nu)}$) and tau neutrino ($m_{\tau\tau}^{(\nu)} \equiv m_{\tau}^{(\nu)}$) , whereas the nondiagonal elements ($m_{e\mu}^{(\nu)} \equiv m_{\mu e}^{(\nu)}, \ m_{e\tau}^{(\nu)} \equiv m_{\tau e}^{(\nu)}, \ m_{\mu\tau}^{(\nu)} \equiv m_{\tau\mu}^{(\nu)}$) characterize the degree of lepton charge nonconservation. {Due to the CPT invariance, the mass matrices for neutrinos and antineutrinos should coincide ($\hat{m}^{(\bar{\nu})} = \hat{m}^{(\nu)}$)} .

- In doing so, the states with the definite lepton charge (“flavor”) $|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle$ are connected with the stationary states $|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle$, being related with the definite masses m_1 , m_2 , m_3, by the following unitary transformation :

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = \hat{U} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix} \qquad (2)$$

- Let us note that, due to *T* invariance , the unitary matrix $\hat{U}$ is real. This means that the inverse matrix $(\hat{U})^{-1}$ coincides with the transposed primary one ($(\hat{U})^{-1}_{ie} = \hat{U}_{ei}$). Thus, the states with the definite lepton charge represent the coherent superpositions of stationary states, and the stationary states represent the superpositions of states with the definite lepton charge, having the same coefficients :

$$|\nu_e\rangle = \sum_{i=1}^{3} U_{ei}\,|\nu_i\rangle \ , \quad |\nu_\mu\rangle = \sum_{i=1}^{3} U_{\mu i}\,|\nu_i\rangle , \quad |\nu_\tau\rangle = \sum_{i=1}^{3} U_{\tau i}\,|\nu_i\rangle \ ; \qquad (3)$$

$$|\nu_i\rangle = U_{ei}\,|\nu_e\rangle + U_{\mu i}\,|\nu_\mu\rangle + U_{\tau i}\,|\nu_\tau\rangle \ . \qquad (4)$$

- The elements of the unitary matrix $\hat{U}$ in Eqs. (3) and (4) are scalar products of the neutrino states with the definite lepton charge and neutrino stationary states :

$$U_{ei} = \langle \nu_e | \nu_i \rangle \;, \quad U_{\mu i} = \langle \nu_\mu | \nu_i \rangle \;, \quad U_{\tau i} = \langle \nu_\tau | \nu_i \rangle \;. \qquad (5)$$

It is obvious that the neutrino stationary states – as the eigenstates of the mass matrix, corresponding to the different masses m_1 , m_2 and m_3 – are mutually orthogonal :

$$\langle \nu_1 | \nu_2 \rangle = \langle \nu_2 | \nu_3 \rangle = \langle \nu_1 | \nu_3 \rangle = 0 \;, \qquad (6)$$

in accordance with the unitarity condition, due to which the following equality holds:

$$\langle \nu_i | \nu_k \rangle = U_{ei}\, U_{ek} + U_{\mu i}\, U_{\mu k} + U_{\tau i}\, U_{\tau k} = \delta_{ik}, \quad i, k = 1,\ 2,\ 3 \;. \qquad (7)$$

3. Neutrino oscillations

- Just the difference of masses of neutrino stationary states is the cause of neutrino oscillations .
- If a neutrino is generated, at a fixed energy *E*, in the state $|\nu_l\rangle$ with the definite lepton charge ("flavor") ($|\nu_l\rangle$ – electron, muon or tau neutrino), then, at the finite distance *L* from the point of generation, it turns into the superposition of the form (see Eqs. (3), (4)) :

$$|\nu_l\rangle_L = \sum_{i=1}^{3} (U_{l\,i} U_{ei} |\nu_e\rangle + U_{l\,i} U_{\mu i} |\nu_\mu\rangle + U_{l\,i} U_{\tau i} |\nu_\tau\rangle)\; e^{i\frac{p_i L}{\hbar}} , \quad (8)$$

where p_i is the momentum of the stationary neutrino $|\nu_i\rangle$ with the mass m_i at the energy *E* . Since neutrinos are in fact ultrarelativistic particles, we may write:

$$p_i = \frac{E}{c} - \frac{m_i^2 c^3}{2\,E} . \quad (9)$$

(In relations (8) and (9) , *c* is the velocity of light in vacuum and $\hbar$ is the Planck constant) .

Then the amplitude of transition of the electron neutrino into the muon one at the distance L from the generation point will have the form :

$$A(|\nu_e\rangle \to |\nu_\mu\rangle)_L = e^{i\frac{EL}{\hbar c}} \Big[U_{e1} U_{\mu 1} \exp\left(-i \frac{m_1^2 c^3 L}{2 \hbar E}\right) + + U_{e2} U_{\mu 2} \exp\left(-i \frac{m_2^2 c^3 L}{2 \hbar E}\right) + U_{e3} U_{\mu 3} \exp\left(-i \frac{m_3^2 c^3 L}{2 \hbar E}\right) \Big] . \qquad (10)$$

Taking into account the unitarity condition, which implies the equality :

$$\left(\sum_i U_{ei} U_{\mu i}\right)^2 = \sum_i U_{ei}^2 U_{\mu i}^2 + 2 \sum_{i<k} \sum_k U_{ei} U_{\mu i} U_{ek} U_{\mu k} = 0 \quad , \qquad (11)$$

the probability of transition of the electron neutrino into the muon one at the distance L from the generation point is as follows :

$$\begin{aligned} W(|\nu_e\rangle \to |\nu_\mu\rangle)_L &= |A(|\nu_e\rangle \to |\nu_\mu\rangle)_L|^2 = \\ &= \sum_i U_{ei}^2 U_{\mu i}^2 + 2 \sum_{i<k} \sum_k U_{ei} U_{\mu i} U_{ek} U_{\mu k} \cos\left(\frac{(m_i^2 - m_k^2) c^3 L}{2 \hbar E}\right) = \\ &= -4 U_{e1} U_{\mu 1} U_{e2} U_{\mu 2} \sin^2\left(\frac{(m_1^2 - m_2^2) c^3 L}{4 \hbar E}\right) - 4 U_{e1} U_{\mu 1} U_{e3} U_{\mu 3} \sin^2\left(\frac{(m_1^2 - m_3^2) c^3 L}{4 \hbar E}\right) - \\ &- 4 U_{e2} U_{\mu 2} U_{e3} U_{\mu 3} \sin^2\left(\frac{(m_2^2 - m_3^2) c^3 L}{4 \hbar E}\right) . \end{aligned} \qquad (12)$$

- In doing so, the arguments of oscillating terms in Eq. (12) may be presented in the form :

$$\frac{1.27\ (m_i^2 - m_k^2)}{E}\ L$$
,

where the masses are given in $\frac{\text{eV}}{c^2}$, the energy – in GeV, and the distance *L* – in kilometers. The respective periods of spatial oscillations are equal to

$$L_0^{(i,k)} = \frac{2\pi\, E}{1.27\ (m_i^2 - m_k^2)} = 4.94\,\frac{E}{m_i^2 - m_k^2}\ \ \text{km}$$
.

Let us remark that if at a given distance *L* the condition $\frac{|\, m_1^2 - m_2^2\, |\, c^3 L}{4\,\hbar\, E} << 1$ is satisfied and, meanwhile, we have

$$\frac{|\, m_1^2 - m_3^2\, |\, c^3 L}{4\,\hbar\, E} \approx \frac{|\, m_2^2 - m_3^2\, |\, c^3 L}{4\,\hbar\, E} \sim 1$$
,

then formula (12) is simplified :

$$W(|\,\nu_e\rangle \to |\,\nu_\mu\rangle\,)_L = -4\,(U_{e1}\,U_{\mu 1} + U_{e2}\,U_{\mu 2})\ U_{e3}\,U_{\mu 3}\,\sin^2\left(\frac{(m_1^2 - m_3^2)\,c^3 L}{4\,\hbar\, E}\right) =$$

$$= 4\,U_{e3}^2\,U_{\mu 3}^2\,\sin^2\left(\frac{(m_1^2 - m_3^2)\,c^3 L}{4\,\hbar\, E}\right) \quad , \tag{13}$$

since, owing to the unitarity condition for the matrix $\hat{U}$,

$$U_{e1}\,U_{\mu 1} + U_{e2}\,U_{\mu 2} = -U_{e3}\,U_{\mu 3} \quad . \tag{14}$$

- In doing so,

$$W(|\nu_e\rangle \to |\nu_e\rangle)_L = 1 - 4U_{e3}^2 U_{\mu 3}^2 \sin^2\left(\frac{(m_1^2 - m_3^2)\, c^3 L}{4\hbar E}\right) \quad . \qquad (15)$$

Formula (15) describes, in particular, the decrease of intensity of the beam of reactor antineutrino with energy around several MeV at comparatively small distances from the reactor on account of the transition of the electron antineutrino into the muon one, which is “sterile” below the threshold of meson production .

Analogously, the probability of transition of the electron neutrino into the tau neutrino at the distance *L* from the point of generation is described by Eq. (12) with the replacements:

$$U_{\mu 1} \to U_{\tau 1}\ ,\quad U_{\mu 2} \to U_{\tau 2}\ ,\quad U_{\mu 3} \to U_{\tau 3}$$

.

- Meantime, the probability of the event that the electron neutrino does not change its “flavor” at the distance L from the point of generation amounts to :

$$W(|\nu_e\rangle \to |\nu_e\rangle)_L = \sum_i U_{ei}^4 + 2\sum_{i<k}\sum_k U_{ei}^2 U_{ek}^2 \cos\left(\frac{(m_i^2 - m_k^2)\,c^3 L}{2\,\hbar\,E}\right) =$$
$$= 1 - 4U_{e1}^2 U_{e2}^2 \sin^2\left(\frac{(m_1^2 - m_2^2)\,c^3 L}{4\,\hbar\,E}\right) - 4U_{e1}^2 U_{e3}^2 \sin^2\left(\frac{(m_1^2 - m_3^2)\,c^3 L}{4\,\hbar\,E}\right) -$$
$$- 4U_{e2}^2 U_{e3}^2 \sin^2\left(\frac{(m_2^2 - m_3^2)\,c^3 L}{4\,\hbar\,E}\right) \quad . \qquad (16)$$

Here, we have taken into account that, in accordance with the unitarity condition,

$$\left(\sum_i U_{ei}^2\right)^2 = 1 \qquad .$$

- Just as one should expect, the unitarity relation

$$U_{ei}\,U_{ek} + U_{\mu i}\,U_{\mu k} + U_{\tau i}\,U_{\tau k} = \delta_{ik}$$

ensures the equality

$$W(|\nu_e\rangle \to |\nu_e\rangle)_L + W(|\nu_e\rangle \to |\nu_\mu\rangle)_L + W(|\nu_e\rangle \to |\nu_\tau\rangle)_L = 1 \quad .$$

- According to the relations (11), (12) and (16), the admixtures of muon and tau neutrinos, averaged over the spatial oscillations (over the energy spectrum at a given distance *L*) , are, respectively, as follows :

$$\overline{W(|\nu_e\rangle \to |\nu_\mu\rangle)} = U_{e1}^2\,U_{\mu 1}^2 + U_{e2}^2\,U_{\mu 2}^2 + U_{e3}^2\,U_{\mu 3}^2 \quad , \qquad (17)$$

$$\overline{W(|\nu_e\rangle \to |\nu_\tau\rangle)} = U_{e1}^2\,U_{\tau 1}^2 + U_{e2}^2\,U_{\tau 2}^2 + U_{e3}^2\,U_{\tau 3}^2 \quad , \qquad (18)$$

and the average intensity of the beam of electron neutrinos, attenuated as a result of spatial oscillations, is proportional to :

$$\overline{W(|\nu_e\rangle \to |\nu_e\rangle)} = U_{e1}^4 + U_{e2}^4 + U_{e3}^4 \quad . \qquad (19)$$

It can be shown that the absolute minimum of the quantity $\overline{W(|\nu_e\rangle \to |\nu_e\rangle)}$ is equal to ⅓ – in accordance with the number of stationary neutrinos $n = 3$.

4. Nondiagonal elements of the mass matrix and their connection with the differences of neutrino masses

- By applying relations (3), the nondiagonal elements of the mass matrix can be expressed through the differences of masses of neutrino stationary states and the elements of the unitary matrix $\hat{U}$. Indeed, it is easy to see that :

$$m_{e\mu}^{(\nu)} = \langle \nu_e | \hat{m}^{(\nu)} | \nu_\mu \rangle = \sum_{i=1}^{3} U_{ei} U_{\mu i} m_i , \quad m_{e\tau}^{(\nu)} = \langle \nu_e | \hat{m}^{(\nu)} | \nu_\tau \rangle = \sum_{i=1}^{3} U_{ei} U_{\tau i} m_i ,$$

$$m_{\mu\tau}^{(\nu)} = \langle \nu_\mu | \hat{m}^{(\nu)} | \nu_\tau \rangle = \sum_{i=1}^{3} U_{\mu i} U_{\tau i} m_i , \qquad (20)$$

where m_i is the neutrino mass in the stationary state $|\nu_i\rangle$, as before .

Taking into account the unitarity condition: $\sum_{i=1}^{3} U_{ei} U_{\mu i} = 0$,

we obtain the following expressions for the mass matrix element :

$$m_{e\mu}^{(\nu)} = U_{e1} U_{\mu 1}(m_1 - m_2) + U_{e3} U_{\mu 3}(m_3 - m_2) , \qquad (21)$$

or

$$m_{e\mu}^{(\nu)} = U_{e2} U_{\mu 2}(m_2 - m_1) + U_{e3} U_{\mu 3}(m_3 - m_1) . \qquad (22)$$

Analogous expressions may be obviously written for the mass matrix elements $m_{e\tau}^{(\nu)}$ and $m_{\mu\tau}^{(\nu)}$.

- In terms of the mixing angles for the neutrino stationary states, which are formally analogous to the Maiani angles introduced for the description of mixing of "lower" quarks *d*, *s* and *b* [Kayser,2008; Goodman,2008; Okun,1990; Maiani,1976], the matrix $\hat{U}$ can be presented in the form :

$$\hat{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} \\ 0 & 1 & 0 \\ -\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (23)$$

According to (23), we have :

$$U_{e1} = \cos\theta_{12}\cos\theta_{13}\,, \quad U_{\mu 1} = -\cos\theta_{12}\cos\theta_{23} - \cos\theta_{12}\sin\theta_{12}\sin\theta_{23}\sin\theta_{13}\,, \quad (24)$$

$$U_{e3} = \sin\theta_{13}\,, \qquad U_{\mu 3} = \sin\theta_{23}\cos\theta_{13} \quad .$$

As a result, Eq. (21) gives :

$$m_{e\mu}^{(\nu)} = \frac{1}{2}[\,(\sin 2\theta_{12}\cos\theta_{23}\cos\theta_{13} + \cos^2\theta_{12}\sin\theta_{23}\sin 2\theta_{13}\,)(m_2 - m_1) \; + \\ + \; \sin 2\theta_{13}\sin\theta_{23}\,(m_3 - m_2) \;] \;. \quad (25)$$

Formula (25), determining the matrix element $m_{e\mu}^{(\nu)}$, incorporates the values of differences of stationary neutrino masses. Meantime, the experimental data on neutrino oscillations contain the information only on the differences of squares of masses.

- If the moduli of differences of mass squares are very small as compared with the square of each of the masses (which seems to be plausible), then the masses of all the three stationary neutrinos may be assumed to be approximately equal to each other:

$$m_1^{(\nu)} \approx m_2^{(\nu)} \approx m_3^{(\nu)} \approx m^{(\nu)} \quad . \qquad (26)$$

In this situation, the moduli of all the differences of masses are very small as compared with the common neutrino mass $m^{(\nu)}$:

$$|m_1 - m_2| << m^{(\nu)}, \quad |m_3 - m_2| << m^{(\nu)} \quad . \qquad (27)$$

- Within this approximation, the differences of stationary neutrino masses are determined according to the formulas { The experimental data on oscillations [Kayser,2008] testify to the fact that $|m_2^2 - m_1^2| << |m_3^2 - m_2^2|$. This means that the difference of masses of the first and second stationary neutrinos is very small as compared with their distinction from the third stationary neutrino, which is itself also relatively small } :

$$m_1 - m_2 = \frac{m_1^2 - m_2^2}{2m^{(\nu)}}, \qquad m_3 - m_2 = \frac{m_3^2 - m_2^2}{2m^{(\nu)}} \qquad (28)$$

Taking into account relations (28), we may rewrite Eq. (25) in the form :

$$m_{e\mu}^{(\nu)} = \frac{1}{4m^{(\nu)}}[\,(\sin 2\theta_{12}\cos\theta_{23}\cos\theta_{13} + \cos^2\theta_{12}\sin\theta_{23}\sin 2\theta_{13}\,)(m_2^2 - m_1^2) + \sin 2\theta_{13}\sin\theta_{23}\,(m_3^2 - m_2^2)\,] \quad . \qquad (29)$$

5. States of charged leptons with the definite masses

- Taking into account the lepton-charge non-conserving interaction, the mass matrix for the family of leptons, including the electron, the negative muon and the τ^- lepton, has the form being analogous to the mass matrix for the neutrino family :

$$\hat{M} = \begin{pmatrix} M_{ee} & M_{e\mu} & M_{e\tau} \\ M_{\mu e} & M_{\mu\mu} & M_{\mu\tau} \\ M_{\tau e} & M_{\tau\mu} & M_{\tau\tau} \end{pmatrix} . \qquad (30)$$

The diagonal elements of the mass matrix $\hat{M}$ are equal to the masses of electron ($M_{ee} \equiv M_e,\ \ L_e = +1,\ L_\mu = 0,\ L_\tau = 0$), negative muon ($M_{\mu\mu} \equiv M_\mu$, $L_e = 0,\ L_\mu = +1,\ L_\tau = 0$) and τ^- lepton ($M_{\tau\tau} \equiv M_\tau$, $L_e = 0,\ L_\mu = 0,\ L_\tau = +1$) , whereas the nondiagonal elements, being responsible for the lepton charge nonconservation, are negligibly small as compared with the electron mass M_e and, all the more, as compared with all the differences of masses $(M_\mu - M_e),\ (M_\tau - M_e),\ (M_\tau - M_\mu)$. Just the same mass matrix corresponds to the family of antileptons, incorporating the positron ($L_e = -1,\ L_\mu = 0,\ L_\tau = 0$), the positive muon ($L_e = 0,\ L_\mu = -1,\ L_\tau = 0$), and the τ^+ lepton ($L_e = 0,\ L_\mu = 0,\ L_\tau = -1$) .

- Due to T invariance, the Hermitian matrix $\hat{M}$ should be symmetric and, hence, real :

$$M_{e\mu} = M_{\mu e}, \quad \mathrm{Im}\, M_{e\mu} = 0 \;; \qquad M_{e\tau} = M_{\tau e}, \quad \mathrm{Im}\, M_{e\tau} = 0 \;; \qquad (31)$$

$$M_{\mu\tau} = M_{\tau\mu}, \quad \mathrm{Im}\, M_{\mu\tau} = 0 \quad .$$

- Within the perturbation theory first-order approximation, the stationary states of leptons represent the superpositions of states with different lepton charges :

$$|e'\rangle = |e\rangle + \varepsilon_{\mu e}\, |\mu\rangle_e + \varepsilon_{\tau e}\, |\tau\rangle_e \quad ,$$

$$|\mu'\rangle = |\mu\rangle + \varepsilon_{e\mu}\, |e\rangle_\mu + \varepsilon_{\tau\mu}\, |\tau\rangle_\mu \quad , \qquad (32)$$

$$|\tau'\rangle = |\tau\rangle + \varepsilon_{e\tau}\, |e\rangle_\tau + \varepsilon_{\mu\tau}\, |\mu\rangle_\tau \quad .$$

- The stationary states, denoted by prime, are related with different masses. In doing so, these masses practically coincide with the masses of leptons :

$$M_{e'} \approx M_e, \quad M_{\mu'} \approx M_\mu, \quad M_{\tau'} \approx M_\tau, \qquad (33)$$

and the coefficients of mixing of states with different lepton charges are expressed through the ratios of nondiagonal elements of the mass matrix to the differences of masses of respective leptons.

Indeed, neglecting the second-order terms over the lepton-charge non-conserving interaction, we find :

$$\varepsilon_{e\mu} = -\varepsilon_{\mu e} = \frac{M_{e\mu}}{M_{\mu} - M_{e}} \quad , \quad |\varepsilon_{e\mu}| << 1 \qquad ;$$

$$\varepsilon_{e\tau} = -\varepsilon_{\tau e} = \frac{M_{e\tau}}{M_{\tau} - M_{e}} \quad , \quad |\varepsilon_{e\tau}| << 1 \qquad ; \qquad (34)$$

$$\varepsilon_{\mu\tau} = -\varepsilon_{\tau\mu} = \frac{M_{\mu\tau}}{M_{\tau} - M_{\mu}} \quad , \quad |\varepsilon_{\mu\tau}| << 1 \qquad .$$

Let us note that, taking into account the small values of mixing coefficients, relations (34) follow also from the expressions being analogous to Eqs. (21) and (22) for neutrinos .

In Eqs. (32), the symbols $|\mu\rangle_e$ and $|\tau\rangle_e$ denote the "muonic" (L_μ = 1) and "tau-leptonic" (L_τ = 1) states included into the stationary superposition with the electron mass M_e , the symbols $|e\rangle_\mu$ and $|\tau\rangle_\mu$ denote the "electronic" (L_e = 1) and "tau-leptonic" (L_τ = 1) states included into the stationary superposition with the muon mass M_μ , and the symbols $|e\rangle_\tau$ and $|\mu\rangle_\tau$ denote the "electronic" (L_e = 1) and "muonic" (L_μ = 1) states included into the stationary superposition with the τ-lepton mass M_τ .

6. Coefficient of mixing of the states with the lepton charges $L_e = 1$ and $L_\mu = 1$ and probability of the decay $\mu^- \to e^- + \gamma$

- Let us estimate the probability of the radiative decay $\mu^- \to e^- + \gamma$, being forbidden under the lepton charge conservation, assuming that this decay occurs on account of "admixture" of the state $|e\rangle_\mu$ with the electronic lepton number $L_e = 1$ and the muon mass to the negative muon . Then the probability of decay $\mu^- \to e^- + \gamma$ per unit time will be as follows :

$$W(\mu^- \to e^- + \gamma) = |\varepsilon_{e\mu}|^2 \, W(|e\rangle_\mu \to e^- + \gamma) \quad , \qquad (35)$$

- where $\varepsilon_{e\mu}$ is the mixing coefficient included in the second formula in Eqs. (32) .

- Meantime, the differential probability of decay of the “heavy electron” with mass M_μ into the ordinary electron with momentum

$$\mathbf{P} = \frac{M_\mu^2 - M_e^2}{2M_\mu} c\,\mathbf{n} \qquad (36)$$

($\mathbf{n}$ is the unit vector along the momentum) and the γ quantum with energy

$$E_\gamma = \frac{M_\mu^2 - M_e^2}{2M_\mu} c^2 \qquad (37)$$

per unit time can be calculated according to the standard formula of quantum electrodynamics [Berestetsky, Lifshitz, Pitaevsky, 1983] :

$$d\,W(|e\rangle_\mu \to e^- + \gamma) = \frac{e^2}{\hbar c}\frac{\omega}{2\pi} K\ |\boldsymbol{\chi}\,\psi_{\mathbf{p}}^+\,\boldsymbol{\alpha}\,\varphi_0|^2\,d\Omega_{\mathbf{n}} \quad , \qquad (38)$$

where $\frac{e^2}{\hbar c} = \frac{1}{137}$ is the fine structure constant, $\omega = \frac{E_\gamma}{\hbar}$ is the frequency of γ quantum, χ is the vector of polarization of γ quantum, $\psi_{\mathbf{p}}$ is the Dirac bispinor describing the ordinary electron with mass M_e and momentum $\mathbf{p}$, α is the four-row Dirac matrix ($\hat{\alpha} = \begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix}$ [Berestetsky, Lifshitz, Pitaevsky, 1983]) , φ_0 is the Dirac bispinor corresponding to the resting “heavy electron” with mass M_μ , $K = 1 - \frac{E_\gamma}{M_\mu c^2}$.

- Passing to the two-row Pauli matrices $\hat{\boldsymbol{\sigma}}$, we may write :

$$\psi_{\mathbf{p}} = \begin{pmatrix} u_0 \\ \dfrac{(\boldsymbol{\sigma}\mathbf{p})\, u_0}{E_e + M_e c^2} \end{pmatrix} \left(\frac{E_e + M_e c^2}{2E_e} \right)^{1/2}, \quad \varphi_0 = \begin{pmatrix} v_0 \\ 0 \end{pmatrix} \quad , \tag{39}$$

where u_0 and v_0 are the two-row spinors, normalized by unity, and

$$E_e = (|\mathbf{p}|^2 c^2 + M_e^2 c^4)^{1/2} = M_\mu c^2 - E_\gamma = \frac{M_\mu^2 + M_e^2}{2M_\mu} c^2 \; . \tag{40}$$

- In doing so, we have :

$$\chi \psi_{\mathbf{p}}^{+} \boldsymbol{\alpha} \varphi_0 = \left(u_0^{+} \frac{(\chi \hat{\sigma})(\hat{\sigma} \mathbf{p})\, c}{E_e + M_e c^2} \; v_0 \right) \left(\frac{E_e + M_e c^2}{2E_e} \right)^{1/2} \quad . \tag{41}$$

Taking into account that the final electron is ultrarelativistic ($E_\gamma \approx E_e \approx \dfrac{M_\mu c^2}{2}, \; K \approx \dfrac{1}{2}$), we obtain the following expression – after averaging the differential probability of emission over the polarizations of the "heavy electron" and summing over the polarizations of the γ quantum and final ordinary electron :

$$d\,W(|e\rangle_\mu \to e^- + \gamma) = \frac{e^2}{\hbar c} \frac{\omega}{4\pi} \left[\frac{1}{2} tr \sum_{\chi} (\hat{\sigma}\chi)(\hat{\sigma}\,\mathbf{n})(\hat{\sigma}\,\mathbf{n})(\hat{\sigma}\chi) \right] \frac{1}{2} d\Omega_{\mathbf{n}} = \frac{1}{8\pi} \frac{e^2}{\hbar c} \frac{M_\mu c^2}{\hbar} d\Omega_{\mathbf{n}} \quad . \tag{42}$$

- The total probability of radiative decay of the "heavy electron" per unit time amounts to :

$$W(|e\rangle_{\mu} \rightarrow e^{-} + \gamma) = \frac{e^{2}}{2\,\hbar c}\,\frac{M_{\mu}\,c^{2}}{\hbar} \quad . \qquad (43)$$

Taking into account the numerical values

$$\left(M_{\mu}\,c^{2} = 105.6\ \text{MeV},\quad \frac{e^{2}}{\hbar c} = \frac{1}{137} \right),$$

the probability of radiative decay of the electronic state with mass M_{μ} (per 1 sec.) equals

$$W(|e\rangle_{\mu} \rightarrow e^{-} + \gamma) = \frac{105.6\cdot 10^{6}\cdot 1.6\cdot 10^{-12}}{2\cdot 1.054\cdot 10^{-27}\cdot 137} = 0.583\cdot 10^{21}\ \sec^{-1} \quad . \qquad (44)$$

According to relations (35) and (38) ,

$$W(\mu^{-} \rightarrow e^{-} + \gamma) = W(\mu^{+} \rightarrow e^{+} + \gamma) =$$

$$= \left(\frac{M_{e\mu}}{M_{\mu} - M_{e}} \right)^{2} \frac{e^{2}}{2\,\hbar c}\,\frac{M_{\mu}\,c^{2}}{\hbar} \approx \frac{M_{e\mu}^{2}}{2\,M_{\mu}}\,\frac{e^{2}}{\hbar c}\,\frac{c^{2}}{\hbar} \quad . \qquad (45)$$

- As follows from the experimental data [Review of Particle Physics, B667, 2008, p. 34, 36] ,

$$W(\mu^- \to e^- + \gamma) < 1.2 \cdot 10^{-11} \; W(\mu^- \to e^- + \overline{\nu}_e + \nu_\mu) \qquad (46)$$

where $W(\mu^- \to e^- + \overline{\nu}_e + \nu_\mu)$ is the probability of decay of the negative muon into the electron, electron antineutrino and muon neutrino per unit time, coinciding practically with the inverse lifetime of the muon :

$$W(\mu^- \to e^- + \overline{\nu}_e + \nu_\mu) = \frac{1}{\tau_\mu} = 0.455 \cdot 10^6 \; \text{sec}^{-1} \quad .$$

Thus, in accordance with the experimental restriction (46), we obtain :

$$W(\mu^- \to e^- + \gamma) < 0.546 \cdot 10^{-5} \; \text{sec}^{-1} \quad . \qquad (47)$$

Taking into account Eqs. (35), (43) and (45), this means that:

$$|\varepsilon_{e\mu}|^2 < 0.936 \cdot 10^{-26}, \quad |M_{e\mu}|^2 < 1.032 \cdot 10^{-10} \; \frac{\text{eV}^2}{c^4},$$

$$M_{e\mu} < 1.016 \cdot 10^{-5} \; \frac{\text{eV}}{c^2} \quad . \qquad (48)$$

7. Hypothesis on the equality of nondiagonal elements of the mass matrices for neutrinos and charged leptons and the estimate of the lower bound of neutrino mass

- Let us suppose that the mixing of ordinary leptons (e , μ, τ) and the mixing of neutrinos (ν_e, ν_μ, ν_τ) are conditioned by the same lepton-charge non-conserving interaction. Under this natural assumption, the non-diagonal elements of the three-row mass matrix for the lepton family should coincide with those of the three-row mass matrix for neutrinos :

$$M_{e\mu} = m_{e\mu}^{(\nu)}, \quad M_{e\tau} = m_{e\tau}^{(\nu)}, \quad M_{\mu\tau} = m_{\mu\tau}^{(\nu)} \quad . \qquad (49)$$

Thus, we will assume that the matrix element $M_{e\mu}$, included into formula (45). may be replaced by the matrix element $m_{e\mu}^{(\nu)}$ corresponding to the neutrino family.

- Then, taking into account Eq. (29) and inequalities (48), we obtain the relation for estimating the lower bound of the neutrino mass :

$$|M_{e\mu}| = |m_{e\mu}^{(\nu)}| = \frac{1}{4m^{(\nu)}} \, | \, (\sin 2\theta_{12} \cos\theta_{23} \cos\theta_{13} + \cos^2\theta_{12} \sin\theta_{23} \sin 2\theta_{13})(m_2^2 - m_1^2) \; + \\ + \; \sin 2\theta_{13} \sin\theta_{23} \, (m_3^2 - m_2^2) \, | \; < 1.016 \cdot 10^{-5} \; \frac{\text{eV}}{c^2} \quad . \tag{50}$$

- According to the experimental data on neutrino oscillations [Review of Particle Physics, B667, 2008, p. 34, 36] ,

$$\sin^2 2\theta_{12} = 0.86^{+0.08}_{-0.04} \; , \quad \sin^2 2\theta_{23} > 0.92 \, , \quad \sin^2 2\theta_{13} < 0.19 \; ;$$

$$|m_2^2 - m_1^2| = (8.0 \pm 0.3) \cdot 10^{-5} \; \frac{\text{eV}^2}{c^4} \; , \quad |m_3^2 - m_2^2| = (1.9 \div 3) \cdot 10^{-3} \; \frac{\text{eV}^2}{c^4} \; .$$

- Assuming, respectively, that

$$\sin^2 2\theta_{12} = 0.86 \quad (\theta_{12} = 34°) \;, \quad \sin^2 2\theta_{23} = 0.92 \quad (\theta_{23} = 36.8°) \,, \quad \theta_{13} = 0° \;,$$

$$|m_2^2 - m_1^2| = 8 \cdot 10^{-5} \; \frac{\text{eV}^2}{c^4}$$

we find :

$$m^{(\nu)} > \frac{0.927 \cdot 0.8 \cdot 8}{4 \cdot 1.016} = 1.46 \; \frac{\text{eV}}{c^2} \quad . \qquad (51)$$

- This value for the lower bound of neutrino mass is in good agreement with the upper limit of antineutrino mass determined in the works by Lobashev et al. [Phys. Lett. B460, 227, 1999]

$$\left(\quad m^{(\nu)} < 2.3 \; \frac{\text{eV}}{c^2} \right)$$

and Kraus et al. [Eur. Phys. J., C40, 447, 2005]

$$\left(m^{(\nu)} < 2.5 \; \frac{\text{eV}}{c^2} \right)$$

within the study of electron spectrum in the tritium β-decay (see also [Review of Particle Physics, B667, 2008, p. 34, 36]) .

8. Concluding remarks

- Let us emphasize that our estimate for the lower bound of neutrino mass $m^{(\nu)}$ is based on the experimental data on neutrino oscillations, on the experimental restriction for the probability of decay $\mu^- \to e^- + \gamma$ per unit time and on the assumption that the nondiagonal element of the neutrino mass matrix $m_{e\mu}^{(\nu)}$, characterizing the mixing of muon and electron neutrino on account of the lepton charge nonconservation, coincides with the nondiagonal element of the mass matrix for ordinary leptons $M_{e\mu}$, characterizing the mixing of negative muon and electron – which seems natural from our point of view .
- Meantime, if $|M_{e\mu}| \neq |m_{e\mu}^{(\nu)}|$, then the value for the lower bound of neutrino mass will change as compared with the magnitude obtained above . In this case, $m^{(\nu)} > 1.46\,|\eta|\,\frac{\text{eV}}{c^2}$, where $\eta = \frac{M_{e\mu}}{m_{e\mu}^{(\nu)}}$ is the ratio of nondiagonal elements of the mass matrices for leptons and neutrinos .

- If, in further experiments, the probability of decay $\mu^- \to e^- + \gamma$ per 1 sec will be determined or the upper limit of this probability will be reduced, this will testify to the fact that the parameter $|\eta| < 1$ – since, otherwise, we would get a contradiction with the experimental data on the upper bound of neutrino mass.

- Under the choice of another set of parameters (taking into account the latest data on nonzero neutrino mixing angle θ_{13}) :

$$|m_2^2 - m_1^2| = 8 \cdot 10^{-5}\ \frac{\text{eV}^2}{c^4},\ \sin^2 2\theta_{12} = 0.86\ (\theta_{12} = 34°),\ |m_3^2 - m_2^2| = 3 \cdot 10^{-3}\ \frac{\text{eV}^2}{c^4},$$

$$\sin^2 2\theta_{13} = 0.095\ \ (\theta_{13} = 9°),\ \ \sin^2 2\theta_{23} = 0.92\ \ (\theta_{23} = 36.8°),$$

we would obtain the estimate

$$m^{(\nu)} > (12.1 \div 15.1)\ |\eta|\ \frac{\text{eV}}{c^2}\ ,$$

which may be in accordance with the presently known upper limits for the probability of decay $\mu^- \to e^- + \gamma$ per 1 sec and neutrino mass only at the ratios of moduli of matrix elements

$$\left|\frac{M_{e\mu}}{m_{e\mu}^{(\nu)}}\right| = |\eta| <\sim 10^{-1}\ .$$

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Thank you !